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Asymptotical Solutions for a Vibrationally Relaxing Gas*

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Abstract. Using the weakly non-linear geometrical acoustics theory, we obtain the small amplitude high frequency asymptotic solution to the basic equations governing one dimensional unsteady planar, spherically and cylindrically symmetric flow in a vibrationally relaxing gas with Van der Waals equation of state. The transport equations for the amplitudes of resonantly interacting waves are derived. The evolutionary behavior of non-resonant wave modes culminating into shock waves is also studied.

Key words: Weakly Non-linear hyperbolic waves, Asymptotic Solution, Resonance, Planar and non-planar Shock Waves, Vibrationally Relaxing Gas.

1 Introduction

When non-equilibrium effects are considered in fluid-dynamic equations, the analysis becomes considerably more complex than the corresponding classical gas dynamic flow because of nonlinear coupling between the relaxing mode and the fluid flow. The occurrence of shock waves in a relaxing gas has been well studied in the past. The propagation of shock waves in a relaxing gas constitutes a problem of great interest for researchers and scientists.

It is well known that at high temperatures, the internal energy of the gas molecules consists of translational, rotational and vibrational components. When the gas is in equilibrium, each internal mode is characterized by the same temperature. A rate of transfer of energy from one mode to another mode can be observed by inducing small changes in any of these temperatures and observing the rate of return to equilibrium. For instance, when gas is compressed by the mechanical action of a piston or by the passage of a shock front, the whole energy goes initially to increase the translational energy, and it is followed by a relaxation from translational mode to rotational mode and

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also from translational mode to vibrational mode until the equilibrium between these modes is re-established. This is called a relaxation process.

A number of problems relating to the effects of nonlinearity in gases with internal relaxation have been studied previously, in particular, by Clarke and McChesney [7], Scott and Johannesen [22], Blythe [4], Ockenden and Spence [19], Parker [20], Radha and Sharma [21] among others. There has been widespread interest in the nonlinear wave phenomena. The work of Whitham [28], Moodie *et al.* [18], He and Moodie ([9, 10]), Shtaras [27], Kalyakin [13], Krylovas and Čiegis ([14, 15]), Sharma and Radha [24], Sharma and Srinivasan [25], Sharma and Arora [23], Arora and Sharma [3], and Arora ([1, 2]) is worth mentioning in the context of nonlinear wave propagation in gas dynamic media.

We use asymptotic method for the analysis of weakly nonlinear hyperbolic waves. Choquet-Bruhat [6] proposed a method to discuss shockless solutions of hyperbolic systems which depend upon a single phase function. Germain [8] has given the general discussion of single phase progressive waves. Hunter and Keller [11] established a general non resonant multi-wave mode theory which has led to several interesting generalizations by Majda and Rosales [16] and Hunter et al [12] to include resonantly interacting multi-wave mode features. Krylovas and Čiegis [14] developed a method of averaging for constructing a uniformly valid asymptotic solution for weakly nonlinear one dimensional gas dynamics systems.

We use the method of weakly non-linear geometrical acoustics to obtain the small amplitude high frequency asymptotic solution to the basic equations governing one dimensional unsteady planar, cylindrically and spherically symmetric flow in vibrationally relaxing gas with Van der Waals equation of state. We derive the transport equations for the amplitudes of resonantly interacting waves.

2 Basic Equations

We consider an unsteady one-dimensional motion in a vibrationally relaxing gas with Van der Waals equation of state. The gas molecules have only one lagging internal mode (i.e., vibrational relaxation) and the various transport effects are negligible. The basic equations can be written as (see [7])

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \frac{m\rho u}{x} = 0,\\
&\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + \frac{1}{\rho}\,\frac{\partial p}{\partial x} = 0,\\
&\frac{\partial p}{\partial t} + u\,\frac{\partial p}{\partial x} + \frac{\gamma p}{1-b\rho}\left(\frac{\partial u}{\partial x} + \frac{mu}{x}\right) + (\gamma-1)\rho Q = 0,\\
&\frac{\partial \sigma}{\partial t} + u\,\frac{\partial \sigma}{\partial x} - Q = 0,
\end{aligned} \tag{2.1}$$

where ρ is the density, u the particle velocity, p the pressure, σ the vibrational energy, t the time, x the spatial coordinate, b the Van der Waals excluded volume, m=0, 1 and 2 correspond, respectively, to planar, cylindrical and spherical symmetry. The quantity Q is the rate of change of vibrational energy that depends on the states ρ, p and σ, and is given by

$$Q = \big(\bar{\sigma}(p,\rho) - \sigma\big)/\tau,$$

where $\bar{\sigma}$ is the equilibrium value of σ defined as $\bar{\sigma} = \sigma_0 + cR(T - T_0)$, T is the translational temperature, R is the specific gas constant, suffix 0 refers to the initial rest condition, and the quantities τ and c, which are respectively the relaxation time and the ratio of vibrational specific heat to the specific gas constant, are assumed to be constant. The Van der Waals equation of state is taken to be of the form

$$p(1 - b\rho) = \rho RT,$$

where b is the Van der Waals excluded gas volume.

