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# Existence and Iteration of a Positive Solution to a Second-Order Quasilinear Problem

**Q. Yao**

*Department of Applied Mathematics,*
*Nanjing University of Finance and Economics*
Nanjing 210003, China
E-mail: yaoqingliu2002@hotmail.com

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**Abstract.** A successive iteration sequence of positive solution is structured for a singular second-order quasilinear problem. The sequence can converge uniformly to the positive solution of the problem. Main tools are the Hammerstein integral equation and the monotone iteration technique on cone. The iteration sequence starts with known constant function and, therefore, is useful in computation analysis.

**Key words:** Singular quasilinear problem, positive solution, successive iteration, existence theorem.

## 1 Introduction

Let $n \geq 3$ be a positive integer and $E \subset [0,1]$ be a closed subset with zero measure. The purpose of this paper is to consider the existence and iteration of positive solution for the following second-order quasilinear problem

$$(P) \quad \begin{cases} u''(t) + \dfrac{n-1}{t} u'(t) + f(t, u(t)) = 0, & t \in [0,1] \backslash E, \\ u'(0) = 0, \quad u(1) = 0. \end{cases}$$

Here, $u^*$ is called a positive solution of the problem $(P)$ if $u^*$ is a solution of $(P)$ and $u^*(t) > 0,\ 0 \leq t < 1$.

Because there exist important applications to the non-Newton flow and the combustion theory (see [1, 2]), the problem $(P)$ has been studied by many authors (see [3, 4, 5, 6, 8, 9, 10, 11, 17]). Among others, some authors considered the computational methods of its solution (see [6, 10, 17]). However these papers mainly discussed the continuous problem $(P)$. Most of the results concentrated on some special kinds of the problem $(P)$, for example when the source term is given as $f(t,u) = h(t)u^{\lambda}e^{-\mu u}$, $0 \leq \lambda < 1$, $0 \leq \mu < +\infty$ (particularly, the Emden-Fowler equation).

In this paper we will use the following assumptions:

(H1) $f : ([0,1]\backslash E) \times [0,+\infty) \to [0,+\infty)$ is continuous.
(H2) There exist constants $0 < k_1 \le k_2$, $0 < \lambda < 1$ and nonnegative functions $j_i \in L^1[0,1] \cap C([0,1]\backslash E)$, $i = 1, 2$ such that

$$(k_1 - j_1(t))u^\lambda \le f(t,u) \le (k_2 + j_2(t))u^\lambda, \quad (t,u) \in ([0,1]\backslash E) \times [0,+\infty),$$

where $\sup\limits_{0\le t\le 1} j_1(t) \le k_1$ and $\int_\mu^\nu (k_1 - j_1(t))dt > 0$ for any $0 \le \mu < \nu \le 1$.
(H3) $f(t,\cdot) : [0,+\infty) \to [0,+\infty)$ is nondecreasing for any $t \in [0,1]\backslash E$.
Therefore, we allow that $f(t,u)$ can be singular on the set $E \times [0,+\infty)$. Particularly, if $E = \{0,1\}$, then $f : (0,1) \times [0,+\infty) \to [0,+\infty)$ is continuous and may be singular at $t = 0, 1$.

Under the assumptions (H1)–(H3), problem $(P)$ may be nonautonomous and the decomposition $f(t,u) = h(t)u^\lambda$ may not exist. To our best knowledge, the existence and iteration of positive solution of the problem $(P)$ under the assumptions (H1)–(H3) are not considered by any author.

By making use of the Hammerstien integral equation (see [7]) and the monotone iterative technique on cone (see [9]), we will construct a successive iteration sequence. In order to prove the convergence of the sequence, we will construct an effective cone. Applying the sequence and the cone, we will obtain not only the existence of a positive solution, but also construct the iteration method for finding this positive solution. The motivation of this work comes from our papers [12, 13, 14, 15, 16].

