Mathematical Modelling and Analysis
Volume 14 Number 3, 2009, pages 313–321
Doi:10.3846/1392-6292.2009.14.313-321
ISSN 1392-6292 print
ISSN 1648-3510 online
www.vgtu.lt/mma/

On Some Limit Properties of Functions with Nonvanishing Divided Differences

E. Kiriyatzkii and J. Kirjackis

Vilnius Gediminas Technical University
Sauletekio av.11, LT-10223, Vilnius, Lithuania
E-mail(*corresp.*): ekira@post.omnitel.net
E-mail: Eduard.Kiriyatzkii@takas.lt

Received November 4, 2008; published online July 15, 2009

Abstract. In present article the sequence of holomorphic in region D functions $F_n(z)$ which have exactly n zeros in this region is considered. The limit properties of the sequence $F_n(z)$, of functions with non-vanishing divided differences of n-th order are investigated. One interpolation property, connected with Chebyshev systems is considered.

Key words: holomorphic functions, divided difference, limit properties, Chebyshev systems.

1 Introduction

Let us define divided difference of n-th order of function holomorphic in some region D by formula

$$\Big[F\left(z\right);z_0,\ldots,z_n\Big]=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{F\left(\xi\right)d\xi}{\left(\xi-z_0\right)\ldots\left(\xi-z_n\right)},$$

where Γ is a simple closed contour, enclosing all the points $z_0,\ldots,z_n$ ([3, 6]). Using the conditions imposed on divided differences of n-th order it is possible to determine special subclasses of univalent and multivalent functions which play an important role in the geometric theory of functions of complex variable. One of such conditions is, for example, the condition $[F\left(z\right);z_0,\ldots,z_n]\neq 0$ for any pairwise distinct $z_0,\ldots,z_n\in D$. If $[F\left(z\right);z_0,z_1]\neq 0$ for all pairs $z_0,z_1\in D$, then $F(z)$ – univalent function in D.

Denote by $K_n(D)$, $n=0,1,2,\ldots$, the class of holomorphic in D functions $F(z)$, for which $[F(z);z_0,\ldots,z_n]\neq 0$ for any pairwise different $z_0,\ldots,z_n\in D$ (see, [5]). For $n=0$ we have a class $K_0(D)$ of non-vanishing in D functions. $K_1(D)$ is a class of univalent in D functions. If $F(z)\in K_n(D)$, $n\geq 2$, then

$\Psi(z) = [F(z); z, z_1, \ldots, z_{n-1}]$ is univalent in D function for any $z_1, \ldots, z_{n-1} \in D$. Note that if $F(z) \in K_n(D)$, then $F^{(n)}(z) \neq 0$ in D.

The rest of the paper is organized as follows. In Section 2, the main problem and the result (Theorem 2) are formulated. For the proof of Theorem 2, we need four auxiliary lemmas, which are presented in Section 3. In Section 4, Theorem 2 is proved. Next we give several corollaries, remarks and examples related to Theorem 2. Final conclusions are given in Section 6.

2 Formulation of the Problem

Let D is a domain of complex plane and Q certain bounded set from the domain D. The set Q may be finite as well as infinite. Let $P_n(z)$ be a polynomial of n-th degree ($n \geq 1$) with leading coefficient equal to one. Denote by $W(Q)$ the set of all such polynomials with roots from the set Q.

Let R be the family of functions $\varphi(z)$, $\varphi(0) = 1$, for which $z^n\varphi(z) \in K_n(E)$, $n = 1, 2, \ldots$, $E = \{|z| < 1\}$. E. G. Kirjackis ([4]) has proved that family R consists only of rational functions

$$\varphi(z) = \frac{1}{1 - az}, \quad |a| \leq 1.$$

Notice that this result was also established by German mathematician St. Ruscheweyh, but using another method [8]. In paper [7] we have proved the following theorem.