We denote $a = (\gamma p/(\rho(1 - b\rho)))^{1/2}$ as the speed of sound, and γ as the specific heat ratio of the gas.

The equations (2.1) may be written in the matrix form as

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} + B(U) = 0, \tag{2.2}$$

where U and $B(U)$ are the column vectors defined by

$$U=(\rho,u,p,\sigma)^T, \quad B(U)=\left(\frac{m\rho u}{x}, 0, \frac{m\gamma p u}{(1-b\rho)x}+(\gamma-1)\rho\frac{(\bar{\sigma}-\sigma)}{\tau}, -\frac{(\bar{\sigma}-\sigma)}{\tau}\right)^T,$$

where

$$\bar{\sigma} = \sigma_0 + c\left(\frac{p(1 - b\rho)}{\rho} - \frac{p_0(1 - b\rho_0)}{\rho_0}\right).$$

$A(U)$ is the 4 x 4 matrix having the components A^{ij}, and the nonzero ones are:

$$A^{11} = A^{22} = A^{33} = A^{44} = u,$$
$$A^{12} = \rho, \quad A^{23} = 1/\rho, \quad A^{32} = \gamma p/(1 - b\rho).$$

System (2.2) being strictly hyperbolic admits four families of characteristics, among them two represent waves propagating in $\pm\, x$ directions with the speed $u \pm a$. The remaining two families form a set of double characteristics representing entropy waves or particle paths propagating with velocity u.

We consider waves propagating into an initial background state $U_0 = (\rho_0, 0, p_0, \sigma_0)^T$. The characteristic speeds at $U = U_0$ are given by $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = a_0$ and $\lambda_4 = -a_0$. The subscript 0 refers to evaluation at $U = U_0$, and is synonymous with the state of equilibrium.

3 Interaction of High Frequency Waves

We denote the left and right eigenvectors of A_0 associated with the eigenvalue λ_i by $L^{(i)}$ and $R^{(i)}$. These eigenvectors satisfy the normalization condition $L^{(i)}R^{(j)} = \delta_{ij}$, $1 \leq i, j \leq 4$, where δ_{ij} represents the Kronecker delta. These eigenvectors are obtained as

$$\begin{aligned}
L^{(1)} &= \Big(1, 0, -\frac{1}{a_0^2}, 0\Big), \quad R^{(1)} = (1, 0, 0, 0)^T, \\
L^{(2)} &= \Big(0, 0, 0, 1\Big), \quad R^{(2)} = (0, 0, 0, 1)^T, \\
L^{(3)} &= \Big(0, \frac{\rho_0}{2a_0}, \frac{1}{2a_0^2}, 0\Big), \quad R^{(3)} = (1, \frac{a_0}{\rho_0}, a_0^2, 0)^T, \\
L^{(4)} &= \Big(0, -\frac{\rho_0}{2a_0}, \frac{1}{2a_0^2}, 0\Big), \quad R^{(4)} = (1, -\frac{a_0}{\rho_0}, a_0^2, 0)^T.
\end{aligned} \tag{3.1}$$

We look for asymptotic solution for (2.2) as $\epsilon \to 0$ of the form

$$U \sim U_0 + \epsilon\, U_1(x, t, \tilde{\theta}) + \epsilon^2\, U_2(x, t, \tilde{\theta}) + O(\epsilon^3), \tag{3.2}$$

where $U_1 = (U_{11}, U_{12}, U_{13}, U_{14})^T$ is a smooth bounded vector, and vector U_2 is bounded in (x, t) in a certain bounded region of interest having at most sub-linear growth in θ as $\theta \to \pm\infty$. Here $\tilde{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)$ represents the "fast variables" characterized by the functions ϕ_i as $\theta_i = \phi_i/\epsilon$, where ϕ_i, $1 \leq i \leq 4$, is the phase of the i-th wave associated with the characteristic speed λ_i. Now we use (3.2) in (2.2), expand A and B in Taylor's series in powers of ϵ about $U = U_0$, replace the partial derivatives $\dfrac{\partial}{\partial X}$ (X being either x or t) by $\dfrac{\partial}{\partial X} + \epsilon^{-1}\sum_{i=1}^{4}\dfrac{\partial \phi_i}{\partial X}\dfrac{\partial}{\partial \theta_i}$, and equate to zero the coefficients of ϵ^0 and ϵ^1 in the resulting expansions, to obtain