## 2 Preliminaries

Consider the Banach space $C[0,1]$ equipped with the norm $\|u\| = \max\limits_{0\le t\le 1} |u(t)|$. Let $G(t,s)$ be the Green function of the problem $(P)$ when $f(t,u) \equiv 0$:

$$G(t,s) = \begin{cases} \dfrac{1}{n-2} s^{n-1}(s^{2-n} - 1), & 0 < t \le s \le 1, \\ \dfrac{1}{n-2} s^{n-1}(t^{2-n} - 1), & 0 < s \le t \le 1, \end{cases}$$

Computing the partial derivative of $G(t,s)$ with respect to $t$, we obtain

$$\frac{\partial}{\partial t} G(t,s) = \begin{cases} 0, & 0 \le t < s \le 1, \\ -s^{n-1}t^{1-n}, & 0 < s \le t \le 1. \end{cases}$$

In this paper, we will use the following constants and symbols:

$$\gamma = \left(\frac{1}{2}\right)^{1/(n-2)}, \quad \beta_1 = \int_0^\gamma s^{n-1} j_1(s)\, ds, \quad \beta_2 = \int_0^\gamma (s - s^{n-1}) j_2(s)\, ds,$$

$$\sigma = \left[\frac{k_1 - n2^{n/(n-2)}\beta_1}{k_2(n-1) + n2^{n/(n-2)}\beta_2}\right]^{1/(1-\lambda)}, \ A = \max_{0\le t\le 1} \int_0^1 G(t,s) ds = \frac{1}{2n},$$

$$B = \max_{0\le t\le 1} \int_0^\gamma G(t,s)\,ds = \frac{n-1}{2n(n-2)\sqrt[n-2]{4}},$$

$C^+[0,1]=\{u \in C[0,1] : u(t)\ge 0,\ 0 \le t \le 1\}$,

$K = \{u \in C^+[0,1] : \min_{0\le t\le\gamma} u(t) \ge \sigma\|u\|\}$, $K[r_1,r_2] = \{u \in K : r_1 \le \|u\| \le r_2\}$.

Obviously, $0 < \sigma < 1$, $0 < B < A$ and $K$ is a cone of nonnegative functions in $C[0,1]$. In addition, we denote

$$M_1 = \max_{0\le t\le 1} \int_0^\gamma G(t,s)j_1(s)\,ds, \quad M_2 = \max_{0\le t\le 1} \int_0^1 G(t,s)j_2(s)\,ds.$$

Define the operator $T$ as follows

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s))\,ds, \quad 0 \le t \le 1, \ \ u \in C^+[0,1].$$

It is easy to see that $T : C^+[0,1] \to C^+[0,1]$ is well defined under the assumptions (H1) and (H2).

**Lemma 1.** *Assume that* (H1) *and* (H2) *hold. Then for any* $u \in K$,

$$(Tu)'(t) = \int_0^1 \frac{\partial}{\partial t}G(t,s)f(s,u(s))\,ds, \quad 0 \le t \le 1.$$

*Proof.* It is easy to see that $|G(t+\Delta t,s) - G(t,s)| \le |\Delta t|$. Let $u \in K$. Then

$$\left|\frac{1}{\Delta t}\left[G(t+\Delta t,s) - G(t,s)\right] f(s,u(s))\right| \le f(s,u(s)).$$

By the assumption (H2),

$$\int_0^1 f(s,u(s))\,ds \le \Big(k_2 + \int_0^1 j_2(s)\,ds\Big)\|u\|^\lambda < \infty.$$

Applying the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}(Tu)'(t) &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left[(Tu)(t+\Delta t) - (Tu)(t)\right]\\ &= \lim_{\Delta t\to 0} \int_0^1 \frac{G(t+\Delta t,s) - G(t,s)}{\Delta t} f(s,u(s))\,ds = \int_0^1 \frac{\partial}{\partial t}G(t,s)f(s,u(s))\,ds.\end{aligned}$$

□

**Lemma 2.** *Assume that* (H1) *and* (H2) *hold. Then the operator* $T$ *has following properties:*

(1) *If* $u \in C^+[0,1]$ *and* $\|u\| > 0$, *then* $(Tu)(t) > 0$, $0 \le t < 1$.

(2) *If* $u \in C^+[0,1]$, *then* $(Tu)(t)$ *is a non-increasing function on* $[0,1]$ *and* $\|Tu\| = (Tu)(0)$.