Theorem 1. *Let $E_x(z_0)$ is a circle $|z - z_0| < x$ and r, ρ be fixed numbers, where $0 < \rho < r$. Suppose τ is a fixed number from $E_\rho(z_0)$. The holomorphic in $E_r(z_0)$ function $\varphi(z)$ satisfies the condition*

$$(z - \tau)^n \varphi(z) \in K_n(E_r(z_0)), \quad n = 0, 1, 2, \ldots,$$

if and only if it takes the form

$$\varphi(z) = \frac{a}{bz + c}, \quad a \neq 0.$$

The purpose of our work is formulation and proof of the following more general assertion.

Theorem 2. *Let $\varphi(z)$ be a holomorphic in D function and bounded set Q from the domain D has no limit points on the boundary of the domain D. Let $P_n(z)$, $n = 1, 2, 3, \ldots$ be a sequence of polynomials from $W(Q)$. Then*

$$P_n(z)\varphi(z) \in K_n(D), \quad n = 1, 2, \ldots \tag{2.1}$$

if and only if function $\varphi(z)$ takes the form

$$\varphi(z) = \frac{a}{bz + c}, \quad a \neq 0. \tag{2.2}$$

For the proof of Theorem 2 we need several lemmas.

3 Auxiliary Lemmas

Lemma 1 [[7]]. *Function $F(z)$ belongs to class $K_n(D)$, $n \geq 1$ if and only if for any polynomial $P(z)$ of degree not greater than $n-1$ the equation $F(z) = P(z)$ has in D not more than n roots and there exists such polynomial for which this equation has exactly n roots in D.*

Lemma 2 [[5]]. *Let n is a fixed natural number and the sequence $F_m(z)$, $m = 1, 2, 3, \dots,$ of functions from $K_n(D)$ uniformly converges inside D to function $F(z)$. Then $F(z)$ belongs to the class $K_n(D)$ or is the polynomial of degree not greater than $n-1$.*

Lemma 3 [[7]]. *Let $u(z)$ is a holomorphic in D function and $(z-\gamma)u(z) \in K_n(D)$, $n \geq 1$. Then $u(z) \in K_{n-1}(D)$.*

Lemma 4. *Let bounded set Q from the domain D has no limit points on the boundary of the domain D. Let*

$$P_n(z) \in W(Q), \quad n = 1, 2, \dots, \tag{3.1}$$

and $\varphi(z) \in K_0(D)$. If

$$P_n(z)\varphi(z) \in K_n(D), \quad n = 1, 2, \dots, \tag{3.2}$$

then there exists a point λ from the D that

$$(z-\lambda)^n \varphi(z) \in K_n(D), \quad n = 1, 2, \dots.$$

Proof. Let us assume that polynomials $P_n(z)$, $n = 1, 2, \dots$ are formed by means of the finite number of roots $q_1, \dots, q_m \in Q$. Since the sequence (3.1) consists of polynomials with increasing degrees, then there are infinitely many polynomials from sequence (3.1), which have the same root, whose multiplicity is unbounded, i.e., it approaches infinity. Let it be the root q (one of roots $q_1, \dots, q_m$). Let us represent the mentioned infinite set of polynomials in the form of sequence $P_{n_k}(z)$, $k = 1, 2, 3, \dots,$ where the polynomial $P_{n_k}(z)$ has a root q of multiplicity ρ_k. According to (3.2) and Lemma 3, we will obtain

$$(z-q)^{\rho_k} \varphi(z) \in K_{\rho_k}(D), \quad k = 1, 2, 3, \dots$$

Since $\rho_k \to \infty$, then by using Lemma 3, we obtain

$$(z-q)^n \varphi(z) \in K_n(D), \quad n = 1, 2, \dots.$$

We proved Lemma 4 on the assumption that polynomials (3.1) were formed by means of the finite number of roots $q_1, \dots, q_m$ from Q. Let us assume now that polynomials (3.1) are formed by means of the infinite set of different roots from Q. Let us represent these polynomials in the form of

$$P_n(z) = (z-a_{n1})(z-a_{n2}) \dots (z-a_{nn}), \quad n = 1, 2, 3, \dots.$$

Let us fix an arbitrary natural number m and will construct the polynomials of degree m:

$$P_m\left(z;n\right)=\left(z-a_{n1}\right)\left(z-a_{n2}\right)\ldots\left(z-a_{nm}\right),\quad n=m,m+1,m+2,\ldots.$$