$$O(\epsilon^0): \quad \sum_{i=1}^{4}\Big(I\frac{\partial \phi_i}{\partial t} + A_0\frac{\partial \phi_i}{\partial x}\Big)\frac{\partial U_1}{\partial \theta_i} = 0, \tag{3.3}$$

$$\begin{aligned}
O(\epsilon^1): \quad \sum_{i=1}^{4}\left(I\frac{\partial \phi_i}{\partial t} + A_0\frac{\partial \phi_i}{\partial x}\right)\frac{\partial U_2}{\partial \theta_i} &= -\frac{\partial U_1}{\partial t} - A_0\frac{\partial U_1}{\partial x} - (U_1.\nabla_U B)_0 \\
&\quad - \sum_{i=1}^{4}\frac{\partial \phi_i}{\partial x}(U_1.\nabla_U A)_0\frac{\partial U_1}{\partial \theta_i},
\end{aligned} \tag{3.4}$$

where I is the 4 x 4 unit matrix and ∇_U is the gradient operator with respect to the dependent variable U. The expressions of $(U_1.\nabla_U B)_0$ and $(U_1.\nabla_U A)_0$ are given as

$$\begin{aligned}
(U_1.\nabla_U B)_0 = \Big(&\frac{U_{12}m\rho_0}{x}, 0, -U_{11}(\gamma-1)\frac{p_0 c}{\tau\rho_0} + U_{12}\frac{m\gamma p_0}{(1-b\rho_0)x} \\
&+ U_{13}\frac{(\gamma-1)c(1-b\rho_0)}{\tau} - U_{14}\frac{(\gamma-1)\rho_0}{\tau}, U_{11}\frac{p_0 c}{\tau\rho_0^2} - U_{13}\frac{c(1-b\rho_0)}{\tau\rho_0} + U_{14}\frac{1}{\tau}\Big)^T,
\end{aligned}$$

$$(U_1.\nabla_U A)_0 = \begin{pmatrix} U_{12} & U_{11} & 0 & 0 \\ 0 & U_{12} & -\frac{U_{11}}{\rho_0^2} & 0 \\ 0 & \left(\frac{\gamma\ b\ U_{11}\ p_0}{(1-b\rho_0)^2} + \frac{\gamma\ U_{13}}{(1-b\rho_0)}\right) & U_{12} & 0 \\ 0 & 0 & 0 & U_{12} \end{pmatrix}.$$

Now since the phase functions ϕ_i, $1 \le i \le 4$, satisfy the eikonal equation

$$\text{Det}\ \left(I\frac{\partial \phi_i}{\partial t} + A_0 \frac{\partial \phi_i}{\partial x}\right) = 0,$$

we choose the simplest phase function of this equation, namely

$$\phi_i(x,t) = x - \lambda_i t, \quad 1 \le i \le 4.$$

It follows from (3.1) that for each phase ϕ_i, $\dfrac{\partial U_1}{\partial \theta_i}$ is parallel to the right eigenvector $R^{(i)}$ of A_0 and thus

$$U_1 = \sum_{i=1}^{4} \sigma_i(x,t,\theta_i) R^{(i)}, \tag{3.5}$$

where $\sigma_i = (L^{(i)} \cdot U_1)$ is a scalar function called the wave amplitude, that depends only on the i-th fast variable θ_i. We assume that $\sigma_i(x,t,\theta_i)$ has zero mean value with respect to the fast variable θ_i, that is,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sigma_i(x,t,\theta_i)\ d\theta_i = 0$$

We then use (3.5) in (3.4) and solve for U_2. To begin with we write

$$U_2 = \sum_{j=1}^{4} m_j R^{(j)},$$

substitute this value in (3.4), and premultiply the resulting equation by $L^{(i)}$ to obtain the system of decoupled inhomogeneous first order partial differential equations:

$$\sum_{j=1}^{4} (\lambda_i - \lambda_j) \frac{\partial m_i}{\partial \theta_j} = -\frac{\partial \sigma_i}{\partial t} - \lambda_i \frac{\partial \sigma_i}{\partial x} - L^{(i)}(U_1 \cdot \nabla B)_0 - \sum_{j=1}^{4} L^{(i)}(U_1 \cdot \nabla A)_0 \frac{\partial U_1}{\partial \theta_j}, \quad 1 \le i \le 4. \tag{3.6}$$

