Mathematical Modelling and Analysis
Volume 14 Number 2, 2009, pages 229–246
Doi:10.3846/1392-6292.2009.14.229-246

ISSN 1392-6292 print
ISSN 1648-3510 online
www.vgtu.lt/mma/

Characteristic Functions for Sturm–Liouville Problems with Nonlocal Boundary Conditions*

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Received September 30, 2008; revised March 25, 2009; published online April 30, 2009

Abstract. This paper presents some new results on a spectrum in a complex plane for the second order stationary differential equation with one Bitsadze–Samarskii type nonlocal boundary condition. In this paper, we survey the characteristic function method for investigation of the spectrum of this problem. Some new results on characteristic functions are proved. Many results of this investigation are presented as graphs of characteristic functions. A definition of constant eigenvalues and the characteristic function is introduced for the Sturm–Liouville problem with general nonlocal boundary conditions.

Key words: Sturm–Liouville problem, nonlocal boundary conditions.

1 Introduction

Problems with nonlocal boundary conditions (NBCs) arise in various fields of mathematical physics [11, 15, 25], biology and biotechnology [32, 43], and in other fields. A. Bitsadze and A. Samarskii considered a certain class of spatial nonlocal problems [2]. Later, in [4, 23, 40], generalizations of Bitsadze–Samarskii type conditions were proposed. J. Cannon [3] investigated integral type NBCs. Linear second order ordinary equations with an integral type NBC have been considered in [29]. The problems with integral NBCs were investigated for parabolic equations [25, 28], for elliptic equations [19, 45], and for hyperbolic equations [1, 14, 46]. Some necessary and sufficient existence and uniqueness conditions for solving stationary differential and discrete problems were obtained in [8, 9]. Numerical methods for problems with NBCs are investigated in [5, 6, 7, 12]. Nonnegative solutions of problems with NBCs were investigated by some authors [13, 20, 24, 30].

* This work was partially supported by the Lithuanian State Science and Studies Foundation within the project on T-73/09 "Methods for Solving Parabolic and Navier–Stokes Differential Equations with Nonlocal Conditions".

The multi-point boundary value problem for the second-order ordinary differential equations was initiated by Ilyin and Moiseev [22]. The completeness of eigenvectors of a nonsymmetric system is closely related to eigenvalue problems for pencils of ordinary differential operators [31]. Ilyin obtained necessary and sufficient properties of a subsystem of eigenfunctions and adjoint functions (as a basis) for Keldysh's bundle of ordinary differential operators [21]. An eigenvalue problem with the NBCs is closely linked to boundary problems for differential equations with NBCs [1, 24, 27, 37].

In [16, 17, 18, 26] eigenvalue problems with NBCs are considered for the points at different ends of the interval. In [10, 38, 41] similar problems are investigated for the operators with a NBC of Bitsadze–Samarskii or integral type. However, the eigenvalue problems for differential operators with NBC are considerably less investigated than in the case of classical boundary conditions.

The goal of this paper is to make a survey of various characteristic functions for the Sturm–Liouville problem with one Bitsadze–Samarskii type NBC. In Section 2, we investigate how the spectrum depends on some boundary condition parameters γ and ξ for this problem. We introduce constant eigenvalue points, complex and real characteristic functions. Some results for such a problem on real eigenvalues are published in [33, 34, 35, 39, 42, 44]. In Section 3, we present a definition of constant eigenvalue points and characteristic functions for Sturm–Liouville problem with general NBCs.

The main results of this article are presented in Section 2.3, Lemma 1, Section 2.5, Lemma 2 and a generalization of constant eigenvalues points and characteristic function is given in Section 3.

2 Characteristic Functions for a Sturm–Liouville Problem with one Bitsadze–Samarskii Type NBC

Let us consider the Sturm–Liouville Problem with Bitsadze–Samarskii type NBC:

$$-u'' = \lambda u, \quad t \in (0,1), \tag{2.1}$$

$$u(0) = 0, \quad u(1) = \gamma u(\xi), \tag{2.2}$$

with the parameters $\gamma \in \overline{\mathbb{C}}$, $\xi \in [0,1]$ and eigenvalues $\lambda \in \mathbb{C}$.

Remark 1. The case of NBC (2.2) is important in the investigation of multidimensional and non-stationary problems, and numerical methods. Characteristic functions for other types of NBCs were investigated by Pečiulytė in her PhD Thesis and in [33, 34]. Theoretical results on the real spectrum are presented in [44].

If $\gamma = 0$, then we have the classical Sturm–Liouville problem. In this case, all eigenvalues of problem (2.1)–(2.2) are positive and algebraically simple:

$$\lambda_k = z_k^2, \quad u_k(t) = \sin(z_k t), \quad z_k = \pi k,\ k \in \mathbb{N} := \{1, 2, \dots\}. \tag{2.3}$$

In the case of $\gamma \neq \infty$ and $\xi = 0$, or $\xi = 1$ and $\gamma \neq 1$ we have the classical case, as well. For $\gamma = \infty$ we have a "boundary" condition $u(\xi) = 0$ instead of

(2.2). So, this case is similar to the classical one for $\xi > 0$:

$$\lambda_l = p_l^2, \quad u_l(t) = \sin(z_l t), \quad p_l = \pi l/\xi, \ l \in \mathbb{N}. \tag{2.4}$$

We denote sets $\hat{\mathcal{Z}} := \{z_k\}_{k=1}^{\infty}$, $\hat{\mathcal{P}} := \{p_l\}_{l=1}^{\infty}$.

The case with one classical boundary condition. If $\gamma = \infty$ and $\xi = 0$, then the second boundary condition (2.2) is the same as the first boundary condition in (2.2). If $\gamma = 1$ and $\xi = 1$, then the second boundary condition (2.2) becomes the identity $0 \equiv 0$. So, these two cases correspond to the problem with one classical boundary condition:

$$-u'' = \lambda u, \quad t \in (0,1), \tag{2.5}$$
$$u(0) = 0. \tag{2.6}$$

The characteristic equation for the second order ordinary equation (2.5) is $-\mu^2 = \lambda$. If $\lambda = 0$, then the general solution of this equation is $u = Ct + C_1$, and functions $u = Ct$ satisfy problem (2.5)–(2.6). If $\lambda \neq 0$, then the characteristic equation has two different roots $\mu = q_{1,2}$ (see Figure 1). Let q be the root with a positive real part when λ is a nonnegative real number, and the root with a positive imaginary part for negative λ. Thus, we have a bijection $\lambda = q^2$ between the domain $\mathbb{C}_q = \{z \in \mathbb{C}\colon -\pi/2 < \arg z \leq \pi/2 \text{ or } z = 0\}$ and the whole complex plane $\mathbb{C}_\lambda = \mathbb{C}$ (see Figure 2).

