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Nodal Solutions of Nonlocal Boundary Value Problems

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Abstract. We study the nonlinear boundary value problem consisting of the second order differential equation on $[a, b]$ and a boundary condition involving a Riemann-Stieltjes integral. By relating it to the eigenvalues of a linear Sturm-Liouville problem with a two-point separated boundary condition, we obtain results on the existence and nonexistence of nodal solutions of this problem. We also discuss the changes of the existence of different types of nodal solutions as the problem changes.

Key words: nodal solutions, nonlinear boundary value problems, integral boundary conditions, eigenvalues of Sturm-Liouville problems, Riemann-Stieltjes integral.

1 Introduction

We are concerned with the nonlocal boundary value problem (BVP) consisting of the equation

$$-(p(t)y')' + q(t)y = w(t)f(y), \quad t \in (a, b), \tag{1.1}$$

and the boundary condition (BC)

$$\begin{aligned} &\cos\alpha \; y(a) - \sin\alpha \; (py')(a) = 0, \quad \alpha \in [0, \pi), \\ &(py')(b) - \int_a^b (py')(s)\, d\xi(s) = 0, \end{aligned} \tag{1.2}$$

where $a, b \in \mathbb{R}$ with $a < b$ and the integral in BC (1.2) is the Riemann-Stieltjes integral with respect to $\xi(s)$ with $\xi(s)$ a function of bounded variation. In the case that $\xi(s) = s$, the Riemann-Stieltjes integral in the second condition of

(1.2) reduces to the Riemann integral. In the case that $\xi(s) = \sum_{i=1}^{m} k_i \chi(s - x_i)$, where $m \geq 1$, $k_i \in \mathbb{R}$, $i = 1, \dots, m$, $\{x_i\}_{i=1}^{m}$ is a sequence of distinct points in $[a, b]$, and $\chi(s)$ is the characteristic function on $[0, \infty)$, i.e.,

$$\chi(s) = \begin{cases} 1, & s \geq 0, \\ 0, & s < 0, \end{cases}$$

the second equation in (1.2) reduces to the multi-point BC

$$(py')(b) - \sum_{i=1}^{m} k_i (py')(x_i) = 0. \tag{1.3}$$

We assume throughout, and without further mention, that the following conditions hold:

(H1) $p, q, w \in C^1[a, b]$ such that $p(t) > 0$, $w(t) > 0$, and $q'(t) + q^* \leq l(q^* - q(t))$ on $[a, b]$ with

$$q^* := \max_{t \in [a,b]} \{q(t), 0\} \quad \text{and} \quad l := \max_{t \in [a,b]} \left\{ \Big(\frac{p'(t) + q^*}{p(t)} \Big)_+, \frac{w'_-(t)}{w(t)} \right\},$$

where $h'_-(t) := \max\{0, -h'(t)\}$ and $h_+(t) := \max\{0, h(t)\}$;

(H2) $f \in C(\mathbb{R})$ such that $yf(y) > 0$ for $y \neq 0$, and f is locally Lipschitz on $(-\infty, 0) \cup (0, \infty)$;

(H3) there exist extended real numbers $f_0, f_\infty \in [0, \infty]$ such that

$$f_0 = \lim_{y \to 0} f(y)/y \quad \text{and} \quad f_\infty = \lim_{|y| \to \infty} f(y)/y.$$

Remark 1. For $p, w \in C^1[a, b]$, the following are examples of the function classes for q satisfying (H1):

(i) $q \in C^1[a, b]$ such that $q'(t) \leq -q^*$ on $[a, b]$. It is easy to see that any non-positive, non-increasing function q belongs to this class. In particular, any non-positive constant belongs to this class.

(ii) $q \in C^1[a, b]$ such that $q'(t) \leq -lq(t)$ on $[a, b]$ with $l \geq 1$. For $c \geq 0$, it is easy to see that

$$q_1(t) = ce^{-kt} \quad \text{for } t \in [0, 1] \text{ with } k \geq l \geq 1 \quad \text{and}$$

$$q_2(t) = -ce^{-kt} \quad \text{for } t \in [0, 1] \text{ with } 0 \leq k \leq l \text{ and } l \geq 1$$

belong to this class.

The existence of positive solutions of BVPs with nonlocal BCs, including three-point, multi-point, and integral BCs, have been studied extensively, see, for example, [1, 4, 5, 6, 7, 8, 9, 15, 16, 24, 26, 28, 29, 31] and the references therein. In recent years, progress has also been made to the study of nodal solutions, i.e., solutions with a specific zero-counting property in (a,b), for nonlinear BVPs consisting of Eq. (1.1) and two-point separated BCs, see [10, 11, 19, 21, 22, 23]. As regard to nonlocal BVPs, results on the existence of nodal solutions have been obtained only for the special BVP consisting of the equation

$$y'' + f(y) = 0, \quad t \in (0,1), \tag{1.4}$$

and the multi-point BC

$$y(0) = 0, \quad y(1) - \sum_{i=1}^{m} k_i y(\eta_i) = 0 \tag{1.5}$$

or the integral BC

$$y(0) = 0, \quad y(1) - g\left(\int_0^1 u(s)ds\right) = 0,$$

see Li and Li [17], Ma [18], Ma and O'Regan [20], Rynne [25], Sun, Xu, and O'Regan [27], and Xu [30]. Among them, [20] and [25] used a standard global bifurcation method to establish the existence of nodal solutions of BVP (1.4), (1.5) by relating it to the eigenvalues of the corresponding linear Sturm-Liouville problem (SLP) with BC (1.5). However, the establishment of these results relies heavily on the direct computation of the eigenvalues and eigenfunctions of the associated multi-point SLP and hence can not be extended to a general BVP with a variable coefficient function w. Moreover, to the best of the authors' knowledge, there is nothing done so far on the existence of nodal solutions of BVPs with the nonlocal BC (1.2).