The characteristic ODEs for the i-th equation in (3.6) are given by

$$\dot{\theta}_j = \lambda_i - \lambda_j \quad \text{for } j \ne i, \quad \dot{\theta}_i = 0, \quad \dot{m}_i = H_i,$$

where

$$H_i(x,t,\theta_1,\theta_2,\theta_3,\theta_4) = -\frac{\partial \sigma_i}{\partial t} - \lambda_i \frac{\partial \sigma_i}{\partial x} - L^{(i)}(U_1 \cdot \nabla B)_0 - \sum_{j=1}^{4} L^{(i)}(U_1 \cdot \nabla A)_0 \frac{\partial U_1}{\partial \theta_j}.$$

We asymptotically average (3.6) along the characteristics and appeal to the sub-linearity of U_2 in θ, which ensures that the expression (3.2) does not contain secular terms. The constancy of θ_i along the characteristics and the vanishing asymptotic mean value of $\dot{m}_i$ along the characteristics implies that the wave amplitudes $\sigma_i, 1 \leq i \leq 4$, satisfy the following system of coupled integro-differential equations

$$\frac{\partial \sigma_i}{\partial t} + \lambda_i \frac{\partial \sigma_i}{\partial x} + a_i \sigma_i + \Gamma^i_{ii} \sigma_i \frac{\partial \sigma_i}{\partial \theta_i} + \sum_{i \neq j \neq k} \Gamma^i_{jk} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \sigma_j(\theta_i + (\lambda_i - \lambda_j)s) \acute{\sigma}_k(\theta_i + (\lambda_i - \lambda_k)s) ds = 0, \tag{3.7}$$

where $\acute{\sigma}_k = \dfrac{\partial \sigma_k}{\partial \theta_k}$ and the coefficients a_i and Γ^i_{jk} are given by

$$a_i = L^{(i)}(R^{(i)} \cdot \nabla B)_0, \quad \Gamma^i_{jk} = L^{(i)}(R^{(j)} \cdot \nabla A)_0 R^{(k)}. \tag{3.8}$$

The interaction coefficients Γ^i_{jk} denote the strength of coupling between the j-th and k-th wave modes ($j \neq k$) that can generate an i-th wave ($i \neq j \neq k$). The coefficients Γ^i_{ii} which refer to the non-linear self-interaction, are non-zero for genuinely non-linear waves and zero for linearly degenerate waves. It is also interesting to note that if all the coupling coefficients $\Gamma^i_{jk}(i \neq j \neq k)$ are zero or the integral in (3.7) vanishes, the waves do not resonate and (3.7) reduces to a system of uncoupled Burgers' equations. The coefficients a_i, Γ^i_{ii} and Γ^i_{jk}, given by (3.7), provide a picture of the non-linear interaction process present in the system under consideration, and can be easily determined in the following form, the non-zero coefficients are given as

$$\begin{aligned}
a_1 &= \frac{(\gamma-1)}{a_0^2} \frac{p_0\ c}{\tau \rho_0}, \quad a_2 = \frac{1}{\tau}, \\
a_3 &= \frac{m a_0}{2x} + \frac{(\gamma-1)}{2a_0^2} \Big(\frac{c a_0^2 (1 - b\rho_0)}{\tau} - \frac{c p_0}{\tau \rho_0} \Big), \\
a_4 &= -\frac{m a_0}{2x} + \frac{(\gamma-1)}{2a_0^2} \Big(\frac{c a_0^2 (1 - b\rho_0)}{\tau} - \frac{c p_0}{\tau \rho_0} \Big), \\
\Gamma^1_{13} &= -\Gamma^1_{14} = \frac{a_0(1 - 2b\rho_0)}{\rho_0(1 - b\rho_0)}, \quad \Gamma^1_{34} = -\Gamma^1_{43} = -\frac{a_0}{\rho_0}\{(1-\gamma) - 2b\rho_0\}, \\
\Gamma^3_{14} &= -\Gamma^4_{13} = -\frac{a_0}{2\rho_0(1 - b\rho_0)}, \quad \Gamma^3_{33} = -\Gamma^4_{44} = \frac{(\gamma+1)}{2} \frac{a_0}{\rho_0(1 - b\rho_0)}.
\end{aligned} \tag{3.9}$$