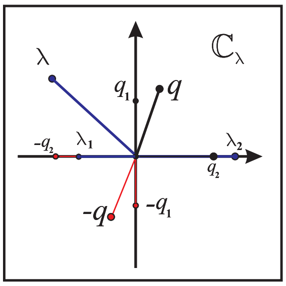


Figure 1. Roots of the equation $q^2 = \lambda$.

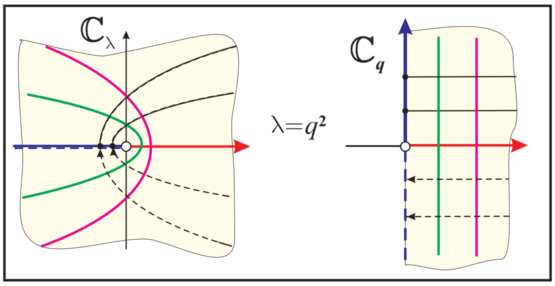


Figure 2. Bijective mapping $\lambda = q^2$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_q$.

If we find such q, then the corresponding eigenvalue is defined as $\lambda = q^2$. Then the roots $q_{1,2} = \pm \mathrm{i}\, q$, and the general solution are given in the following form

$$u = C_1 \mathrm{e}^{\mathrm{i}\, q\, t} + C_2 \mathrm{e}^{-\mathrm{i}\, q\, t} = C_3 \sin(qt) + C_4 \cos(qt).$$

All functions $u = C \sin(qt)$ satisfy equation (2.5) and boundary condition (2.6). In both cases ($q = 0$ and $q \neq 0$), we can write a formula for the solution

$$u = C \sin(qt)/q = C \sinh(-\mathrm{i}\, qt)/(-\mathrm{i}\, q), \quad q \in \mathbb{C}_q, \tag{2.7}$$

where $C \in \mathbb{C}$ is an arbitrary constant [44]. So all $\lambda \in \mathbb{C}$ are eigenvalues of problem (2.5)–(2.6).

2.1 Constant eigenvalues

Let us return to problem (2.1)–(2.2) and consider that $0 < \xi < 1$ and $\gamma \in \mathbb{C}$. If we substitute function (2.7) into the second boundary condition (2.2), then we get the equality

$$C\big(\sin q/q - \gamma\sin(\xi q)/q\big) = 0. \tag{2.8}$$

There exists a nontrivial solution (eigenfunction) if q is the root of the function

$$f(q) := \gamma\sin(\xi q)/q - \sin q/q. \tag{2.9}$$

In the case $\lambda = q = 0$, we get the equality $\gamma\xi - 1 = 0$. So, the eigenvalue $\lambda = 0$ exists if and only if $\gamma = 1/\xi$ [42, 44].

Let us consider the case $q \neq 0$. If $\sin(\xi q) = 0$ and $\sin q = 0$, then equality (2.9) is valid for all $\gamma \in \mathbb{C}$. In this case, we have constant eigenvalues (that do not depend on the parameter γ) $\lambda = q^2$, where q is a constant eigenvalue point, see [42, 44]. It is the root of the system

$$\begin{cases} \sin q = 0, \\ \sin(\xi q) = 0. \end{cases} \tag{2.10}$$

In this paper, we suppose that m and n ($n > m > 0$) are positive coprime integer numbers. Constant eigenvalues exist only for rational $\xi = r = \frac{m}{n} \in [0, 1]$, and those eigenvalues are equal to $\lambda_k = c_k^2$, $c_k = \pi k$, $k \in n\mathbb{N} := \{n, 2n, 3n, \ldots\}$ [44], and $c_k \in \mathcal{C} := \hat{\mathcal{Z}} \cap \hat{\mathcal{P}}$ are positive real numbers.

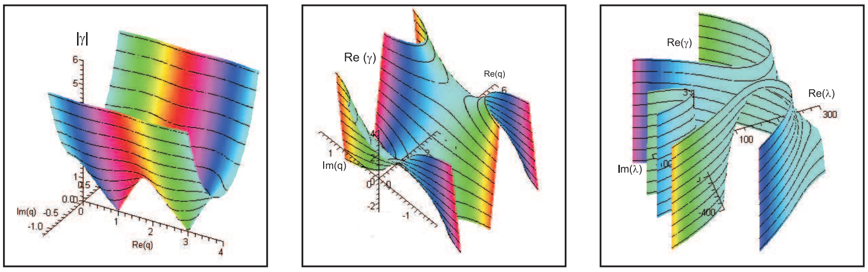
2.2 Complex characteristic function

All nonconstant eigenvalue points (that depend on the parameter γ) are γ-points of the meromorphic function

$$\gamma = \gamma_c(q) := \sin q/\sin(\xi q), \quad \gamma_c\colon \mathbb{C}_q \to \overline{\mathbb{C}}. \tag{2.11}$$

Note that, if γ is fixed, then the γ-point is the root of the equation $\gamma_c(q) = \gamma$. So, nonconstant eigenvalue is defined as $\lambda = q^2(\gamma)$. We call this function a complex characteristic function. The graphs of the functions $|\gamma_c(q)|$, $\operatorname{Re}\gamma_c(q)$, and $\operatorname{Re}\gamma_c(\sqrt{\lambda})$ are presented in Figure 3 for $\xi = 1/2$. The function $\gamma_c(\pi q)$ is drawn in the graphs instead of the function $\gamma_c(q)$. In this case, zeroes of the function are the points k, $k \in \mathbb{N}$. We can investigate these functions only for $\operatorname{Im} q \geq 0$ and $\operatorname{Re} q \geq 0$ because $\bar{\gamma}_c(q) = \gamma_c(\bar{q})$ [44].

Remark 2 [Case of rational $\xi = m/n$]. The complex characteristic function for $\xi = m/n$ is periodical in the real direction $\gamma_c(q + 2\pi n) = \gamma_c(q)$, $n \in \mathbb{N}$ and it is even with respect to the line $\operatorname{Re} q = \pi n$ in the domain $\operatorname{Re} q \in [0, 2\pi n]$, i.e., $\gamma_c(\pi n - q) = \gamma_c(\pi n + q)$. Thus, we can consider this function in the domain $\operatorname{Re} q \in [0, \pi n]$. Moreover, the complex characteristic function is an even (odd) function with respect to the line $\operatorname{Re} q = \pi n/2$ in the domain $\operatorname{Re} q \in [0, \pi n]$ for $m, n \in \mathbb{N}_{odd}$ ($m \in \mathbb{N}_{odd}, n \in \mathbb{N}_{even}$ or $m \in \mathbb{N}_{even}, n \in \mathbb{N}_{odd}$) [44]. Finally, we can restrict to the domain $\operatorname{Re} q \in [0, \pi n/2]$ for investigating some properties of this function.