In view of the fact that eigenvalues are easy to calculate for all two-point linear self-adjoint SLPs using standard software packages such as those in [2], in this paper, we obtain results on the existence and nonexistence of nodal solutions of BVP (1.1), (1.2) by relating it to the eigenvalues of an associated linear SLP with a two-point separated BC rather than a nonlocal BC. The shooting method and a generalized energy function play key roles in the proofs. We also discuss the changes of the existence of different types of nodal solutions when some parameters in the problem change, more precisely, when the interval $[a,b]$ shrinks, when the functions w, p, and q increase in certain directions, and when the boundary condition angle α changes. Note that our results are for the general BVP (1.1), (1.2) with variable w, p, and q, a separated BC at the left endpoint prescribed by an arbitrary α, and a BC given by a Riemann-Stieltjes integral with respect to $\xi(s)$.

2 Existence and Nonexistence of Nodal Solutions

We study the nodal solutions of BVP (1.1), (1.2) in the following classes.

Definition 1. A solution y of BVP (1.1), (1.2) is said to belong to class $\mathcal{S}_n^\gamma$ for $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $\gamma \in \{+, -\}$ if

(i) y has exactly n zeros in (a,b),

(ii) $\gamma y(t) > 0$ in a right-neighborhood of a.

Our results on the existence and nonexistence of nodal solutions of BVP (1.1), (1.2) are established utilizing the eigenvalues of the linear SLP consisting of the equation

$$-(p(t)y')' + q(t)y = \lambda w(t)y, \quad t \in (a, b), \tag{2.1}$$

and the two-point BC

$$\begin{aligned} &\cos\alpha\ y(a) - \sin\alpha\ (py')(a) = 0, \quad \alpha \in [0, \pi), \\ &y(b) = 0. \end{aligned} \tag{2.2}$$

It is well known that SLP (2.1), (2.2) has an infinite number of eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ satisfying

$$-\infty < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \quad \text{and} \quad \lambda_n \to \infty \text{ as } n \to \infty,$$

and any eigenfunction associated with λ_n has n simple zeros in (a, b), see [32, Theorem 4.3.2].

Note that the function $\xi(s)$ given in BC (1.2) is of bounded variation on $[a, b]$. Thus, there are two nondecreasing functions $\xi_1(s)$ and $\xi_2(s)$ such that

$$\xi(s) = \xi_1(s) - \xi_2(s), \quad s \in [a, b]. \tag{2.3}$$

In the following we assume (2.3) holds. We now present our main results with the proofs given later in this section after several technical lemmas are derived. The first theorem is about the existence of certain types of nodal solutions.

Theorem 1. *Assume either (i) $f_0 < \lambda_n$ and $\lambda_{n+1} < f_\infty$, or (ii) $f_\infty < \lambda_n$ and $\lambda_{n+1} < f_0$, for some $n \in \mathbb{N}_0$. Suppose*

$$1 - \int_a^b \sqrt{\frac{p(s)}{p(b)}}\ e^{l(b-a)/2}\ d\big(\xi_1(s) + \xi_2(s)\big) > 0. \tag{2.4}$$

Then BVP (1.1), (1.2) *has two solutions $y_{n,\gamma} \in \mathcal{S}_{n+1}^{\gamma}$ for $\gamma \in \{+, -\}$.*

Remark 2. (a) Note that for the multi-point case, i.e., BVP (1.1), (1.2) with the second condition in (1.2) replaced by (1.3), we have that $\xi(s) = \sum_{i=1}^{m} k_i \chi(s - x_i)$. Thus $\xi(s) = \xi_1(s) - \xi_2(s)$ with

$$\xi_1(s) = \sum_{i=1}^{m} (k_i)_+ \chi(s - x_i) \ \text{ and } \ \xi_2(s) = \sum_{i=1}^{m} (k_i)_- \chi(s - x_i),$$

where $(k_i)_\pm = \max\{\pm k_i, 0\}$. Hence $\xi_1(s) + \xi_2(s) = \sum_{i=1}^{m} |k_i| \chi(s - x_i)$. It is easy to see that condition (2.4) then becomes

$$1 - \sum_{i=1}^{m} |k_i| \sqrt{\frac{p(x_i)}{p(b)}}\ e^{l(b-a)/2} > 0.$$

(b) When $\xi_i \in C^1[a,b]$ for $i = 1, 2$, condition (2.4) becomes

$$1 - \int_a^b \sqrt{\frac{p(s)}{p(b)}}\ e^{l(b-a)/2}\ \Big(\xi_1'(s) + \xi_2'(s)\Big)ds > 0.$$

In particular, if $p(t) \equiv 1$, $q(t) \equiv 0$, $w(t) > 0$ is increasing, and $\xi(t) = t$, then it is reduced to $b - a < 1$.

As a consequence of Theorem 1, we have the following corollary on the existence of an infinite number of different types of nodal solutions for a special case of BVP (1.1), (1.2).

Corollary 1. Consider the special case that $p(t) \equiv 1$ and $q(t) \equiv 0$ on $[a,b]$. Assume (2.4) holds and

$$\text{either } f_0 = 0 \text{ and } f_\infty = \infty, \quad \text{or} \quad f_\infty = 0 \text{ and } f_0 = \infty.$$

Then there exists $\alpha^* \in (\pi/2, \pi)$ such that

(i) if $\alpha \in [0, \alpha^*)$, then BVP (1.1), (1.2) has a solution $y_n^\gamma \in \mathcal{S}_{n+1}^\gamma$ for each $n \geq 0$ and $\gamma \in \{+,-\}$;

(ii) if $\alpha \in [\alpha^*, \pi)$, then BVP (1.1), (1.2) has a solution $y_n^\gamma \in \mathcal{S}_{n+1}^\gamma$ for each $n \geq 1$ and $\gamma \in \{+,-\}$.

Remark 3. The number α^* in the above theorem can be explicitly computed using the fundamental solutions of (2.1) see [3, Theorem 2.2] for details.

The next theorem is about the nonexistence of certain types of nodal solutions.