(3) $T : C^+[0,1] \to C^+[0,1]$ *is completely continuous.*

*Proof.* (1) By the expression of $G(t,s)$, we have $G(t,s) > 0,\ 0 < t,s < 1$. Let $u \in C^+[0,1]$ and $\|u\| > 0$, then there exist $0 \le \mu < \nu \le 1$ such that $u(t) > 0$, $\mu < t < \nu$. By the assumption (H2),

$$f(t,u(t)) \ge (k_1 - j_1(t))u^\lambda(t), \quad k_1 - j_1(t) \ge 0, \quad 0 \le t \le 1$$

and $\int_\mu^\nu (k_1 - j_1(t))\,dt > 0$. Applying these facts, we get, for $0 \le t < 1$,

$$(Tu)(t) \ge \int_0^1 G(t,s)(k_1 - j_1(s))u^\lambda(s)\,ds \ge \int_\mu^\nu G(t,s)(k_1 - j_1(s))u^\lambda(s)\,ds > 0.$$

(2) Direct computations give that

$$(Tu)(1) = \frac{1}{n-2}\int_0^1 s^{n-1}(1-1)f(s,u(s))\,ds = 0.$$

By Lemma 1 and the expression of $\frac{\partial}{\partial t}G(t,s)$, we have

$$(Tu)'(t) = -\frac{1}{t^{n-1}}\int_0^t s^{n-1}f(s,u(s))\,ds \le 0, \quad 0 \le t \le 1.$$

Thus, $(Tu)(t)$ is a non-increasing function on $[0,1]$ and $\|Tu\| = (Tu)(0)$.

(3) Let us assume that $r > 0$ and $V(r) = \{u \in C^+[0,1] : \|u\| \le r\}$. Let denote $\zeta_m(t) = \min\{j_2(t), m\}$, then $\lim\limits_{m\to\infty} \int_0^1 [j_2(t) - \zeta_m(t)]\,dt = 0$. Define

$$f_m(t,u) = \begin{cases} f(t,u), & f(t,u) \le (k_2 + \zeta_m(t))u^\lambda, \\ (k_2 + \zeta_m(t))u^\lambda, & f(t,u) \ge (k_2 + \zeta_m(t))u^\lambda; \end{cases}$$

$$(T_m u)(t) = \int_0^1 G(t,s)f_m(s,u(s))\,ds, \quad 0 \le t \le 1, \ \ u \in C^+[0,1].$$

Then $f_m : ([0,1]\backslash E) \times [0,r] \to [0,+\infty)$ is continuous and bounded. Let

$$M(r,m) = \sup\{f_m(t,u) : (t,u) \in ([0,1]\backslash E) \times [0,r]\}.$$

If $u \in V(r)$, then $0 \le u(t) \le r,\ 0 \le t \le 1$. Thus, $f_m(t,u(t)) \le M(r,m),\ t \in [0,1]\backslash E$. So, for $0 \le t \le 1$, we obtain

$$\begin{aligned} |(T_m u)'(t)| = \frac{1}{t^{n-1}}\int_0^t s^{n-1}f_m(s,u(s))\,ds &\le \frac{M(r,m)}{t^{n-1}}\int_0^t s^{n-1}\,ds \\ &\le \frac{M(r,m)t}{n} \le \frac{M(r,m)}{n}. \end{aligned}$$

By the differential mean value theorem, we have

$$|(T_m u)(t_2) - (T_m u)(t_1)| \le \frac{M(r,m)}{n}|t_2 - t_1|, \quad 0 \le t_1, t_2 \le 1.$$

It implies that $T_m(V(r))$ is an equicontinuous set in $C[0,1]$. On the other hand,

$$\begin{aligned}\sup\{\|T_m u\|: \ u\in V(r)\} &= \sup_{u\in V(r)}\max_{0\le t\le 1}\int_0^1 G(t,s)f_m(s,u(s))\,ds\\ &\le M(r,m)\max_{0\le t\le 1}\int_0^1 G(t,s)ds \le M(r,m)/(2n).\end{aligned}$$