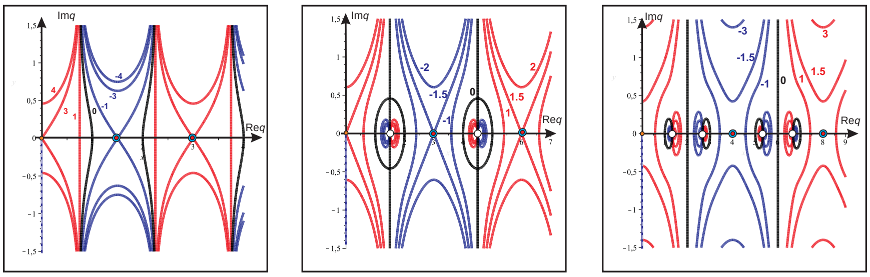
a) $|\gamma_c(q)|,\ q \in \mathbb{C}_q$ b) $\operatorname{Re}\gamma_c(q),\ q \in \mathbb{C}_q$ c) $\operatorname{Re}\gamma_c(\sqrt{\lambda}),\ \lambda \in \mathbb{C}$

Figure 3. The complex characteristic function $\gamma_c(\pi q)$ for $\xi = 1/2$.

All zeroes $z \in \mathcal{Z} := \hat{\mathcal{Z}} \setminus \mathcal{C}$ and poles $p \in \mathcal{P} := \hat{\mathcal{P}} \setminus \mathcal{C}$ of the function $\gamma_c(q)$, lie in the positive part of the real axis [44]. We also use the notation $\overline{\mathcal{P}} = \mathcal{P} \cup \{p_0 := 0, p_\infty := +\infty\}$. Note that, for $\xi = 1/n$, $n \in \mathbb{N}$, there are no poles, i.e., the complex characteristic function $\gamma_c(q)$ is an entire function. So, there are no poles in the whole interval $(p_0; p_\infty) = (0; +\infty)$ in this case, while in the other cases we have an infinite sequence of poles [44].

2.3 Complex-real characteristic function

In this Section, we describe characteristic functions for the Sturm–Liouville problem (2.1)–(2.2) with real $\gamma \in \mathbb{R}$. In this case, we must investigate only the real part of the complex characteristic function $\operatorname{Re}\gamma_c(q)$ (see, Figure 3b) and have an additional condition $\operatorname{Im}\gamma_c(q) = 0$. Some information on the function $\operatorname{Re}\gamma_c(q)$ can be presented as a surfacecontours on the graph of this function (see, Figure 3b) for $\xi = 1/2$) or contour lines on the plane $\mathbb{C}_q$ ((see, Figure 4 for $\xi = 1/3, 2/3, 3/4$).



$\xi = \frac{1}{3},\ \gamma = 0, \pm 1, \pm 3, \pm 4$ $\xi = \frac{2}{3},\ \gamma = 0, \pm 1, \pm 1.5, \pm 2$ $\xi = \frac{3}{4},\ \gamma = 0, \pm 1, \pm 1.5, \pm 3$

Figure 4. Contour lines of the function $\operatorname{Re}\gamma_c(\pi q)$ for various ξ.

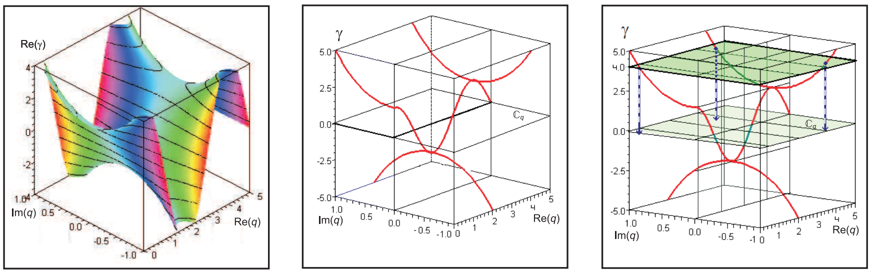
We call the restriction $\operatorname{Im}\gamma_c(q) = 0$ of the complex characteristic function a complex-real characteristic function. This restricted function $\gamma(q)$ is defined on some subset (net) $\mathcal{N} := \gamma^{-1}(\mathbb{R}) := \{q \in \mathbb{C}_q \colon \operatorname{Im}\gamma_c(q) = 0\} \subset \mathbb{C}_q$ and $\gamma = \gamma_c|_{\mathcal{N}} : \mathcal{N} \to \mathbb{R}$. In the general case, the subset $\mathcal{N}$ is a union of curves. A complex-real characteristic function describes complex eigenvalue points (and complex eigenvalues) for real γ.

Let $q = x + \mathrm{i}y$. Then the condition $\operatorname{Im}\gamma_c(q) = 0$ is equivalent to

$$\frac{\cos x \sin(\xi x) \sinh y \cosh(\xi y) - \sin x \cos(\xi x) \cosh y \sinh(\xi y)}{\sin^2(\xi x)\cosh^2(\xi y) + \cos^2(\xi x)\sinh^2(\xi y)} = 0 \qquad (2.12)$$

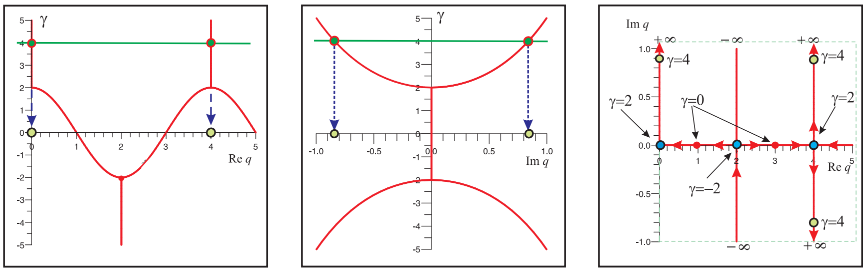
and

$$\gamma_c|_{\mathcal{N}} = \frac{\sin x \sin(\xi x) \cosh y \cosh(\xi y) + \cos x \cos(\xi x) \sinh y \sinh(\xi y)}{\sin^2(\xi x)\cosh^2(\xi y) + \cos^2(\xi x)\sinh^2(\xi y)}. \qquad (2.13)$$



a) complex b) complex-real c) eigenvalue points, $\gamma = 4$

Figure 5. Characteristic functions for $\xi = 1/2$ in $\mathbb{C}_q$.



a) $(\operatorname{Re} q, \gamma)$ b) $(\operatorname{Im} q, \gamma)$ c) $\mathcal{N} \subset \mathbb{C}_q$

Figure 6. Projections of the complex-real characteristic function $\gamma(\pi q)$ for $\xi = 1/2$.