Theorem 2. *(i) Assume $f(y)/y \leq \lambda_n$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP* (1.1)*,* (1.2) *has no solution in $\mathcal{S}_i^\gamma$ for all $i \geq n+1$ and $\gamma \in \{+,-\}$.*

(ii) Assume $f(y)/y \geq \lambda_n$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP (1.1)*,* (1.2) *has no solution in $\mathcal{S}_i^\gamma$ for all $i \leq n$ and $\gamma \in \{+,-\}$.*

To prove Theorem 1, we need some preliminaries. The lemmas below are on the initial value problems (IVPs) associated with Eq. (1.1) and are simple generalizations of [11, Corollary 3.1, Lemmas 4.1, 4.2, 4.4, and 4.5] originally for the case where $p(t) \equiv 1$ with essentially the same proofs. The first one is on the global existence of solutions of IVPs associated with Eq. (1.1).

Lemma 1. *Any initial value problem associated with Eq.* (1.1) *has a unique solution which exists on the whole interval $[a,b]$. Consequently, the solution depends continuously on the initial condition.*

For $\gamma \in \{+,-\}$, let $y(t,\rho)$ be the solution of Eq. (1.1) satisfying

$$y(a) = \gamma\rho\sin\alpha \quad \text{and} \quad (py')(a) = \gamma\rho\cos\alpha, \tag{2.5}$$

where $\rho > 0$ is a parameter. Let $\theta(t,\rho)$ be the Prüfer angle of $y(t,\rho)$, i.e., $\theta(\cdot,\rho)$ is a continuous function on $[a,b]$ such that

$$\tan\theta(t,\rho) = y(t,\rho)/(py')(t,\rho) \quad \text{and} \quad \theta(a,\rho) = \alpha.$$

By Lemma 1, $\theta(t,\rho)$ is continuous in ρ on $(0,\infty)$ for any $t \in [a,b]$.

The next two lemmas provide some estimates for the Prüfer angle.

Lemma 2. *(i) Assume $f_0 < \lambda_n$ for some $n \in \mathbb{N}_0$. Then there exists $\rho_* > 0$ such that $\theta(b,\rho) < (n+1)\pi$ for all $\rho \in (0,\rho_*)$.*

(ii) Assume $\lambda_n < f_\infty$ for some $n \in \mathbb{N}_0$. Then there exists $\rho^* > 0$ such that $\theta(b,\rho) > (n+1)\pi$ for all $\rho \in (\rho^*,\infty)$.

Lemma 3. *(i) Assume $f_\infty < \lambda_n$ for some $n \in \mathbb{N}_0$. Then there exists $\rho^* > 0$ such that $\theta(b,\rho) < (n+1)\pi$ for all $\rho \in (\rho^*,\infty)$.*

(ii) Assume $\lambda_n < f_0$ for some $n \in \mathbb{N}_0$. Then there exists $\rho_* > 0$ such that $\theta(b,\rho) > (n+1)\pi$ for all $\rho \in (0,\rho_*)$.

Proof of Theorem 1. We first prove it for the case where $f_0 < \lambda_n$ and $\lambda_{n+1} < f_\infty$. Without loss of generality we assume $\gamma = +$. The case with $\gamma = -$ can be proved in the same way. Let $y(t,\rho)$ be the solution of Eq. (1.1) satisfying (2.5) with $\gamma = +$ and $\theta(t,\rho)$ its Prüfer angle. By Lemma 2, there exist $0 < \rho_* < \rho^* < \infty$ such that $\theta(b,\rho) < (n+1)\pi$ for all $\rho \in (0,\rho_*)$ and $\theta(b,\rho) > (n+2)\pi$ for all $\rho \in (\rho^*,\infty)$. By the continuity of $\theta(t,\rho)$ in ρ, there exist $\rho_* \le \rho_{n+1} < \rho_{n+2} \le \rho^*$ such that

$$\begin{aligned}
&\theta(b,\rho_{n+1}) = (n+1)\pi, \quad \theta(b,\rho_{n+2}) = (n+2)\pi,\\
&(n+1)\pi < \theta(b,\rho) < (n+2)\pi \quad \text{for} \quad \rho_{n+1} < \rho < \rho_{n+2}.
\end{aligned} \tag{2.6}$$

Then, for all $t \in [a,b]$ and all $\rho > 0$, we define an energy function $E(t,\rho)$ for $y(t,\rho)$ by

$$E(t,\rho) = \frac{1}{2p(t)}\Big(p(t)y'(t,\rho)\Big)^2 + \frac{1}{2}\Big(q^* - q(t)\Big)y^2(t,\rho) + w(t)F\Big(y(t,\rho)\Big).$$

where $F(y) = \int_0^y f(s)ds$. By (H1) and (H2), $F(y) \ge 0$ on $\mathbb{R}$ yielding $E(t,\rho) \ge 0$ on $[a,b]$. For ease of notation, in the following, we use $p = p(t)$, $q = q(t)$, $w = w(t)$, $y = y(t,\rho)$, $E = E(t,\rho)$. Then, by (1.1) and (H1), we find that

$$\begin{aligned}
E' &= -\frac{p'}{2p^2}(py')^2 - \frac{1}{2}q'y^2 + q^*yy' + w'F(y)\\
&\ge -\frac{p'}{2p^2}(py')^2 - \frac{1}{2}q'y^2 - \frac{q^*}{2}(y^2 + y'^2) + w'F(y)\\
&= -\frac{(p'+q^*)}{2p^2}(py')^2 - \frac{1}{2}(q'+q^*)y^2 + w'F(y) \ge -(\frac{p'+q^*}{p})_+(\frac{1}{2p}(py')^2) - l(\frac{1}{2}(q^*-q)y^2)\\
&\quad - \frac{w'_-}{w}wF(y) \ge -l\frac{(py')^2}{2p} - l(\frac{1}{2}(q^*-q)y^2) - lwF(y) = -lE(t,\rho).
\end{aligned}$$