According to Lemma 3 we have

$$P_m\left(z;n\right)\varphi\left(z\right)\in K_n\left(D\right),\quad n=m,m+1,m+2,\ldots.$$

Since the sequence of polynomials

$$P_m(z;n),\quad n=m,m+1,m+2,\ldots.$$

uniformly bounded inside the region D, then according to the principle of the condensation [2] let us extract from it a subsequence, which uniformly converges inside D to the polynomial $P_m^*\left(z\right)=\left(z-b_1\right)\ldots\left(z-b_m\right)$. But then the sequence of functions

$$P_m\left(z;n\right)\varphi\left(z\right),\quad n=m,m+1,m+2,\ldots$$

uniformly converges to the function $P_m^*(z)\varphi(z)$. By using Lemma 2 we obtain

$$P_m^*\left(z\right)\varphi\left(z\right)\in K_m\left(D\right),\quad m=1,2,3,\ldots$$

If the sequence $b_1,b_2,b_3,\ldots$ contains only finite number of terms, then according to the first assumption about roots, Lemma 4 is proven. Let the sequence $b_1,b_2,b_3,\ldots$ contain infinitely many of pairwise different terms. Then from the sequence $b_1,b_2,b_3,\ldots$ by virtue of its boundedness let us extract a subsequence $c_1,c_2,c_3,\ldots$, which converges to a certain number g, $g\in Q\subset D$. Let us now take arbitrarily a natural number m and construct a sequence of polynomials of degree m:

$$P_{mn}\left(z\right)=\left(z-c_n\right)\left(z-c_{n+1}\right)\ldots\left(z-c_{n+m-1}\right),\quad n=1,2,3,\ldots. \tag{3.3}$$

Using Lemma 3 and taking into account (3.3), we conclude that

$$P_{mn}\left(z\right)\varphi\left(z\right)\in K_m\left(D\right).$$

Taking n to infinity and using Lemma 2 we will get

$$\left(z-g\right)^m\varphi\left(z\right)\in K_m\left(D\right).$$

Since m is arbitrary natural number we can write

$$\left(z-g\right)^n\varphi\left(z\right)\in K_n\left(D\right),\quad n=1,2,3,\ldots.$$

Thus the Lemma 4 is proved. $\square$

4 Proof of Theorem 2

Let $P_n(z) \in W(Q)$, $n = 1, 2, 3, \ldots$ and $\varphi(z) \in K_0(D)$. Let furthermore

$$P_n(z)\,\varphi(z) \in K_n(D), \quad n = 1, 2, 3, \ldots$$

By Lemma 4 there exists a point z_0 in D for which

$$(z - z_0)^n\,\varphi(z) \in K_n(D), \quad n = 1, 2, \ldots \tag{4.1}$$

Thus, we have obtained that if the function $\varphi(z)$ satisfies the condition (3.2) in Theorem 2, then it satisfies also the condition (4.1). Let us choose such value of r that the circle $|z - z_0| < r$ completely belongs to D. Since $\varphi(z) \in K_0(D)$ and the condition (4.1) is fulfilled, then by Theorem 1 the function is of the form (2.2).

It remains to show that any holomorphic in D function $\varphi(z)$ of the form (2.2) satisfies the condition (2.1). Indeed, for $n \geq 1$ the equation $P_n(z)\varphi(z) = T(z)$, where $T(z)$ is the arbitrary polynomial of degree not higher than $n-1$, cannot have more than n roots in region D. But then, from Lemma 1 it follows that

$$P_n(z)\,\varphi(z) \in K_n(D), \quad n = 1, 2, 3, \ldots$$

The Theorem 2 is completely proved.

5 Discussion of Theorem 2

Remark 1. Remind that the set $W(Q)$ consists of polynomials with roots from the set Q, $Q \subset D$. In Theorem 2 we have assumed that the set Q does not have limit points on the boundary of the region D. On the assumption that the set Q has limit point on the boundary of D, Theorem 2 is not always valid. We will prove this fact with the aid of the following examples.