We get the eigenvalue points as γ-values of the complex-real characteristic function $\gamma(q)$ (see, Figure 5). Projections of the complex-real characteristic function are shown in Figure 6. The net $\mathcal{N}$ is a projection into the domain $\mathbb{C}_q$. We add arrows that show how the eigenvalue points are moving, i.e., the direction in which the parameter γ is growing. In Figure 8, we see the parts (on three planes) of the complex-real characteristic function graph where there exist complex eigenvalue points. The case of a positive imaginary axis ($\operatorname{Re} z = 0$, $\operatorname{Im} z > 0$) corresponds to negative eigenvalues. Projections into plane $\mathbb{C}_q$ may give various curves.

We call the point $q_c \in \mathbb{C}_q$ such that $\gamma'(q_c) = 0$, a critical point.

Lemma 1. *There exist only real critical points for problem* (2.1)–(2.2).

Proof. The condition $\gamma'(q) = 0$ is equivalent to

$$\bigl(\cos q \sin(\xi q) - \xi \sin q \cos(\xi q)\bigr)/\sin^2(\xi q) = 0. \qquad (2.14)$$

So, if $q \in \mathbb{C}_q$ is a critical point, then two equalities are valid:

$$\sin q = \gamma \sin(\xi q), \quad \cos q = \xi\gamma \cos(\xi q). \qquad (2.15)$$

Let us eliminate $\sin(\xi q)$ and $\cos(\xi q)$ and express $\sin^2 q$ as follows:

$$\sin^2 q = \sin^2 x \cosh^2 y - \cos^2 x \sinh^2 y + \mathrm{i} \sin(2x) \sinh(2y)/2 = \frac{1-\gamma^2\xi^2}{1-\xi^2}. \qquad (2.16)$$

Since the right-hand side of (2.16) is real for real γ, the critical points are real (the case $\sinh(2y) = 0$) or if $q = x + \mathrm{i}y$, where $\sin(2x) = 2\sin x \cos x = 0$. In the case $\cos x = 0$ and $|\gamma| \geqslant 1$, we have $\cosh^2 y = (1-\gamma^2\xi^2)/(1-\xi^2) \leqslant 1$, i.e., $y = 0$ as well. For real $\gamma \in (-1, 1)$, all eigenvalue points are real numbers and there are no critical points (see, [44, Lemma 5]). So, critical points may exist only for $q = x + \mathrm{i}y$, where $\sin x = 0$. If we eliminate $\sin q$ and $\cos q$, then we can derive that critical points may exist only for $q = x + \mathrm{i}y$, where $\sin(\xi x) = 0$, analogously. If we join these two conditions, then we can make a conclusion that critical points may exist only for $\xi = m/n \in \mathbb{Q}$ and $q = x + \mathrm{i}y$, where $\sin x = 0$ and $\sin(\xi x)$, i.e., $x = 2\pi nk$, $k \in \mathbb{N}$ (constant eigenvalue point). For $\xi = m/n$ the function $\gamma_c(q)$ is periodical in the real direction

$$\gamma_c(2\pi nk + \mathrm{i}y) = \gamma_c(\mathrm{i}y) = \sin(\mathrm{i}y)/\sin(\mathrm{i}y\xi) = \sinh y/\sinh(y\xi).$$

The derivative of the function $\sinh y/\sinh(y\xi)$ is negative for $y < 0$ and positive for $y > 0$ (see, [36, Theorem 1]). Consequently, all critical points exist only for $y = 0$. $\square$

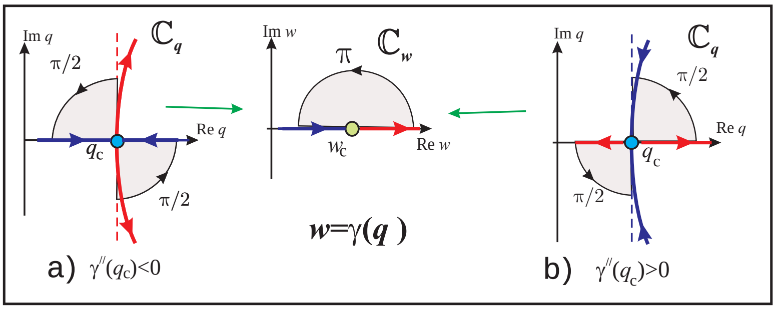


Figure 7. Projection of a complex-real characteristic function to $\mathbb{C}_q$ near the critical point q_c; $w = \gamma_c(q)$ is the complex-real characteristic function, w_c is the critical value.

A few curves of the domain $\mathcal{N}$ can intersect only at the critical point q_c (see, Figure 7). It is valid that $\gamma''(q_c) \neq 0$ for problem (2.1)–(2.2) (see, [44, Corollary 7]). So, two symmetrical "complex" curves are orthogonal with the real axis and eigenvalue points leave or enter the real axis orthogonally in this problem.

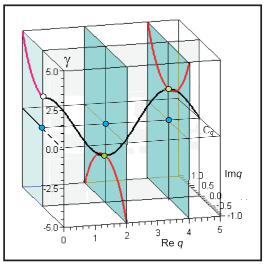
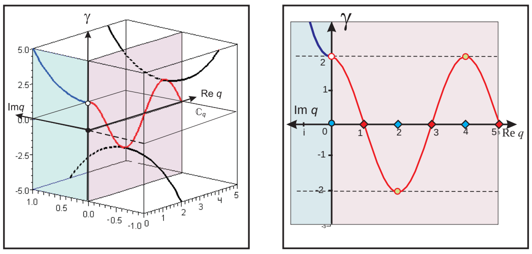


Figure 8. Complex-real characteristic function for complex and negative eigenvalues in the case $\xi = 1/2$.



a) b)

Figure 9. a) Complex-real characteristic function for real eigenvalues in the case $\xi = 1/2$. b) The graph of the real characteristic function.

2.4 Real characteristic function

If we take q only in the rays $q = x \geq 0$ and $q = -\mathrm{i}x$, $x \leq 0$ (see, Figure 9) instead of $q \in \mathbb{C}_q$, then we get positive eigenvalues in case of the ray $q = x > 0$, and we get negative eigenvalues for the ray $q = -\mathrm{i}x$, $x < 0$. The point $q = x = 0$ corresponds to $\lambda = 0$. For the function $\gamma_c : \mathbb{C}_q \to \overline{\mathbb{R}}$ we obtain its two restrictions on those rays: $\gamma_+(x) = \gamma_c(x + \mathrm{i}0)$ for $x \geq 0$, and $\gamma_-(x) = \gamma_c(0 - \mathrm{i}x)$ for $x \leq 0$. The function γ_+ corresponds to the case of nonnegative eigenvalues, while the function γ_- corresponds to that of nonpositive eigenvalues. Let us use the notation