Thus, $E'(t,\rho) + lE(t,\rho) \geq 0$ for all $t \in [a,b]$ and $\rho > 0$. By solving this inequality, we obtain that

$$E(s,\rho) \leq E(b,\rho)e^{l(b-s)}, \quad s \in [a,b]. \tag{2.7}$$

We observe that for $\rho = \rho_{n+1}$ and $\rho = \rho_{n+2}$

$$E(s,\rho) \geq \frac{1}{2p(s)}\Big[p(s)y'(s,\rho)\Big]^2, \quad s \in [a,b], \quad E(b,\rho) = \frac{1}{2p(b)}\Big[p(b)y'(b,\rho)\Big]^2.$$

Thus, for $\rho = \rho_{n+1}$, $\rho = \rho_{n+2}$, and $s \in [a,b]$,

$$|(py')(s,\rho)| \leq \sqrt{2p(s)E(s,\rho)} \ \text{ and } \ |(py')(b,\rho)| = \sqrt{2p(s)E(b,\rho)}. \tag{2.8}$$

Define

$$\Gamma(\rho) = (py')(b,\rho) - \int_a^b (py')(s,\rho)d\xi(s).$$

Assume $n = 2k-1$ with $k \in \mathbb{N}_0$. Since $(py')(b,\rho_{2k}) > 0$ and $(py')(b,\rho_{2k+1}) < 0$, by (2.3), (2.7), (2.8), and (2.4) we have

$$\begin{aligned}
&\Gamma(\rho_{2k}) = (py')(b,\rho_{2k}) - \int_a^b (py')(s,\rho_{2k})d\xi(s) \geq \Big|(py')(b,\rho)\Big| \\
&- \int_a^b |(py')(s,\rho_{2k})| d\big(\xi_1(s)+\xi_2(s)\big) \\
&\geq \sqrt{2p(b)E(b,\rho_{2k})} - \int_a^b \sqrt{2p(s)E(s,\rho_{2k})} d(\xi_1(s)+\xi_2(s)) \\
&\geq \sqrt{2p(b)E(b,\rho_{2k})} - \int_a^b \sqrt{2p(s)E(b,\rho_{2k})e^{l(b-a)}} d\big(\xi_1(s)+\xi_2(s)\big) \\
&= \sqrt{2p(b)E(b,\rho_{2k})}\Big(1 - \int_a^b \sqrt{p(s)/p(b)} e^{l(b-a)/2} d\big(\xi_1(s)+\xi_2(s)\big)\Big) > 0, \\
&\Gamma(\rho_{2k+1}) = (py')(b,\rho_{2k+1}) - \int_a^b (py')(s,\rho_{2k+1})d\xi(s) \\
&\leq -\Big|(py')(b,\rho_{2k+1})\Big| + \int_a^b (|(py')(s,\rho_{2k+1})| d\big(\xi_1(s)+\xi_2(s)\big) \\
&\leq -\sqrt{2p(b)E(b,\rho_{2k+1})} + \int_a^b \sqrt{2p(s)E(s,\rho_{2k+1})}\ d\big(\xi_1(s)+\xi_2(s)\big) \\
&\leq -\sqrt{2p(b)E(b,\rho_{2k+1})} + \int_a^b \sqrt{2p(s)E(b,\rho_{2k+1})e^{l(b-a)}}\ d\big(\xi_1(s)+\xi_2(s)\big) \\
&= -\sqrt{2p(b)E(b,\rho_{2k+1})}\Big(1 - \int_a^b \sqrt{p(s)/p(b)}\ e^{l(b-a)/2}\ d\big(\xi_1(s)+\xi_2(s)\big)\Big) < 0.
\end{aligned}$$

By the continuity of $\Gamma(\rho)$, there exists $\bar{\rho} \in (\rho_{2k}, \rho_{2k+1})$ such that $\Gamma(\bar{\rho}) = 0$. Similarly, for $n = 2k$ with $k \in \mathbb{N}_0$, there exists $\bar{\rho} \in (\rho_{2k+1}, \rho_{2k+2})$ such that

$\Gamma(\bar\rho) = 0$. In both cases, it follows from (2.6) that $(n+1)\pi < \theta(b, \bar\rho) < (n+2)\pi$. Since

$$\theta'(t,\rho) = \frac{1}{p(t)} \cos^2\theta(t,\rho) + w(t)\frac{f(y(t,\rho))y(t,\rho)}{r^2(t,\rho)} - q(t)\sin^2\theta(t,\rho) \qquad (2.9)$$

for $t \in [a, b]$, where $r = (y^2 + py')^{1/2}$, we have that $\theta(\cdot, \rho)$ is strictly increasing at the points t where $\theta(t, \rho) = 0 \pmod \pi$. We note that $y(t) = 0$ if and only if $\theta(t, \rho) = 0 \pmod \pi$. Thus, y has exactly $n + 1$ zeros in (a, b). Initial condition (2.5) implies that $y(t, \bar\rho) > 0$ in a right-neighborhood of a. Therefore, $y(t, \bar\rho) \in \mathcal{S}^+_{n+1}$.