Example 1. Let $\mathit{\Pi} = \{\mathrm{Re} z > 0\}$ and

$$P_n(z) = \left(z - \frac{1}{n}\right)^n, \quad n = 1, 2, 3, \ldots.$$

It is clear that the set Q has limit point $\zeta = 0$, which belongs to boundary of the region $\mathit{\Pi}$. The polynomial $h(z) = z + 1$ satisfies the condition

$$\left(z - \frac{1}{n}\right)^n h(z) \in K_n(\mathit{\Pi}), \; n = 1, 2, 3, \ldots$$

In fact, for any $n \geq 1$

$$\left[\left(z - \frac{1}{n}\right)(z+1); z_0, \ldots, z_n\right] = z_0 + \ldots + z_n \neq 0, \quad \forall z_0, \ldots, z_n \in \mathit{\Pi}.$$

However, polynomial $h(z) = z + 1$ is not of the form (2.2) as it asserts Theorem 2.

Example 2. Let

$$\varphi(z) = \frac{2-z}{(1-z)^2}.$$

Let us note that $\varphi(z)$ is univalent in unit disc $E = |z| < 1$. Let the set Q consists of numbers $z_0 = 0$, $z_n = -(n-3)/(n+1)$, $n = 1, 2, 3, \ldots$. Let us take the sequence of the polynomials

$$P_n(z) = z\left(z + \frac{n-3}{n+1}\right)^{n-1}, \ n = 1, 2, 3, \ldots$$

from the $W(Q)$. The set Q has a limit point $z = -1$, on the circumference $|z| = 1$. Let us prove that

$$P_n(z)\,\varphi(z) \in K_n(E), \quad n = 1, 2, 3, \ldots. \tag{5.1}$$

For $n = 1$ assertion (5.1) is true, because function $z\varphi(z)$ is univalent, that is $z\varphi(z) \in K_1(E)$. It is not difficult to establish identity

$$z\Big(z + \frac{n-3}{n+1}\Big)^{n-1}\frac{2-z}{\Big(1-z\Big)^2} = P^*_{n-1}\Big(z\Big) + \frac{2^{n-2}\Big(n-1\Big)^{n-1}}{\Big(n+1\Big)^{n-2}}\frac{z+\frac{1-n}{1+n}}{\Big(1-z\Big)^2}$$

for $n \ge 2$. Using elementary properties of divided differences we get

$$\Big[P_n\Big(z\Big)\varphi\Big(z\Big); z_0, \ldots, z_n\Big] = 2^{n-2}\Big(\frac{n-1}{n+1}\Big)^{n-1}\Big[\frac{z+\frac{1-n}{1+n}}{\Big(1-z\Big)^2}; z_0, \ldots z_n\Big]$$

$$= 2^{n-2}\Big(\frac{n-1}{n+1}\Big)^{n-1}\Big(-1 + \frac{2}{n+1}\sum_{m=0}^{n}\frac{1}{1-z_m}\Big)\prod_{m=0}^{n}\frac{1}{1-z_m} \neq 0,$$

$$\forall z_0, \ldots, z_n \in E.$$

Thus relation (5.1) is correct, but function $\varphi(z)$ is not linear-fractional of form (2.2).

Remark 2. We assumed until now that $W(Q)$ consists of the polynomials with roots from the set Q and $Q \subset D$. Let us assume now that the set consists of the polynomials whose all roots belong the set Q, however, these roots do not belong to region D. The following example shows that in this case Theorem 2 also will not be valid.

Example 3. Let

$$f(z) = \frac{z}{(1-z)^2}. \tag{5.2}$$

It is not difficult to check that

$$\Big[f(z); z_0, \ldots, z_n\Big] = \Big(-1 + \sum_{m=0}^{n}\frac{1}{1-z_m}\Big)\prod_{m=0}^{n}\frac{1}{1-z_m} \neq 0, \ \forall z_0, \ldots, z_n \in E.$$

Consequently $f(z) \in K_n(E)$, $n = 1, 2, \ldots$. Next we will use the following lemma.