It follows that $\underline{T_m(V(r))}$ is a bounded set in $C[0,1]$. By the Arzela-Ascoli theorem, the set $\overline{T_m(V(r))}$ is compact and $T_m: V(r)\to C^+[0,1]$ is completely continuous. Since

$$\begin{aligned}&\sup_{u\in V(r)}\|Tu - T_m u\| = \sup_{u\in V(r)}\ \max_{0\le t\le 1}\int_0^1 G(t,s)[f(s,u(s)-f_m(s,u(s))]\,ds\\ &\quad\le \sup_{u\in V(r)}\max_{0\le t\le 1}\int_0^1 G(t,s)(j_2(s)-\zeta_m(s))u^\lambda(s)\,ds\\ &\quad\le r^\lambda \max_{0\le t,s\le 1} G(t,s)\int_0^1 (j_2(s)-\zeta_m(s))\,ds\to 0\ \ (m\to\infty).\end{aligned}$$

It implies that the completely continuous operators $\{T_m\}_{m=1}^\infty$ uniformly converges to $T$ on the bounded set $V(r)$. Thus, $T: V(r)\to C^+[0,1]$ is completely continuous. By the arbitrariness of $r>0$, $T:\ C^+[0,1]\to C^+[0,1]$ is completely continuous. $\square$

**Lemma 3.** *Assume that* (H1) *and* (H2) *hold. Then* $T: K\to K$.

*Proof.* Let $u\in K$. Then $Tu\in C^+[0,1]$ by Lemma 2(3). We need to prove that $\min\limits_{0\le t\le\gamma}(Tu)(t)\ge\sigma\|Tu\|$. We will use the useful fact: if $\beta\ge\alpha>0,\ \tau\ge 0$, then $\frac{\alpha+\tau}{\beta+\tau}\ge\frac{\alpha}{\beta}$. From Lemma 2 (2) and the definition of constant $\sigma$, we have

$$\begin{aligned}\frac{\min\limits_{0\le t\le\gamma}(Tu)(t)}{\|Tu\|} &= \frac{(Tu)(\gamma)}{(Tu)(0)}\\ &= \frac{\int_0^\gamma s^{n-1}(\gamma^{2-n}-1)f(s,u(s))\,ds + \int_\gamma^1 s^{n-1}(s^{2-n}-1)f(s,u(s))\,ds}{\int_0^\gamma s^{n-1}(s^{2-n}-1)f(s,u(s))\,ds + \int_\gamma^1 s^{n-1}(s^{2-n}-1)f(s,u(s))\,ds}\\ &\ge \frac{\int_0^\gamma s^{n-1}(\gamma^{2-n}-1)f(s,u(s))\,ds}{\int_0^\gamma s^{n-1}(s^{2-n}-1)f(s,u(s))\,ds} \ge \frac{\int_0^\gamma s^{n-1}(k_1-j_1(s))u^\lambda(s)\,ds}{\int_0^\gamma (s-s^{n-1})(k_2+j_2(s))u^\lambda(s)\,ds}\\ &\ge \frac{\left[\min\limits_{0\le t\le\gamma}u(t)\right]^\lambda\left[k_1\int_0^\gamma s^{n-1}\,ds - \int_0^\gamma s^{n-1}j_1(s)\,ds\right]}{\left[\max\limits_{0\le t\le 1}u(t)\right]^\lambda\left[k_2\int_0^\gamma (s-s^{n-1})\,ds + \int_0^\gamma (s-s^{n-1})j_2(s)\,ds\right]}\\ &= \frac{\sigma^\lambda\|u\|^\lambda(k_1\frac{1}{n}(\frac{1}{2})^{n/(n-2)}-\beta_1)}{\|u\|^\lambda\big(k_2\big[\frac12(\frac12)^{n/(n-2)}-\frac1n(\frac12)^{n/(n-2)}\big]+\beta_2\big)} = \frac{(k_1-n2^{n/(n-2)}\beta_1)\sigma^\lambda}{k_2(n-1)+n2^{n/(n-2)}\beta_2} = \sigma.\end{aligned}$$