The proof for the case where $f_\infty < \lambda_n$ and $\lambda_{n+1} < f_0$ is essentially the same as above except that the discussion is based on Lemma 3 instead of Lemma 2. □

Proof of Corollary 1. Consider the SLP consisting of Eq. (2.1) with $p(t) \equiv 1$, $q(t) \equiv 0$, and the BC

$$\begin{aligned} \cos\alpha\ y(a) - \sin\alpha\ y'(a) &= 0, \qquad \alpha \in [0, \pi), \\ \cos\beta\ y(b) - \sin\beta\ y'(b) &= 0, \qquad \beta \in (0, \pi]. \end{aligned}$$

Denote by $\lambda_n(\alpha, \beta)$ the nth eigenvalue of this problem for $n \in \mathbb{N}_0$. It is easy to see that $\lambda_0(\pi/2, \pi/2) = 0$. In fact, $y_0(t) \equiv 1$ is an associated eigenfunction. From [14, Theorem 4.2] and [12, Lemma 3.32], we see that $\lambda_0(\alpha, \beta)$ is a continuous function of (α, β) on $[0, \pi) \times (0, \pi]$, and is strictly decreasing in α and strictly increasing in β. Furthermore, for any $\beta \in (0, \pi]$,

$$\lim_{\alpha\to\pi-} \lambda_0(\alpha, \beta) = -\infty \quad \text{and} \quad \lim_{\alpha\to\pi-} \lambda_{n+1}(\alpha, \beta) = \lambda_n(0, \beta) \text{ for } n \in \mathbb{N}_0.$$

This shows that $\lambda_0(\pi/2, \pi) > 0$, and hence there exists $\alpha^* \in (\pi/2, \pi)$ such that $\lambda_0(\alpha, \pi) > 0$ for $\alpha \in [0, \alpha^*)$, and $\lambda_0(\alpha, \pi) \le 0$ and $\lambda_1(\alpha, \pi) > 0$ for $\alpha \in [\alpha^*, \pi)$. Note that $\beta = \pi$ if and only if $y(b) = 0$. Then the conclusion follows from Theorem 1. □

Proof of Theorem 2. (i) Assume to the contrary that BVP (1.1), (1.2) has a solution $y \in \mathcal{S}^\gamma_i$ for some $i \ge n+1$ and $\gamma \in \{+, -\}$. Let $\tilde w(t) = w(t)f(y(t))/y(t)$. Then $\tilde w(t)$ is continuous on $[a, b]$ by the continuous extension since $f_0 < \infty$. Let $\theta(t)$ be the Prüfer angle of $y(t)$ with $\theta(a) = \alpha$. Then $\theta(t)$ satisfies Eq. (2.9) and $\theta(b) > i\pi$. Note, from the assumption that $\tilde w(t) \le \lambda_n w(t) \le \lambda_{i-1} w(t)$ on $[a, b]$, we have that for $t \in [a, b]$

$$\begin{aligned} \theta'(t) &= \frac{1}{p(t)} \cos^2\theta(t) + [\tilde w(t) - q(t)]\sin^2\theta(t,\rho) \\ &\le \frac{1}{p(t)} \cos^2\theta(t,\rho) + [\lambda_{i-1} w(t) - q(t)]\sin^2\theta(t,\rho). \end{aligned}$$

Let $u(t)$ be an eigenfunction of SLP (2.1), (2.2) associated with the eigenvalue λ_{i-1} and $\phi(t)$ its Prüfer angle with $\phi(a) = \alpha$. Then

$$\phi'(t) = \frac{1}{p(t)} \cos^2\phi(t) + [\lambda_{i-1} w(t) - q(t)]\sin^2\phi(t)$$

and $\phi(b) = i\pi$. By the theory of differential inequalities, we find that $\theta(b) \le \phi(b) = i\pi$. We have reached a contradiction.

(ii) It is similar to (i) and hence omitted. $\square$

3 Dependence of Nodal Solutions on the problem

In this section, we investigate the changes of the existence of different types of nodal solutions of BVP (1.1), (1.2) as the problem changes. Our work is based on the following lemma for the dependence of the nth eigenvalue of SLP (2.1), (2.2) on the problem which can be excerpted from [13, Theorems 2.2 and 2.3], [14, Theorem 4.2], and [12, Lemma 3.32].

Lemma 4. *For any $n \in \mathbb{N}_0$, we have the following conclusions.*

(a) Consider the nth eigenvalue of SLP (2.1)*,* (2.2) *as a function of b for $b \in (a, \infty)$, denoted by $\lambda_n(b)$. Then $\lambda_n(b) \to \infty$ as $b \to a^+$.*

(b) Consider the nth eigenvalue of SLP (2.1)*,* (2.2) *as a function of w for $w \in C^1[a,b]$, denoted by $\lambda_n(w)$. Then $\lambda_n(w)$ is decreasing as long as it is positive, i.e., for $w_1, w_2 \in C^1[a,b]$ such that $w_1(t) \le w_2(t)$ for $t \in [a,b]$, we have $\lambda_n(w_1) \ge \lambda_n(w_2)$ as long as $\min\{\lambda_n(w_1), \lambda_n(w_2)\} \ge 0$.*

(c) Consider the nth eigenvalue of SLP (2.1)*,* (2.2) *as a function of q for $q \in C^1[a,b]$, denoted by $\lambda_n(q)$. Then $\lambda_n(q)$ is increasing, i.e. for $q_1, q_2 \in C^1[a,b]$ such that $q_1(t) \le q_2(t)$ for $t \in [a,b]$, we have $\lambda_n(q_1) \le \lambda_n(q_2)$.*

(d) Consider the nth eigenvalue of SLP (2.1)*,* (2.2) *as a function of $1/p$ for $1/p \in C^1[a,b]$, denoted by $\lambda_n(1/p)$. Then $\lambda_n(1/p)$ is decreasing, i.e. for $1/p_1, 1/p_2 \in C^1[a,b]$ such that $1/p_1(t) \le 1/p_2(t)$ for $t \in [a,b]$, we have $\lambda_n(1/p_1) \ge \lambda_n(1/p_2)$.*

(e) Consider the nth eigenvalue of SLP (2.1)*,* (2.2) *as a function of the boundary condition angle α, denoted by $\lambda_n(\alpha)$. Then $\lambda_n(\alpha)$ is a continuous and decreasing function on $[0, \pi)$. Furthermore,*

$$\lim_{\alpha \to \pi^-} \lambda_0(\alpha) = -\infty \quad \text{and} \quad \lim_{\alpha \to \pi^-} \lambda_{n+1}(\alpha) = \lambda_n(0) \quad \text{for} \quad n \ge 1.$$

The first result is about the changes as the interval $[a,b]$ shrinks, more precisely, as $b \to a^+$. We discuss both the cases when one of f_0 and f_∞ is infinite and when both of them are finite.