The resonance equations (3.7) can now be written as

$$\frac{\partial\sigma_1}{\partial t}+a_1\sigma_1=0,\quad \frac{\partial\sigma_2}{\partial t}+a_2\sigma_2=0 \tag{3.10}$$

$$\frac{\partial\sigma_3}{\partial t}+a_0\frac{\partial\sigma_3}{\partial x}+a_3\sigma_3+\mathit{\Gamma}\sigma_3\frac{\partial\sigma_3}{\partial\theta_3}-\lim_{P\to\infty}\frac{1}{2P}\int_{-P}^{P}K\Big(x,t,\frac{\theta_3+\phi}{2}\Big)\sigma_4(x,t,\phi)d\phi=0,$$

$$\frac{\partial\sigma_4}{\partial t}-a_0\frac{\partial\sigma_4}{\partial x}+a_4\sigma_4-\mathit{\Gamma}\sigma_4\frac{\partial\sigma_4}{\partial\theta_4}+\lim_{P\to\infty}\frac{1}{2P}\int_{-P}^{P}K\Big(x,t,\frac{\theta_4+\phi}{2}\Big)\sigma_3(x,t,\phi)d\phi=0,$$

where $\mathit{\Gamma}=\mathit{\Gamma}^3_{33}=-\mathit{\Gamma}^4_{44}=(\gamma+1)a_0/(2\rho_0(1-b\rho_0))$, and the kernel K is defined as

$$K\Big(x,t,\frac{\theta+\phi}{2}\Big)=\frac{\mathit{\Gamma}^3_{14}}{2}\frac{\partial\sigma_1}{\partial\theta_1}\Big(x,t,\frac{\theta+\phi}{2}\Big).$$

Let the initial value of σ_j be $\sigma_j|_{t=0}=\sigma^0_j(x,\theta_j)$. Hence $(3.10)_{1,2}$ gives

$$\sigma_1(x,t,\theta_1)=\sigma^0_1(x,\theta_1)e^{-a_1t},\quad \sigma_2(x,t,\theta_2)=\sigma^0_2(x,\theta_2)e^{-a_2t},$$

and subsequently the system (3.10) reduces to a pair of equations for the wave fields σ_3 and σ_4 coupled through the linear integral operator involving the kernel

$$K(x,t,\theta)=\frac{\mathit{\Gamma}^3_{14}}{2}e^{-a_1t}\frac{\partial\sigma^0_1}{\partial\theta_1}(x,\theta). \tag{3.11}$$

If the initial data $\sigma^0_j(x,\theta)$ are 2π periodic functions of the phase variable θ, then the pair of resonant asymptotic equations in system (3.10) becomes

$$\frac{\partial\sigma_3}{\partial t}+a_0\frac{\partial\sigma_3}{\partial x}+a_3\sigma_3+\mathit{\Gamma}\sigma_3\frac{\partial\sigma_3}{\partial\theta_3}-\frac{1}{2\pi}\int_{-\pi}^{\pi}K\Big(x,t,\frac{\theta_3+\phi}{2}\Big)\sigma_4(x,t,\phi)\ d\phi=0, \tag{3.12}$$

$$\frac{\partial\sigma_4}{\partial t}-a_0\frac{\partial\sigma_4}{\partial x}+a_4\sigma_4-\mathit{\Gamma}\sigma_4\frac{\partial\sigma_4}{\partial\theta_4}+\frac{1}{2\pi}\int_{-\pi}^{\pi}K\Big(x,t,\frac{\theta_4+\phi}{2}\Big)\sigma_3(x,t,\phi)\ d\phi=0,$$

where K is given by (3.11).

The system of equations obtained in equation (3.10) is (mathematically) the same system obtained in [16], for the compressible Euler equations of Gas Dynamics. The thing that changes is the physical meaning of the various parameters (coupling kernel K in the integral term, wave velocities, etc.) because of the effect of vibrational relaxation. The behavior of this system (including shocks even in the resonant case) was studied extensively in [17].

More details (for the special case when the wave interactions occur within a closed pipe) can be found in [26]. Shocks in the resonant case within a bounded environment seem to be very important, as they lead the solutions to asymptote to very non-trivial nonlinear standing wave like states.