The proof is complete. $\square$

## 3 The Main Result

Let $a \geq [k_2 A + M_2]^{1/(1-\lambda)}$ be a positive constant. In this paper, we construct the following successive iteration sequence

$$u_0(t) \equiv a, \quad u_m(t) = \int_0^1 G(t,s) f(s, u_{m-1}(s)) ds, \quad 0 \leq t \leq 1, \ m = 1, 2, \ldots.$$

The main result is given in Theorem 1. It shows that the problem $(P)$ has a positive solution $u^* \in C[0,1]$ and the sequence $\{u_m(t)\}_{m=1}^{\infty}$ converges uniformly to $u^*(t)$ on $[0,1]$ under the assumptions (H1)–(H3).

**Theorem 1.** *Assume that* (H1)–(H3) *hold. Then problem* $(P)$ *has one positive solution* $u^* \in K \cap C^1[0,1] \cap C^2([0,1] \backslash E)$ *such that*

$$\|u^*\| \leq a, \qquad \lim_{m \to \infty} \|\tilde{u}_m - u^*\| = 0.$$

*Proof.* Let $0 < b \leq [\sigma^{\lambda}(k_1 B - M_1)]^{1/(1-\lambda)}$. Obviously, $b < a$. By Lemma 2 and 3, the operator $T : K[a,b] \to K$ is completely continuous. If $u \in K[a,b]$, then $\sigma b \leq \sigma \|u\| \leq \min\limits_{0 \leq t \leq \gamma} u(t) \leq \max\limits_{0 \leq t \leq 1} u(t) = \|u\| \leq a$. It follows that

$$\begin{aligned}
\|Tu\| &\leq \max_{0 \leq t \leq 1} \int_0^1 G(t,s)(k_2 + j_2(s)) u^{\lambda}(s)\, ds \\
&\leq \|u\|^{\lambda} \left[ k_2 \max_{0 \leq t \leq 1} \int_0^1 G(t,s) ds + \max_{0 \leq t \leq 1} \int_0^1 G(t,s) j_2(s)\, ds \right] \leq a^{\lambda}(k_2 A + M_2) \leq a, \\
\|Tu\| &\geq \max_{0 \leq t \leq 1} \int_0^{\gamma} G(t,s)[k_1 - j_1(t)] u^{\lambda}(s)\, ds \\
&\geq \sigma^{\lambda} \|u\|^{\lambda} \left[ k_1 \max_{0 \leq t \leq 1} \int_0^{\gamma} G(t,s) ds - \max_{0 \leq t \leq 1} \int_0^{\gamma} G(t,s) j_1(s) ds \right] \geq \sigma^{\lambda} b^{\lambda}(k_1 B - M_1) \geq b.
\end{aligned}$$

Thus, $T : K[a,b] \to K[a,b]$. By the definition of $u_m$, $u_m = T u_{m-1}$, $m = 1, 2, \ldots$. Since $u_0 \in K[a,b]$, we have $u_1 = T u_0 \in K[a,b]$. It follows that

$$u_1(t) \leq \|u_1\| \leq a = u_0(t), \quad 0 \leq t \leq 1.$$

By the assumption (H3), for $0 \leq t \leq 1$,

$$u_2(t) = (T u_1)(t) = \int_0^1 G(t,s) f(s, u_1(s))\, ds \leq \int_0^1 G(t,s) f(s, u_0(s))\, ds = u_1(t)$$

and $u_2 = T u_1 \in K[a,b]$. By induction, we assert that

$$u_{m+1}(t) \leq u_m(t), \quad 0 \leq t \leq 1, \quad u_m \in K[a,b], \quad m = 1, 2, \ldots.$$

Since $T : K[a,b] \to K[a,b]$ is completely continuous, there exist a subsequence $\{u_{m_k}\}_{k=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ and a function $u^* \in K[a,b]$ such that $u_{m_k} \to u^*$ in $C[0,1]$. Since $u_{m+1}(t) \leq u_m(t)$, $0 \leq t \leq 1$, $m = 0, 1, 2, \ldots$, we see that $u_m \to u^*$ in $C[0,1]$.