Theorem 3. *Let Eq.* (1.1) *and BC* (1.2) *be fixed and let* (2.4) *hold.*

(i) Assume either $f_0 < \infty$ and $f_\infty = \infty$, or $f_\infty < \infty$ and $f_0 = \infty$. Then for any $n \in \mathbb{N}_0$, there exists $b_n > a$ such that for any $b \in (a, b_n)$ and for any $i \ge n$, BVP (1.1)*,* (1.2) *has a solution $y_i^\gamma \in \mathcal{S}_{i+1}^\gamma$ for $\gamma \in \{+,-\}$.*

(ii) Assume $f_0 < \infty$ and $f_\infty < \infty$. Then for any $n \in \mathbb{N}_0$, there exists $b_n > a$ such that for any $b \in (a, b_n)$ and for any $i \ge n+1$, BVP (1.1)*,* (1.2) *has no solutions in $\mathcal{S}_i^\gamma$ for $\gamma \in \{+,-\}$.*

Proof. (i) Without loss of generality assume $f_0 < \infty$ and $f_\infty = \infty$. Let $\lambda_n(b)$ be defined as in Lemma 4 (a). By Lemma 4 (a), for any $n \in \mathbb{N}_0$, there exists $b_n > a$ such that for any $b \in (a, b_n)$, we have $f_0 < \lambda_n(b) < f_\infty$ and hence $f_0 < \lambda_i(b) < f_\infty$ for all $i \geq n$. Then the conclusion follows from Theorem 1.

(ii) By Lemma 4 (a), for any $n \in \mathbb{N}$, there exists $b_n > a$ such that for any $b \in (a, b_n)$, we have that $\lambda_n(b) > f^* := \sup\{f(y)/y : y \in (0, \infty)\}$. Then the conclusion follows from Theorem 2 (i). □

We then present a result on the nonexistence of certain types of nodal solutions of BVP (1.1), (1.2) as the function w increases in a given direction. More precisely, let $s \geq 0$ and $h \in C^1[a, b]$ such that $h(t) > 0$ on $[a, b]$, and consider the equation

$$-(p(t)y')' + q(t)y = \big[w(t) + sh(t)\big] f(y). \tag{3.1}$$

Theorem 4. *Let the interval $[a, b]$ and BC (1.2) be fixed and let (2.4) hold. Assume $f(y)/y \geq f_* > 0$ for all $y \neq 0$. Then for any $n \in \mathbb{N}_0$, there exists $s_n \geq 0$ such that for any $s > s_n$ and for any $i \leq n$, BVP (3.1), (1.2) has no solution in $\mathcal{S}_i^\gamma$ for $\gamma \in \{+, -\}$.*

Proof. For $s \geq 0$ and $i \in \mathbb{N}_0$, we denote by $\lambda_i(s)$ the ith eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + q(t)y = \lambda\big[w(t) + sh(t)\big] y$$

and BC (2.2). Let $h_* = \min\{h(t)/w(t) : t \in [a, b]\}$, and denote by $\mu_i(s)$ the ith eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + q(t)y = \mu(1 + sh_*)w(t)y$$

and BC (2.2). Since

$$w(t) + sh(t) \geq (1 + sh_*)w(t) \quad \text{for } s \geq 0,$$

by Lemma 4 (b),

$$\lambda_i(s) \leq \mu_i(s) \ \text{ for all } s \geq 0 \text{ and } i \geq 0, \ \text{whenever } \lambda_i(s) \geq 0. \tag{3.2}$$

Note that for $i \geq 0$, $\mu_i(s)(1 + sh_*) = \mu_i(0)$, we have

$$\mu_i(s) = \frac{\mu_i(0)}{1 + sh_*} \to 0 \ \text{ as } s \to \infty.$$

This together with (3.2) implies that $\lambda_i(s) < f_*$ as $s \to \infty$. Then, for any $n \in \mathbb{N}_0$, there exists $s_n \geq 0$ such that $\lambda_n(s) < f_*$ for $s > s_n$. Therefore, the conclusion follows from Theorem 2 (ii). □

The next result is on the nonexistence and existence of certain types of nodal solutions of BVP (1.1), (1.2) as the function q changes in a given direction. More precisely, let $s \in \mathbb{R}$ and $h \in C^1[a, b]$ such that $h(t) > 0$ on $[a, b]$, and consider the equation

$$-(p(t)y')' + \big[q(t) + sh(t)\big]y = w(t)f(y). \tag{3.3}$$

Theorem 5. *Let the interval* $[a, b]$ *and BC* (1.2) *be fixed and let* (2.4) *hold.*

(i) For any $n \in \mathbb{N}_0$*, there exists* $s_n \leq 0$ *such that for any* $s < s_n$ *and for any* $i \leq n$*, BVP* (3.3)*,* (1.2) *has no solutions in* $\mathcal{S}_i^\gamma$ *for* $\gamma \in \{+, -\}$*.*

(ii) Assume either $f_0 < \infty$ *and* $f_\infty = \infty$*, or* $f_\infty < \infty$ *and* $f_0 = \infty$*. Then for any* $n \in \mathbb{N}_0$*, there exists* $s_n \geq 0$ *such that for any* $s > s_n$ *and for any* $i \geq n$*, BVP* (3.3)*,* (1.2) *has two solutions* $y_{i,\gamma} \in \mathcal{S}_{i+1}^\gamma$ *for* $\gamma \in \{+, -\}$*.*

(iii) Assume $f_0 < \infty$ *and* $f_\infty < \infty$*. Then for any* $n \in \mathbb{N}_0$*, there exists* $s_* \geq 0$ *such that for any* $s > s_*$*, BVP* (3.3)*,* (1.2) *has no solution in* $\mathcal{S}_i^\gamma$ *for all* $i \geq n + 1$ *and* $\gamma \in \{+, -\}$*.*

Proof. For $s \in \mathbb{R}$ and $i \in \mathbb{N}_0$, we denote by $\lambda_i(s)$ the ith eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + \big[q(t) + sh(t)\big]y = \lambda w(t)y$$

and BC (2.2). Let $h_* = \min\{h(t)/w(t) : t \in [a, b]\}$, and denote by $\mu_i(s)$ the ith eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + \big[q(t) + sh_*(t)w(t)\big]y = \mu w(t)y \tag{3.4}$$

and BC (2.2).