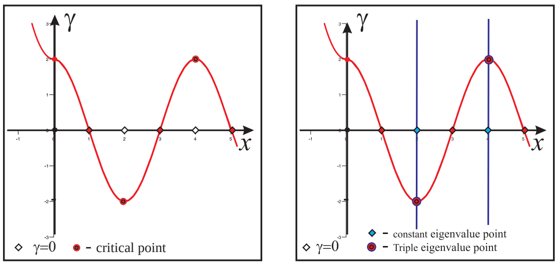
$$\{f_1(x); f_2(x)\} := \begin{cases} f_1(x), & \text{for } x < 0, \\ f_2(x), & \text{for } x \geq 0. \end{cases}$$

All the real eigenvalues $\lambda = \{-x^2; x^2\}$ are investigated using a real characteristic function $\gamma : \mathbb{R} \to \overline{\mathbb{R}}$: $\gamma(x) := \{\gamma_-(x); \gamma_+(x)\}$ (see, Figure 9). We use the same notation of this function as for the complex-real characteristic function. Such a restriction of the characteristic function is very useful for investigating real eigenvalues. For the complex characteristic function (2.11), the real characteristic function can be written as:

$$\gamma(x) = \{\sinh x / \sinh(\xi x); \sin x / \sin(\xi x)\}, \quad \xi \in (0, 1).$$

The cases $\xi = 1/2$ and $\xi = 1/4$ were described in [42].

We draw constant eigenvalues as vertical lines which intersect with the x-axis at the constant eigenvalue points. We call the union of the graph of the real characteristic function and all constant eigenvalue lines a generalized real characteristic function (see, Figure 10). Then, for each γ_*, the constant function $\gamma \equiv \gamma_*$ (horizontal line) intersects the graph of the real characteristic function $\gamma(x)$ or constant eigenvalue lines at some points. Now we have all real eigenvalue points x_i for this γ_* (see, Figure 11). Usually, we enumerate the eigenvalues in such a way: $x_k(0) = \pi k$, $k \in \mathbb{N}$, i.e., using the classical case. Eigenvalues (and $x_k(\gamma)$) are continuously dependent on the parameter γ.



a) real   b) generalized real

Figure 10. Real characteristic functions, $\xi = 1/2$.

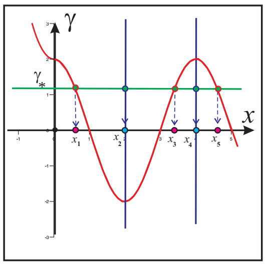
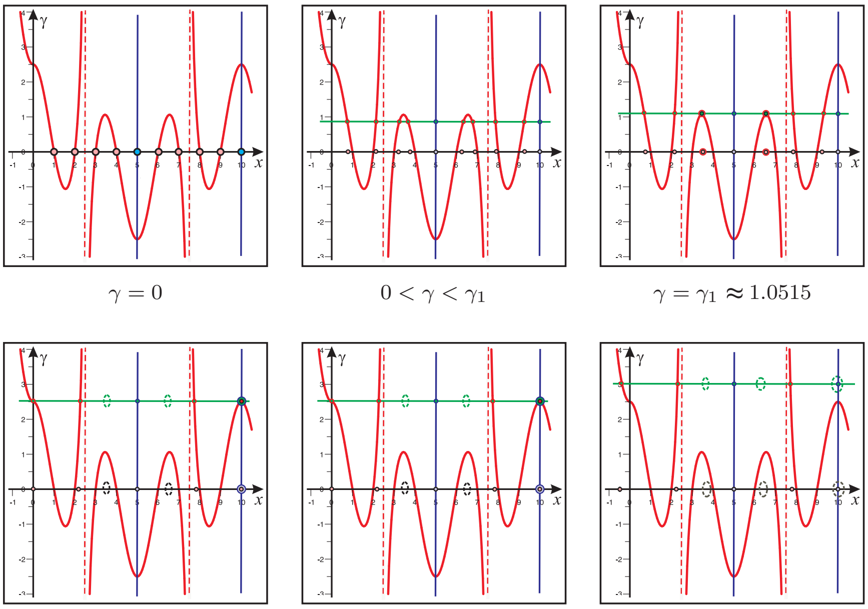


Figure 11. Generalized real characteristic function and real eigenvalue points x_i.

Remark 3. We can define a generalized complex-real characteristic function analogously, i.e., add vertical lines at constant eigenvalue points (see, Figure 15, $\xi = 0.5$) to the graph of the complex-real characteristic function.



$\gamma = 0$ $0 < \gamma < \gamma_1$ $\gamma = \gamma_1 \approx 1.0515$

$\gamma_1 < \gamma < 2.5$ $\gamma = 2.5$ $\gamma > 2.5$

Figure 12. Generalized real characteristic functions $\gamma(x\pi; \xi)$ for $\xi = 2/5$.

Example 1 [$\xi = 2/5$]. In Figure 12, we see how the eigenvalue points are changing subject to a real value of the parameter γ in the case $\xi = 2/5$. For $\gamma_* = 0$ (the classical case), we have $x_k = \pi k$, $\lambda_k = x_k^2$, $k \in \mathbb{N}$, for $0 < \gamma_* < \gamma_1 \approx 1.0515$ all eigenvalues are positive and simple. If $\gamma_* = \gamma_1$, then some eigenvalue points combine into one point and we have double eigenvalues. For $\gamma_1 < \gamma_* < 2.5$, some eigenvalues are complex (we draw them as circles), other eigenvalues are positive and simple. In the case $\gamma_* = 2.5$, we have simple positive eigenvalues (constant and nonconstant), and one zero eigenvalue $\lambda_1 = 0$ as well as multiple (triple) eigenvalues, in addition. In this case, there exist complex eigenvalues, too. For $\gamma_* > 2.5$, we obtain simple positive eigenvalues, complex eigenvalues, and one negative eigenvalue $\lambda_1 = -x_1^2$. The real positive eigenvalues approach poles as $\gamma_* \to +\infty$. We can investigate the negative γ_* values analogously.

Real eigenvalues for Sturm–Liouville problems were investigated in [33, 34, 44]. Graphs of the generalized functions $\gamma(x)$ for various ξ are shown in the Figure 13.