Lemma 5 [see [6]]. *Let linear-fractional function*

$$\xi = \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

maps domain D onto domain D^*. If $F(\xi) \in K_n(D^*)$, $n = 1, 2, \ldots,$ then

$$(cz+d)^{n-1} F\left(\frac{az+b}{cz+d}\right) \in K_n(D).$$

Function $\xi = (z+x)/(1+xz)$, $0 < x < 1$, maps one-to-one unit disc E onto itself. Using Lemma 5 and elementary properties of divided differences we get that

$$\left(z + \frac{1}{x}\right)^n \frac{z+x}{(1-z)^2} \in K_n(E), \quad n = 1, 2, \ldots.$$

Next we form a set Q from numbers $-1/x$. Then $W(Q)$ consists of polynomials

$$P_n(z) = \left(z + \frac{1}{x}\right)^n$$

with roots from the set Q and these roots do not belong to the unit disc E. It is clear from (5.3) that Theorem 2 does not take place.

Example 4. Let $P_n(z) = (1+z)^n$ and $\varphi(z) = 1 + z$. It is not clear that

$$P_n(z)\,\varphi(z) \in K_n(D), \quad n = 1, 2, \ldots$$

Here the polynomial $P_n(z)$ has the root $z_0 = -1$ which belongs to the circumference $|z| = 1$, and the function $\varphi(z)$ is not a linear-fractional function. In this case Theorem 2 is also not valid.

Example 5. Let us give one additional interesting in our view example, related to the sequence of the polynomials. Consider function $\psi(z) = e^z$. For any $z_0, \ldots, z_n \in E$ the formula

$$\Big[\psi(z); z_0, \ldots, z_n\Big] = \int_0^{t_1}\int_0^{t_1} \cdots \int_0^{t_{n-1}} \psi^{(n)}(\zeta)\, dt_1 \ldots dt_n,$$

$$0 \le t_1 \le 1, \quad 0 \le t_2 \le t_1, \ldots,\ 0 \le t_n \le t_{n-1},$$

$$\zeta = z_0 + t_1(z_1 - z_0) + \ldots + t_n(z_n - z_{n-1}) \in E$$

is valid ([6]). Since $\mathrm{Re}\left\{\psi^{(n)}(\zeta)\right\} > 0$, $\forall \zeta \in E$, then

$$\mathrm{Re}\left\{\Big[\psi(z); z_0, \ldots, z_n\Big]\right\} = \int_0^{t_1}\int_0^{t_1} \cdots \int_0^{t_{n-1}} \mathrm{Re}\left\{\psi^{(n)}(\zeta)\right\} dt_1 \ldots dt_n > 0,$$

$$\forall z_0, \ldots, z_n \in E.$$

Therefore $\psi(z) \in K_n(E)$, $n = 1, 2, \ldots$. Applying the Lemma 5 we obtain

$$\left(1 + \bar{\zeta} z\right)^{n-1} e^{z+\zeta/1+\bar{\zeta} z} \in K_n(E), \quad \forall \zeta \in E,\ n = 1, 2, \ldots.$$

Corollary 1. Let set Q from the domain D has no limit points on the boundary of the domain D. Let $P_n(z)$, $n = 1, 2, 3, \dots$ is a sequence of polynomials from the set $W(Q)$. The condition

$$P_n(z)\,h(z) \in K_n(D), \quad n = 1, 2, 3, \dots$$

for the polynomial $h(z)$ is satisfied if and only if $h(z) \equiv a$, where $a \neq 0$.

DEFINITION 1. Let $u_0(z), u_1(z), \dots, u_n(z)$ be a set of linearly independent and holomorphic in domain D functions. If for any complex $c_0, c_1, \dots, c_n$ (except the case where $c_9, c_1, \dots, c_n$ are all zeros) equation

$$c_0 u_0(z) + c_1 u_1(z) + ... + c_n u_n(z) = 0$$

has not more than n roots in domain D, then the system of functions

$$\{u_0(z),\ u_1(z),\ \dots,\ u_n(z)\}$$

is called Chebyshev system in domain D.