(i) Since for $s \leq 0$,

$$q(t) + sh(t) \leq q(t) + sh_* w(t),$$

by Lemma 4 (c), $\lambda_i(s) \leq \mu_i(s)$ for all $s \leq 0$ and $i \geq 0$. Note that Eq. (3.4) yields

$$-(p(t)y')' + q(t)y = (\mu - sh_*)w(t)y.$$

Thus, for $s \leq 0$ and $i \geq 0$, $\mu_i(0) = \mu_i(s) - sh_*$, which implies that

$$\mu_i(s) = \mu_i(0) + sh_* \to -\infty \quad \text{as } s \to -\infty,$$

and hence $\lambda_i(s) \to -\infty$ as $s \to -\infty$ for all $i \geq 0$. Then, for any $n \in \mathbb{N}_0$ there exists $s_n \leq 0$ such that $\lambda_n < 0$ for all $s < s_n$. Therefore, the conclusion follows from Theorem 2 (ii).

(ii) Without loss of generality, assume $f_0 < \infty$ and $f_\infty = \infty$. Similar to the argument in (i), we have $\lambda_i(s) \to \infty$ as $s \to \infty$ for all $i \geq 0$. Then for any $n \in \mathbb{N}_0$ there exists $s_n \geq 0$ such that for any $s > s_n$ we have $f_0 < \lambda_n(s) < f_\infty$ and hence $f_0 < \lambda_i(s) < f_\infty$ for $i \geq n$. Therefore, the conclusion follows from Theorem 1.

(iii) As we can see from (ii), for any $n \in \mathbb{N}_0$, there exists $s_* \geq 0$ such that for all $s > s_*$ we have $\lambda_n(s) > f^* := \sup\{f(y)/y : y \in (0, \infty)\}$. Thus, the conclusion follows from Theorem 2 (i). □

Similar to Theorem 4, we show a result on the nonexistence of certain types of nodal solutions of BVP (1.1), (1.2) as the function $1/p(t)$ increases in a certain direction. More precisely, let $s \geq 0$ and $h \in C[a,b]$ such that $h(t) > 0$ on $[a,b]$, and consider the equation

$$-\Big(\frac{1}{1/p(t) + sh(t)}y'\Big)' + q(t)y = w(t)f(y). \tag{3.5}$$

Theorem 6. *Let the interval $[a,b]$ and BC (1.2) be fixed and let (2.4) hold. Define $\hat{q} := \max\{q(t)/w(t) : t \in [a,b]\}$ and assume $f(y)/y \geq f_* > \hat{q}$ for all $y \neq 0$. Then for any $n \in \mathbb{N}_0$, there exists $s_n \geq 0$ such that for any $s > s_n$, BVP (3.5), (1.2) has no solution in $\mathcal{S}_i^\gamma$ for all $i \leq n$ and $\gamma \in \{+,-\}$.*

Proof. For $s \geq 0$ and $i \in \mathbb{N}_0$, we denote by $\lambda_i(s)$ the ith eigenvalue of the SLP consisting of the equation

$$-\Big(\frac{1}{1/p(t) + sh(t)}y'\Big)' + q(t)y = \lambda w(t)y$$

and BC (2.2) with an eigenfunction $u_i(t,s)$. Let $\theta_i(t,s)$ be the Prüfer angle of $u_i(t,s)$ satisfying $\theta_i(a,s) = \alpha$. Then

$$\theta_i'(t,s) = \Big[\frac{1}{p(t)} + sh(t)\Big]\cos^2\theta_i(t,s) + \big[\lambda_i w(t) - q(t)\big]\sin^2\theta_i(t,s). \tag{3.6}$$

By Lemma 4 (d), $\lambda_i(s)$ is decreasing and hence

$$\lim_{s\to\infty}\lambda_i(s) = \lambda_i^* \in [-\infty,\infty).$$

We show that $\lambda_i^* < f_*$ and then the conclusion follows from Theorem 2.2 (ii). Assume the contrary, i.e., $\lambda_i^* \geq f_*$. Let $w_* = \min\{w(t) : t \in [a,b]\}$. By (3.6),

$$\begin{aligned}
\theta_i'(t,s) &\geq \Big[\frac{1}{p(t)} + sh(t)\Big]\cos^2\theta_i(t,s) + [\lambda_i^* w(t) - q(t)]\sin^2\theta_i(t,s) \\
&= \Big[\frac{1}{p(t)} + sh(t)\Big]\cos^2\theta_i(t,s) + \big[\lambda_i^* - q(t)/w(t)\big]w(t)\sin^2\theta_i(t,s) \\
&\geq \Big[\frac{1}{p(t)} + sh(t)\Big]\cos^2\theta_i(t,s) + \big[f_* - \hat{q}\big]w_*\sin^2\theta_i(t,s).
\end{aligned}$$

Let $\phi(t,s)$ be the solution of the equation

$$\phi'(t,s) = \Big[\frac{1}{p(t)} + sh(t)\Big]\cos^2\phi(t,s) + \big[f_* - \hat{q}\big]w_*\sin^2\phi(t,s) \tag{3.7}$$

satisfying $\phi(a,s) = \alpha$. By the theory of differential inequalities, we have $\phi(t,s) \leq \theta_i(t,s)$. In particular,

$$\phi(b,s) \leq \theta_i(b,s) = (i+1)\pi. \tag{3.8}$$

We observe from (3.7) that $\phi(t,s)$ is strictly increasing in t and s, and $0 < \phi(t,s) \leq (i+1)\pi$ for $t \in [a,b]$ and $s \geq 0$. Let $\phi^*(t) = \lim_{s\to\infty}\phi(t,s)$. Then $0 < \phi^*(t) \leq (i+1)\pi$ for $t \in [a,b]$. We claim that

$$\phi^*(t) \not\equiv k\pi + \pi/2 \quad \text{on} \quad (a,b] \quad \text{for any} \quad 0 \leq k \leq i. \tag{3.9}$$