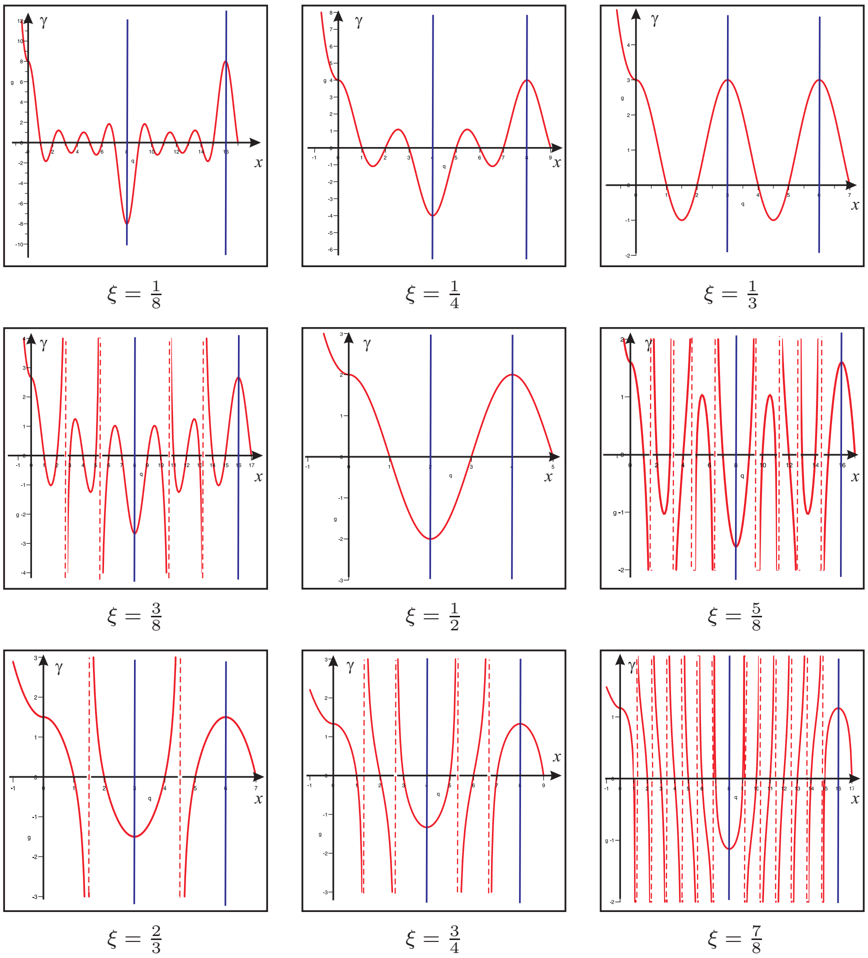


Figure 13. Generalized real functions $\gamma(x\pi)$ for various ξ.

Example 2 [Case $\xi = (n-1)/n$, $n \in \mathbb{N}$]. We consider the characteristic function in the interval $x \in [0, \pi n]$ (see, Remark 2). The point $x = \pi n = p_{n-1} = c_1$ is a unique constant eigenvalue point in this interval. In each interval (p_{k-1}, p_k), $k = 1, \ldots, n-1$ we have exactly one zero of the real characteristic function.

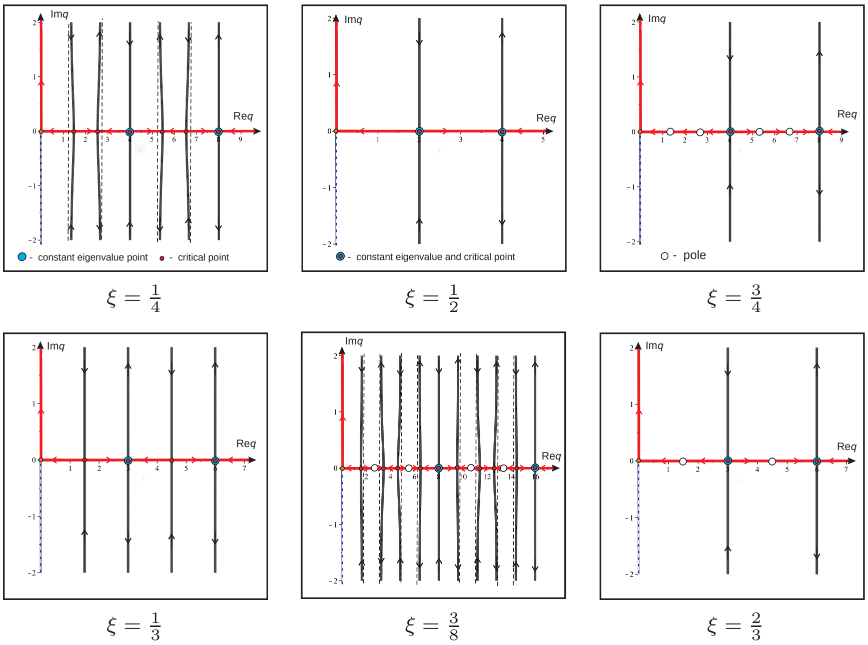


Figure 14. Domain $\mathcal{N}$ for various ξ.

Then we obtain (see,[44, Corollary 13, Corollary 15]) that the characteristic function is a decreasing function in each interval (p_{k-1}, p_k), $k = 1, \dots, n-1$, i.e., there are no critical points for $x \in (0, \pi n)$.

Example 3 [$\xi = (n-2)/n$, $n \in \mathbb{N}_{odd}$]. We consider the characteristic function in the interval $x \in [0, \pi n]$ (see, Remark 2) once more. The point $x = \pi n = p_{n-1} = c_1$ is a unique constant eigenvalue point in this interval. In each interval (p_{k-1}, p_k), $k = 1, \dots, n-1$, $k \neq k_0 := (n-1)/2$ we have exactly one zero of the real characteristic function. So, the characteristic function is a decreasing function in each interval. In the interval (p_{k_0-1}, p_{k_0}) we have two zeroes and one critical point $x = \pi n/2$ (see,[44, Corollary 15]), i.e., there is only one critical point $x = \pi n/2$.

2.5 Complex eigenvalues

Theoretical investigation of complex eigenvalues is a very difficult problem even for simple nonlocal boundary problem such as (2.1)–(2.2). Therefore, we present only simple properties of a complex part of the spectrum for this problem. The domains $\mathcal{N}$ for various ξ are shown in Figure 14. Since $\bar{\gamma}_c(z) = \gamma_c(\bar{z})$, for a complex-real characteristic function, the equality $\gamma(z) = \gamma(\bar{z})$ is valid. Thus, in this subsection we investigate a complex-real characteristic function in the domain

$$\mathbb{C}_q^+ := \{q = x + \mathrm{i}y \colon x > 0, y > 0\}.$$

The points of the domain $\mathcal{N}$ satisfy the equality

$$M(x, y) := \cos x \sin(\xi x) \sinh y / \sinh(\xi y) - \sin x \cos(\xi x) \cosh y / \cosh(\xi y) = 0$$

and $\sinh y/\sinh(\xi y) > 0$, $\cosh y/\cosh(\xi y) > 0$ for $q \in \mathbb{C}_q^+$. In Figure 14, we see that some parts of the domain $\mathcal{N}$ are on the lines. We have such lines if x is a solution of the system:

$$\sin x \cos(\xi x) = 0, \quad \cos x \sin(\xi x) = 0. \tag{2.17}$$

The solutions of this system exist only for rational $\xi = m/n$:

$$x = c_k = \pi n k_1, k_1 \in \mathbb{N} \tag{2.18}$$

$$x = \tilde{c}_k = \pi n(k_2 - 1/2), k_2 \in \mathbb{N}, m, n \in \mathbb{N}_{odd}. \tag{2.19}$$