Particularly, if for any complex $c_0, c_1, \dots, c_n$ equation

$$c_0 1 + c_1 z + \dots + c_{n-1} z^{n-1} + U(z) = 0$$

has in D not more than n roots, then the system $1, z, \dots, z^{n-1}, U(z)$ forms the Chebyshev system in D (for example [1].

Lemma 6. *The condition $U(z) \in K_n(D)$, $n \geq 1$ is necessary and sufficient for the system $1, z, \dots, z^{n-1}, U(z)$ to be Chebyshev system in D* ([6, 8]).

Corollary 2. Let $\varphi(z)$ be a holomorphic in D function and a bounded set Q from the domain D has no limit points on the boundary of the domain D. Let $P_n(z) \in W(Q)$, $n = 1, 2, \dots$. If the functions $1, z, \dots, z^{n-1}, P_n(z)\,\varphi(z)$ for any n, $n \geq 1$ form a Chebyshev system in D, then

$$\varphi(z) = \frac{a}{bz + c}, \quad a \neq 0. \tag{5.3}$$

In fact, by Lemma 6 $P_n(z)\,\varphi(z) \in K_n(D)$ for any $n \geq 1$ and using Theorem 2 we get (5.3).

Lemma 7. *For function $U(z)$ to be interpolated by polynomial of $n-1$-th degree not more than in n points it is necessary and sufficient that $U(z) \in K_n(D)$.*

The proof follows from the Lemma 6.

Corollary 3. Let $\varphi(z)$ be a holomorphic in D function and a bounded set Q from the domain D has no limit points on the boundary of the domain D. Let $P_n(z) \in W(Q)$, $n = 1, 2, \dots$. If for any number $n \geq 1$ any rational function

$$\frac{b_0 + b_1 z + \dots + b_{n-1} z^{n-1}}{P_n(z)} \tag{5.4}$$

interpolates holomorphic in D function $\varphi(z)$ at most in n points from D, then

$$\varphi(z) = \frac{a}{bz + c}, \qquad a \neq 0.$$

Indeed, by condition every rational function (5.4) interpolates the function $\varphi(z)$ at most in n points. Therefore, every polynomial $b_0+b_1z+\ldots+b_{n-1}z^{n-1}$ interpolates the function $P_n(z)\varphi(z)$ at most in n points. Then by Lemma 7 we have that $P_n\,(z)\,\varphi\,(z)\in K_n\,(D)$ for any $n\geq 1$. Using Theorem 2 we obtain (5.3).

6 Conclusions

Lemmas 6, 7 and Corollaries 2, 3 indicate a close connection between the special Chebyshev systems and the interpolation of functions by correct proper fractions. In the present paper article, the classes $K_n\,(D)\,,\,n=0,1,2,\ldots$ are considered in the aggregate which enables us to have sufficiently complete information on the limit property of a sequence composed of functions, belonging to different classes.

References

[1] M. Bellalij, Y. Saad and H. Sadok. Analysis of some Krylov subspace methods for normal matrices via approximation theory and convex optimization. *El. Trans. Numer. Anal.*, **33**:17–30, 2008.

[2] G.M. Goluzin. *Geometric theory of functions of a complex variable.* Amer. Math. Soc., Providence, R. I., 1969.

[3] I.I. Ibragimov. *Methods of Interpolation of Functions and some Applications.* Nauka, Moscow, 1971. (in Russian)

[4] E. G. Kirjackis. On a class of univalent functions. *Lith. Math. J.*, **16**(2):111–116, 1976. (in Russian)

[5] E. G. Kirjackis. Some properties of functions with nonzero divided difference. *Lith. Math. J.*, **19**(1):68–80, 1979.

[6] E.G. Kirjackis. *Multivalent functions and divided difference.* Technika, Vilnius, 1995. (in Russian)

[7] J. Kirjackis and E. Kirjackis. On one primitive interpolation process in class of analytic functions. *Lith. Math. J.*, **44**:158–163, 2004. (in Russian)

[8] St. Ruscheweyh. On functions with nonvanishing divided differences. *Math. Nachr.*, **73**:143–146, 1976.