If not, for any $a_1 \in (a, b]$ and $\epsilon > 0$, there exists $s^* > 0$ such that for $s \geq s^*$,

$$\phi(a_1, s) \in (k\pi + \pi/2 - \epsilon, k\pi + \pi/2),$$

which yields that

$$\phi(t, s) \in (k\pi + \pi/2 - \epsilon, k\pi + \pi/2) \quad \text{for} \quad t \in [a_1, b].$$

This implies that

$$0 < \phi(b, s) - \phi(a_1, s) < \epsilon. \tag{3.10}$$

However, from (3.7), we see that for s sufficiently large,

$$\phi'(t, s) \geq \frac{1}{2}(f_* - \hat{q})w_* \quad \text{for} \quad t \in [a_1, b].$$

This contradicts (3.10) and hence verifies (3.9).

It is easy to see that $\phi(t, s) \to \phi^*(t)$ uniformly on $[a_1, b]$ as $s \to \infty$. Thus, $\phi^*(t)$ is continuous on $[a_1, b]$. From (3.9), we can find a nontrivial closed interval $[c, d] \subset [a, b]$ such that $\cos^2 \phi^*(t) \geq \nu > 0$ for $t \in [c, d]$. Then from (3.7),

$$\phi'(t, s) \geq \left[\frac{1}{p(t)} + sh(t)\right] \nu \to \infty \quad \text{uniformly for} \quad t \in [c, d] \quad \text{as} \quad s \to \infty.$$

Therefore,

$$\phi(b,s) \geq \phi(d,s) \geq \phi(c,s) + \int_c^d \left[\frac{1}{p(t)} + sh(t)\right] \nu ds \geq \int_c^d \left[\frac{1}{p(t)} + sh(t)\right] \nu ds \to \infty \quad \text{as} \quad s \to \infty.$$

This contradicts (3.8) and hence completes the proof. □

The last result is on the existence of certain types of nodal solutions of BVP (1.1), (1.2) as the boundary condition angle α changes.

Theorem 7. *Let Eq.* (1.1) *and the interval* $[a, b]$ *be fixed and let* (2.4) *hold. Assume either* $f_0 = 0$ *and* $f_\infty = \infty$, *or* $f_\infty = 0$ *and* $f_0 = \infty$. *For* $n \in \mathbb{N}_0$ *denote* $\lambda_n(\alpha)$ *the nth eigenvalue of the SLP* (2.1), (2.2). *Suppose* k *is the first nonnegative integer such that* $\lambda_k(\alpha^*) > 0$ *for some* $\alpha^* \in (0, \pi)$. *Then*

(i) for $\alpha \in [0, \alpha^*)$, *BVP* (1.1), (1.2) *has a solution* $y_n^\gamma \in \mathcal{S}_{n+1}^\gamma$ *for all* $n \geq k$ *and* $\gamma \in \{+, -\}$;

(ii) for $\alpha \in [\alpha^*, \pi)$, *BVP* (1.1), (1.2) *has a solution* $y_n^\gamma \in \mathcal{S}_{n+1}^\gamma$ *for all* $n \geq k+1$ *and* $\gamma \in \{+, -\}$.

Proof. By assumption, $\lambda_k(\alpha^*) > 0$. Then Lemma 4 (e) shows that $\lambda_k(\alpha) > 0$ for $\alpha \in [0, \alpha^*]$; and for $\alpha \in (\alpha^*, \pi)$, $\lambda_k(\alpha) < 0$ and $\lambda_{k+1}(\alpha) > 0$. Therefore, the conclusion follows from Theorem 2.1. □

Remark 4. Theorems 3–7 show that we can "create" or "eliminate" certain types of nodal solutions by changing the interval $[a, b]$, the coefficient functions q, p, w, and the boundary condition angle α. Since the eigenvalues of SLP (2.1), (2.2) can be easily computed using computer software such as that in [2], we are able to construct specific BVPs (1.1), (1.2) which have or do not have nodal solutions in $\mathcal{S}_n^\gamma$ for a prescribed $n \in \mathbb{N}_0$.

4 Conclusions

Second order nonlocal BVPs arise not only from theoretical interests, but also from a wide range of applications in physics and applied mathematics. Different from two-point BVPs, the theory for BVPs with nonlocal BCs has not been well established. In particular, only a few papers can be found so far on the existence of nodal solutions for nonlocal BVPs.

In this paper, by relating BVP (1.1), (1.2) to the eigenvalues of an associated linear SLP with a two-point separated BC, we provide sufficient conditions for the existence and nonexistence of nodal solutions of BVP (1.1), (1.2). One of the advantages of our results lies in the fact that eigenvalues are easy to compute for two-point linear SLPs using standard software packages, while the existence of eigenvalues of general linear SLPs with nonlocal BCs still remain unsolved, not to mention their computations. The main approach in our work is by utilizing the shooting method and a general energy function. This is in contrast with most existing works where the bifurcation method serves as a major tool. We also study how the existence of nodal solutions depends on the problem including the interval $[a, b]$, the coefficient functions p, q, and w, and the α in the BC. Here it is worthwhile to notice that little progress has been made in the current literature on the dependence of nodal solutions on the problem, even for BVPs with two-point BCs, except those recently done by the authors.

It will be a subject of future efforts to extend and generalize the results in this paper to higher order BVPs, BVPs with p-Laplacian, and system BVPs which involve nonlocal BCs.

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