In other cases, we have graphs of the function $y = \varphi(x)$ obtained by solving the following implicit equation

$$f(y) := \sinh y/\sinh(\xi y) \cdot \cosh(\xi y)/\cosh y = \sin x/\sin(\xi x) \cdot \cos(\xi x)/\cos x. \tag{2.20}$$

Here $f(y)$ is a decreasing function for $y > 0$ and $f(0) = 1/\xi$, $f(+\infty) = 1$, because

$$f'(y) := -\frac{\xi \sinh(2y\xi)}{2\sinh^2(\xi y)\cosh^2 y}\Big(\frac{\sinh(2y)}{\sinh(2y\xi)} - \frac{1}{\xi}\Big) < 0. \tag{2.21}$$

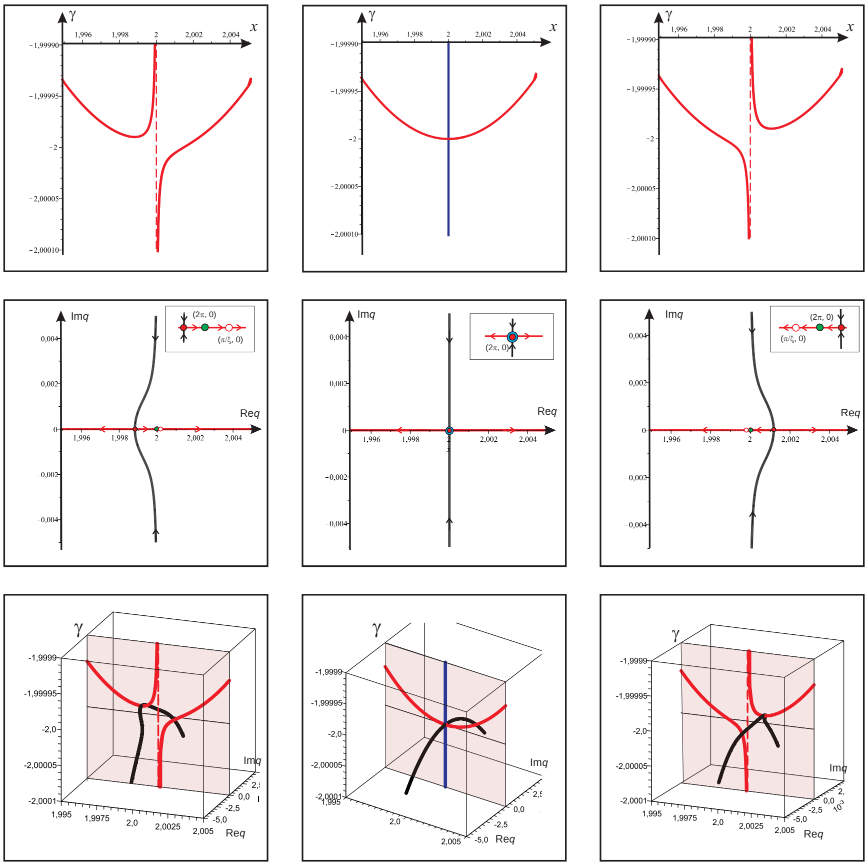
The positiveness of $\sinh(2y)/\sinh(2y\xi) - 1/\xi$ is evident (see, [36, Theorem 1],[44]). As we see in Figure 14 for some ξ, we have only lines in $\mathbb{C}_q^+$.

Lemma 2. *The eigenvalue points of problem* (2.1)–(2.2) *belong to the set of lines in* $\mathbb{C}_q^+$ *if* $\xi = (n-1)/n$, $n \in \mathbb{N}$ *or* $\xi = (n-1)/n$, $n \in \mathbb{N}$ *or* $\xi = (n-2)/n$, $m, n \in \mathbb{N}_{odd}$ *and all the points of those lines in* $\mathbb{C}_q^+$ *are eigenvalue points of problem* (2.1)–(2.2)*. For other rational* $\xi = m/n$*, eigenvalue points belong to the lines* $x = c_k = \pi n k_1$*,* $k_1 \in \mathbb{N}$ *or* $x = \tilde{c}_k = \pi n(k_2 - 1/2)$*,* $k_2 \in \mathbb{N}$*,* $m, n \in \mathbb{N}_{odd}$*. However, there exist eigenvalue points that belong to the graphs of some functions* $\operatorname{Im} q = \varphi(\operatorname{Re} q)$*. In the case of irrational* ξ*, all the eigenvalue points belong to such graphs.*

Proof. The proof follows from Example 2, Example 3, from the property that complex-real characteristic function is even with respect to the line $\operatorname{Re} q = \pi n/2$ in the case $\xi = (n-2)/n$, $m, n \in \mathbb{N}_{odd}$, from formulae (2.20)–(2.21) and from the property that all critical points are real. □

2.6 Dynamics of complex eigenvalues

As shown in the previous subsection, the behavior of eigenvalue points is quite simple for fixed ξ. Zeroes of the characteristic function are fixed for all ξ. The poles depend on ξ: $p_l(\xi) = \pi l/\xi$. A qualitative view of the domain $\mathcal{N}$ with respect to ξ changes when the pole and zero meet at the constant eigenvalue point. If ξ is growing, then the pole moves to the left. In Figure 15, we see a typical situation near the constant eigenvalue point.



$\xi = 0.499999999$ $\xi = 0.5$ $\xi = 0.500000001$

Figure 15. Complex-real characteristic functions (real, domain $\mathcal{N}$ and complex-real). Their dependence on the parameter ξ in the neighborhood of the constant eigenvalue point $\xi = 0.5$ and $q = 2\pi$.