4 Non-Linear Geometrical Acoustics Solution

The asymptotic solution (3.2) of hyperbolic system (2.2) satisfying small amplitude oscillating initial data

$$U(x,\ 0) = U_0 + \epsilon\ U_1^0(x,\ x/\epsilon) + O(\epsilon^2), \tag{4.1}$$

is non-resonant if $U_1^0(x,\ x/\epsilon)$ are smooth functions with a compact support [16]. The issue of how to deal with shocks when resonances occur is studied in Cehelsky and Rosales [5]. The characteristic equations are

$$\frac{d\theta_j}{dx} = \frac{\Gamma\sigma_j}{a_0}, \qquad \frac{dt}{dx} = \frac{e_j}{a_0}, \tag{4.2}$$

where

$$e_j = \begin{cases} +1, & \text{if } j = 3, \\ -1, & \text{if } j = 4. \end{cases}$$

In terms of the characteristic equations, the decoupled equations $(3.10)_3$ and $(3.10)_4$ can be written as

$$\frac{d\sigma_j}{dt} = -a_j\ \sigma_j, \quad j = 3, 4.$$

Integration along the rays $s_j = x - e_j\ a_0\ t =$ constant yields:

$$\sigma_j = \sigma_j^0(s_j,\ \xi_j)\ e^{-a_j t}, \tag{4.3}$$

where the function σ_j^0 is obtained from the initial condition (4.1), and the fast variable ξ_j parameterizes the set of characteristic curves $(4.2)_1$.

Thus, we obtain from (4.2)

$$\xi_j = \theta_j + \frac{\Gamma_{jj}^j}{a_j}\Big(\sigma_j - \sigma_j^0\Big). \tag{4.4}$$

The solution of (2.2), satisfying (4.1), where $U_1^0(x, x/t)$ has compact support, is obtained as

$$\begin{aligned} \rho(x,t) &= \rho_0 + \epsilon\sigma_1^0(x, x/\epsilon)e^{-a_1 t} + \epsilon\big(\sigma_3^0(x - a_0 t, \xi_3)e^{-a_3 t} \\ &\qquad + \sigma_4^0(x + a_0 t, \xi_4)e^{-a_4 t}\big) + O(\epsilon^2), \\ u(x,t) &= \epsilon\frac{a_0}{\rho_0}\Big(\sigma_3^0(x - a_0 t, \xi_3)e^{-a_3 t} - \sigma_4^0(x + a_0 t, \xi_4)e^{-a_4 t}\Big) + O(\epsilon^2), \\ p(x,t) &= p_0 + \epsilon a_0^2\Big(\sigma_3^0(x - a_0 t, \xi_3)e^{-a_3 t} + \sigma_4^0(x + a_0 t, \xi_4)e^{-a_4 t}\Big) + O(\epsilon^2), \\ \sigma(x,t) &= \sigma_0 + \epsilon\sigma_2^0(x, x/\epsilon)e^{-a_2 t} + O(\epsilon^2), \end{aligned} \tag{4.5}$$

where a_1, a_2, a_3, a_4 are given in (3.9) with $a_0^2 = \dfrac{\gamma p_0}{\rho_0(1 - b\rho_0)}$, the fast variables $\xi_j (1 \le j \le 4)$ are given in (4.4), and the initial values for σ_i, $(1 \le i \le 4)$ are obtained from the solution (4.5) specified at $t = 0$ as

$$\sigma_1^0(x, x/\epsilon) = \rho_1^0(x, x/\epsilon) - \frac{1}{a_0^2} p_1^0(x, x/\epsilon), \quad \sigma_2^0(x, x/\epsilon) = \sigma_1^0(x, x/\epsilon),$$
$$\sigma_3^0(x, \xi_3) = \Big(\frac{\rho_0}{2a_0}\Big) u_1^0(x, \xi_3) + \Big(\frac{1}{2a_0^2}\Big) p_1^0(x, \xi_3),$$
$$\sigma_4^0(x, \xi_4) = -\Big(\frac{\rho_0}{2a_0}\Big) u_1^0(x, \xi_4) + \Big(\frac{1}{2a_0^2}\Big) p_1^0(x, \xi_4).$$

This is the complete solution of (2.2) and (4.1); any multivalued overlap in this solution is resolved by introducing shocks into the solution. In (4.5) if we substitute the values of a_1, a_2, a_3 and a_4 given in (3.9) with $a_0^2 = \dfrac{\gamma p_0}{\rho_0(1 - b\rho_0)}$, we obtain the dependence of ρ, u, p and σ upon b and m explicitly. For $b = 0$ these equations yield the results for the ideal gas.