Putting $m \to \infty$ in the equality $u_{m+1} = Tu_m$, we get $Tu^* = u^*$. Since $u^* \in K[a,b]$, we have $0 < b \le \|u^*\| \le a$. By Lemma 2 (1), $u^*(t) = (Tu^*)(t) > 0$, $t \in [0,1)$. Since $Tu^* = u^*$, we have

$$u^*(t) = \int_0^1 G(t,s) f(s, u^*(s))\, ds, \quad 0 \le t \le 1.$$

By the definition $G(1,s) = 0$, $0 < s \le 1$. So $u^*(1) = 0$. By Lemma 1, for $0 \le t \le 1$.

$$(u^*)'(t) = \int_0^1 \frac{\partial}{\partial t} G(t,s) f(s, u^*(s))\, ds = -\frac{1}{t^{n-1}} \int_0^t s^{n-1} f(s, u^*(s))\, ds.$$

By the definition $\frac{\partial}{\partial t} G(0,s) = 0$, $0 < s \le 1$. So $(u^*)'(0) = 0$. Noticing that $[0,1] \backslash E$ is an open subset in $[0,1]$, we get

$$(u^*)''(t) = -\frac{n-1}{t}(u^*)'(t) - f(t, u^*(t)), \quad t \in [0,1] \backslash E.$$

In other words, $u^*(t)$ is a positive solution of the problem $(P)$ and $u^* \in K \cap C^1[0,1] \cap C^2([0,1]\backslash E)$. The proof is completed. $\square$

*Remark 1.* By the proof of Theorem 1, the iterative sequence $\{u_m\}_{m=0}^{\infty}$ is non-increasing, that is $a = u_0(t) \ge u_1(t) \ge \ldots \ge u_m(t) \ge \ldots \ge u^*(t)$, $0 \le t \le 1$.

*Remark 2.* By the assumption (H2), $f(t,0) \equiv 0$, $t \in [0,1] \backslash E$. So the problem $(P)$ has a trivial solution. The purpose of Theorem 1 is to find a positive solution other than this trivial solution of $(P)$.

## 4 An Example

The following example illustrates that our result is useful for the computational analysis.

**Example 1**. Let $\rho > 0$. Consider the quasilinear boundary value problem

$$\begin{cases} u''(t) + \dfrac{n-1}{t} u'(t) + \rho \Big[ \dfrac{1+u}{4+u} + \dfrac{1}{\sqrt{t(1-t)}} \Big] \sqrt{u(t)} = 0, \quad 0 < t < 1, \\ u'(0) = 0, \quad u(1) = 0. \end{cases}$$

Here, $f(t,u) = \rho \left[ \frac{1+u}{4+u} + \frac{1}{\sqrt{t(1-t)}} \right] \sqrt{u}$. Obviously, $f(t,u)$ is an increasing function in $u$. Moreover,

$$\frac{\rho}{4}\sqrt{u} \le f(t,u) \le \rho \Big[ 1 + \frac{1}{\sqrt{t(1-t)}} \Big] \sqrt{u}, \quad (t,u) \in [0,1] \times [0, +\infty).$$

In this example, $E = \{0,1\}$, $\lambda = 0.5$, $k_1 = \rho/4$, $k_2 = \rho$, $j_1(t) \equiv 0$, $j_2(t) = \rho/\sqrt{t(1-t)}$. Noticing that $G(t,s) \le 1/(n-2)$, we can choose

$$a \ge \Big[ \frac{\rho}{2n} + \frac{\rho}{n-2} \int_0^1 \frac{1}{\sqrt{t(1-t)}}\, dt \Big]^2 = \rho^2 \Big[ \frac{1}{2n} + \frac{\pi}{n-2} \Big]^2.$$

By Theorem 1, the problem has a positive solution $u^* \in C^1[0,1] \cap C^2(0,1)$ and $u^*(t)$ can be approximated uniformly by the sequence $\{u_m(t)\}_{m=1}^{\infty}$ on $[0,1]$. In this example, $f(t,u)$ is singular at $t=0,\ t=1$ and can not be decomposed to the form $f(t,u)=h(t)u^{\lambda}$.

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