3 The Sturm–Liouville Problem with General NBCs

Let us consider a Sturm–Liouville problem with the following NBCs:

$$-(p(t)u')' + q(t)u = \lambda u, \quad t \in (0,1), \tag{3.1}$$

$$\langle l_0, u(t)\rangle = 0, \tag{3.2}$$

$$\langle l_1, u(t)\rangle = \gamma\langle k, u(t)\rangle, \tag{3.3}$$

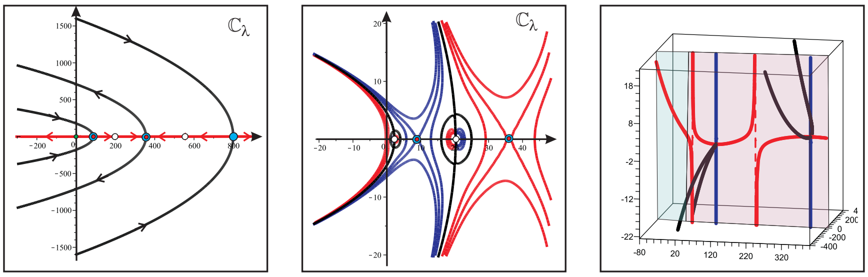
where $p(t) \geqslant p_0 > 0$, $p \in C^1[0,1]$, $q \in C[0,1]$, l_0, l_1 and k are linear functionals. For example, the functional k can describe multi-point or integral NBCs:

$$\langle k, u(t)\rangle = \sum_{j=1}^{n} \big(\varkappa_j u(\xi_j) + \kappa_j u'(x_j)\big), \quad \langle k, u(t)\rangle = \int_0^1 \varkappa(t)u(t)\mathrm{d}t,$$

and the functionals l_i, $i = 0, 1$ can describe local (classical) boundary conditions

$$\langle l_0, u(t)\rangle = \alpha_0 u(0) + \beta_0 u'(0), \quad \langle l_1, u(t)\rangle = \alpha_1 u(1) + \beta_1 u'(1),$$

where the parameters $|\alpha_i| + |\beta_i| > 0$, $i = 0, 1$.



a) domain $\mathcal{N}_\lambda$ b) contour lines c) complex-real function

Figure 16. Complex-real function $\tilde{\gamma} : \mathbb{C}_\lambda \to \mathbb{R}$, case $\xi = 2/3$.

Let $\varphi_0(t;\lambda)$ and $\varphi_1(t;\lambda)$ be two independent solutions of equation (3.1). For example, we can find such a solution by solving initial value problems with the conditions: $u(0) = 1$, $u'(0) = 0$ and $u(0) = 0$, $u'(0) = 1$. Let us denote

$$D_s^t(\lambda) := D_s^t[\varphi_0, \varphi_1](\lambda) = \begin{vmatrix} \varphi_0(t;\lambda) & \varphi_1(t;\lambda) \\ \varphi_0(s;\lambda) & \varphi_1(s;\lambda) \end{vmatrix},$$

$$\langle k_1 \cdot k_2, D_s^t(\lambda)\rangle := \begin{vmatrix} \langle k_1, \varphi_0(t;\lambda)\rangle & \langle k_1, \varphi_1(t;\lambda)\rangle \\ \langle k_2, \varphi_0(s;\lambda)\rangle & \langle k_2, \varphi_1(s;\lambda)\rangle \end{vmatrix}.$$

All solutions of equation (3.1) are of the form $u{=}C_0\varphi_0(t;\lambda){+}C_0\varphi_1(t;\lambda)$. There exists a nontrivial solution of problem (3.1)–(3.3) if and only if $\Psi(\lambda)\gamma{=}\Phi(\lambda)$, where $\Psi := \langle l_0 \cdot k, D_s^t(\lambda)\rangle$, $\Phi := \langle l_0 \cdot l_1, D_s^t(\lambda)\rangle$. Both functions $\Psi(\lambda)$ and $\Phi(\lambda)$ are entire functions for $\lambda \in \mathbb{C}$.

We can find constant eigenvalues for problem (3.1)–(3.3) as the roots of the system

$$\begin{cases} \Psi(\lambda) = 0, \\ \Phi(\lambda) = 0. \end{cases}$$

The complex characteristic function

$$\tilde{\gamma}_c := \Phi(\lambda)/\Psi(\lambda), \quad \tilde{\gamma}_c \colon \mathbb{C}_\lambda \to \overline{\mathbb{C}} \tag{3.4}$$

is a meromorphic function and describes nonconstant eigenvalues. We can introduce the plane $\mathbb{C}_q$ only for the differential operator $-u''$. For this operator the characteristic function (3.4) is more complicated in the plane $\mathbb{C}_\lambda$ than the characteristic function in the plane $\mathbb{C}_q$. The domain $\mathcal{N}_\lambda := \tilde{\gamma}_c^{-1}(\mathbb{R}) := \{q \in \mathbb{C}_\lambda \colon \operatorname{Im} \tilde{\gamma}_c(\lambda) = 0\} \subset \mathbb{C}_\lambda$, $\tilde{\gamma}_c$, contour lines and the complex-real function $\tilde{\gamma}_c$ are presented in Figure 16 for $\xi = 2/3$ (see, Figure 4, Figure 13, Figure 14 for $\xi = 2/3$). In this case (see, Lemma 2), the lines become parabolas. The contour lines and the complex-real function are complicated, too.

4 Conclusions

In this paper the spectrum for Sturm–Liouville problem with one Bitsadze–Samarskii type nonlocal boundary condition was investigated.

- The new type of characteristic function (complex-real characteristic function) was introduced for problem with real parameter γ. Properties of the complex part of spectrum were investigated with the help of this characteristic function. We prove that projections of this function onto domain $\mathbb{C}_q$ are curves with intersections only in the real part of the spectrum. Moreover, these curves are often defined by lines (or parabolas in $\mathbb{C}_\lambda$). We find the conditions when characteristic curves are defined by lines and also describe cases when we have only lines as characteristic curves.

- We investigate all characteristic functions (complex, complex-real and real characteristic functions) for this problem. We summarize some results which are published in the other articles on real eigenvalues, constant eigenvalues points, critical points. All definitions and examples are illustrated by graphs. We can do such an investigation for the other problems with one nonlocal boundary condition. So, our investigation of the problem with one Bitsadze–Samarskii type nonlocal boundary condition describes a technique for investigation of such problems.

- Theoretical investigation of complex eigenvalues is a very difficult problem even for this simple test problem with one Bitsadze–Samarskii type nonlocal boundary condition. Therefore, we present only few simple properties of a complex part of the spectrum for this problem and give results of computational analysis as graphs of the complex characteristic functions and their projections for some values of the parameter ξ.

- We investigate a qualitative view of the domain $\mathcal{N}$ with respect to the dynamics of ξ when the pole and zero meet at the constant eigenvalue point and show that a dynamical view is still relative simple. We note that such a simple behavior is valid only for this problem. In the case of some other nonlocal boundary conditions the situation can be more complicated even for simple integral BC (in this case the constant eigenvalue point is the pole of characteristic function) or two-points nonlocal BC (in this case higher order real critical points can exist).

We give a definition of the constant eigenvalues and the complex characteristic function for the Sturm–Liouville problem with one general type nonlocal boundary condition. By using characteristic functions we can investigate a very wide class of problems with nonlocal boundary conditions, analogously to the problem with one Bitsadze–Samarskii type nonlocal boundary condition.

Acknowledgement

The authors are grateful to Prof. Mifodijus Sapagovas for his valuable comments and discussions.

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