5 Shock Waves

A shock wave may be initiated in the flow region, and once it is formed, it will propagate by separating the portions of the continuous region. Following [11] it can be shown that the shock location θ_j^s satisfies the relation

$$\frac{d\theta_j^s}{dt} = \frac{1}{2} \Gamma_{jj}^j \Big(\sigma_j^{(-)} + \sigma_j^{(+)}\Big), \quad j = 3, 4,$$

which is the shock speed in $\theta_j - t$ plane. Here $\sigma_j^{(-)}$ and $\sigma_j^{(+)}$, respectively, are the values of σ_j just ahead and behind the shock. We have $\sigma_j^{(-)} = 0$ for the undisturbed region ahead of the shock. Now we use (4.3) and drop the superscripts on θ_j^s and σ_j^+ to obtain

$$\frac{d\theta_j}{dt} = \frac{\Gamma_{jj}^j}{2} \sigma_j^0(s_j, \xi_j) e^{-a_j t} \tag{5.1}$$

Now using (5.1) and (4.4) we obtain the following equation which determines the shock path parametrically,

$$\theta_j = \xi_j - \frac{2}{\sigma_j^0} \int_0^{\xi_j} \sigma_j^0(\tilde{t})\, d\tilde{t}. \tag{5.2}$$

If $\sigma_j^0 \neq 0$ then the shock forms right at the origin.

6 Conclusion

The method of weakly non-linear geometrical acoustics is used to obtain the small amplitude high frequency asymptotic solution to the basic equations governing one dimensional unsteady planar, cylindrically and spherically symmetric flow in a vibrationally relaxing gas with Van der Waals equation of state.

We used the weakly nonlinear geometrical acoustics theory to analyze the resonant waves interaction, and derived the transport equations for the wave amplitudes along the rays of the governing system; these transport equations constitute a system of inviscid Burger's' equations with quadratic nonlinearity coupled through a linear integral operators with a known kernel. The coefficients appearing in the transport equations provide a measure of coupling between the various modes and set a qualitative picture of the interaction process involved therein.

The system of equations obtained in equation (3.10) is (mathematically) the same system obtained in [24] and [16], for the compressible Euler equations of gas dynamics. The thing that changes is the physical meaning of the various parameters (coupling kernel K in the integral term, wave velocities, etc.) because of the presence of vibrational relaxation. The behavior of this system (including shocks even in the resonant case) was studied extensively in [17].

It is observed that the wave fields associated with the particle paths do not interact with each other; however they do interact with an acoustic wave field to produce resonant contribution towards the other acoustic field. The acoustic wave fields may or may not interact, but in either case their net contribution, which is directed towards the entropy field, is always zero.

In our analysis the governing system of Euler equations reduces to a pair of resonant asymptotic equations for the acoustic wave fields. For a non-resonant multi wave mode case, proposed by Hunter and Keller [11], the reduced system of transport equations gets decoupled with vanishing integral average terms, and the occurrences of shocks in the acoustic wave fields are analyzed. It is found that in a contracting piston motion having spherical symmetry, a shock is always formed before the formation of a focus no matter how small is the initial wave amplitude; this is in contrast with the corresponding cylindrical situation where a shock forms before the focus only if the initial amplitude exceeds a critical value.

The dependence of ρ, u, p and σ on the parameter m and the Van der Waals excluded volume b is clear from (4.5) upon substitution of the values of a_1, a_2, a_3 and a_4 given in (3.9) with $a_0^2 = \dfrac{\gamma p_0}{\rho_0(1-b\rho_0)}$. For $b = 0$ the equations (4.5) yield the results for the ideal gas; these results are in agreement with earlier results (Whitham [28] and Krylovas and Čiegis [14]). Krylovas and Čiegis [14] developed a method of averaging for constructing a uniformly valid asymptotic solution for weakly nonlinear one dimensional gas dynamics systems, and they solved numerically a system similar to (3.12). The shock amplitude depends very strongly on the internal energy that exists in the relaxing mode, and depends on the wave geometry through the parameter m.

References

[1] Rajan Arora. Similarity solutions and evolution of weak discontinuities in a van der Waals gas. *The Canadian Applied Mathematics Quarterly*, **13**:297–311, 2005.

[2] Rajan Arora. Non-planar shock waves in a magnetic field. *Computers and*

Mathematics with Applications (Elsevier), **56**:2686–2691, 2008. (doi:10.1016/j.camwa.2008.03.056)

[3] Rajan Arora and V.D. Sharma. Convergence of strong shock in a van der Waals gas. *SIAM J. Applied Mathematics*, **66**:1825–1837, 2006. (doi:10.1137/050634402)

[4] P.A. Blythe. Nonlinear wave propagation in a relaxing gas. *J. Fluid Mech.*, **37**:31–50, 1969. (doi:10.1017/S0022112069000401)

[5] P. Cehelsky and R.R. Rosales. Resonantly interacting weakly nonlinear hyperbolic waves in the presence of shocks: A single space variable in a homogeneous, time independent medium. *Studies in Applied Mathematics*, **74**:117–138, 1986.

[6] V. Choquet-Bruhat. Ondes asymptotique et approchees pour systemes d'equations aux derivees partielles nonlineaires. *J. Math. Pures Appl.*, **48**:119–158, 1969.

[7] J. F. Clarke and M. McChesney. *Dynamics of Relaxing Gases*. Butterworths, London, 1976.

[8] P. Germain. Progressive waves. In *14th Prandtl Memorial Lecture, Jarbuch, der DGLR*, pp. 11–30, 1971.

[9] Y. He and T.B. Moodie. Two-wave interactions for weakly nonlinear hyperbolic waves. *Stud. Appl. Math.*, **88**:241–267, 1993.

[10] Y. He and T.B. Moodie. Geometrical optics and post shock behaviour for nonlinear conservation laws. *Applicable Analysis*, **57**:145–176, 1995. (doi:10.1002/cpa.3160360502)

[11] J.K. Hunter and J. Keller. Weakly nonlinear high frequency waves. *Comm. Pure Appl. Math.*, **36**:547–569, 1983. (doi:10.1002/cpa.3160360502)

[12] J.K. Hunter, A. Majda and R. Rosales. Resonantly interacting weakly nonlinear hyperbolic waves II. Several space variables. *Stud. Appl. Math.*, **75**:187–226, 1986.

[13] L.A. Kalyakin. Long wave asymptotics. Integrable equations as asymptotic limits of non-linear systems. *RUSS. MATH. SURV.*, **44**(1):3–42, 1989. translation from *Uspehi Mat. Nauk*, 1989. **44**(1), (in Russian)

[14] A. Krylovas and R. Čiegis. Asymptotical analysis of one dimensional gas dynamics equations. *Math. Model. Anal.*, **6**(1):117–128, 2001.

[15] A. Krylovas and R. Čiegis. Review of numerical asymptotic averaging for weakly nonlinear hyperbolic waves. *Math. Model. Anal.*, **9**(3):209–222, 2004.

[16] A. Majda and R. Rosales. Resonantly interacting weakly nonlinear hyperbolic waves. *Stud. Appl. Math.*, **71**:149–179, 1984.

[17] A.J. Majda, R. Rosales and M. Schonbek. A canonical system of integro-differential equations arising in resonant nonlinear acoustics. *Studies in Applied Mathematics*, **79**:205–262, 1988.

[18] T.B. Moodie, Y. He and D.W. Barclay. Wavefront expansions for nonlinear hyperbolic waves. *Wave Motion*, **14**:347–367, 1991. (doi:10.1016/0165-2125(91)90030-R)

[19] H. Ockenden and D.A. Spence. Nonlinear wave propagation in a relaxing gas. *J. Fluid Mech.*, **39**, 1969.

[20] D.F. Parker. Propagation of rapid pulses through a relaxing gas. *Phys. Fluids*, **15**:256–262, 1972. (doi:10.1063/1.1693902)

[21] Ch. Radha and V. D. Sharma. Propagation and interaction of waves in a relaxing gas. *Philos. Trans. Roy. Soc. London A*, **352**:169–195, 1995. (doi:10.1098/rsta.1995.0062)

[22] W.A. Scott and N.H. Johannesen. Spherical nonlinear wave propagation in a vibrationally relaxing gas. *Proc. R. Soc. London*, **Ser. A 382**:103–134, 1982. (doi:10.1098/rspa.1982.0092)

[23] V.D. Sharma and Rajan Arora. Similarity solutions for strong shocks in an ideal gas. *Stud. Appl. Math.*, **114**:375–394, 2005. (doi:10.1111/j.0022-2526.2005.01557.x)

[24] V.D. Sharma and Ch. Radha. Similarity solutions for converging shock in a relaxing gas. *Int. J. Engg. Science*, **33**:535–553, 1995. (doi:10.1016/0020-7225(94)00086-7)

[25] V.D. Sharma and Gopala Krishna Srinivasan. Wave interaction in a non-equilibrium gas flow. *Int. J. Non-linear Mechanics*, **40**:1031–1040, 2005. (doi:10.1016/j.ijnonlinmec.2005.02.003)

[26] M. Shefter and R.R. Rosales. Quasi-periodic solutions in weakly nonlinear gas dynamics. part I. Numerical results in the inviscid case. *Studies in Applied Mathematics*, **103**:279–337, 1999. (doi:10.1111/1467-9590.1034137)

[27] A.L. Shtaras. Asymptotic integration of weakly nonlinear partial differential equations(russian, english). *Sov. Math. Dokl.*, **18**:1462–1466, 1977. 1978; translation from *Dokl. Akad. Nauk SSSR* 237, 525–528, 1977

[28] G.B. Whitham. *Linear and Nonlinear Waves.* Wiley, New York, 1974.