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Conservative Numerical Method for a System of Semilinear Singularly Perturbed Parabolic Reaction-Diffusion Equations*

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Abstract. On a vertical strip, a Dirichlet problem is considered for a system of two semilinear singularly perturbed parabolic reaction-diffusion equations connected only by terms that do not involve derivatives. The highest-order derivatives in the equations, having divergent form, are multiplied by the perturbation parameter ε^2; $\varepsilon \in (0, 1]$. When $\varepsilon \to 0$, the parabolic boundary layer appears in a neighbourhood of the strip boundary.

Using the integro-interpolational method, conservative nonlinear and linearized finite difference schemes are constructed on piecewise-uniform meshes in the x_1-axis (orthogonal to the boundary) whose solutions converge ε-uniformly at the rate $\mathcal{O}\big(N_1^{-2} \ln^2 N_1 + N_2^{-2} + N_0^{-1}\big)$. Here $N_1 + 1$ and $N_0 + 1$ denote the number of nodes on the x_1-axis and t-axis, respectively, and $N_2 + 1$ is the number of nodes in the x_2-axis on per unit length.

Key words: boundary value problem, vertical strip, system of semilinear equations, parabolic reaction-diffusion equations, perturbation parameter ε, parabolic boundary layer, conservative difference schemes, nonlinear and linearized difference schemes, piecewise-uniform mesh, ε-uniform convergence..

1 Introduction

Boundary value problems for systems of singularly perturbed partial differential equations in which the highest-order derivatives are multiplied by a small

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(perturbation) parameter ε often occur, for example, in modeling and analysis of heat- and mass- transfer processes when the thermal conductivity and diffusion coefficients are small and (or) the rate of reactions is large. When the parameter tends to zero, boundary layers appear in a neighborhood of the boundary.

Boundary value problems for linear systems of elliptic and parabolic equations on a strip were considered, for example, for reaction-diffusion equations in [10], and for convection-diffusion equations in [11]. A boundary value problem on a rectangle for a system of linear parabolic reaction-diffusion equations have been considered in [15, 17] and for a system of linear elliptic reaction-diffusion equations with two perturbation parameters have been considered in [13, 16].

In mathematical modeling quite often, differential equations are written in divergent form (see, e.g., [3]). Such a form of differential equations allows us to construct conservative finite difference schemes for which conservation laws hold. For parabolic equations in divergent form, conservative finite difference schemes were constructed in [8]. These schemes converge in the maximum norm at the rate $\mathcal{O}\big(N^{-2} + N_0^{-1}\big)$, where $N + 1$ and $N_0 + 1$ is the number of nodes in spatial and temporal variables. When constructing such schemes the integro-interpolational method is applied.

In the present paper, special finite difference approximations of a Dirichlet problem are considered on a strip for a system of two semilinear singularly perturbed parabolic reaction-diffusion equations. The highest-order derivatives in the differential equations having divergent form are multiplied by the perturbation parameter ε^2; the parameter ε takes arbitrary values in the open-closed interval (0, 1]. For $\varepsilon = 0$, the system of second-order equations degenerates into a system of ordinary differential equations. The equations in the system are connected by terms that do not involve derivatives. When ε tends to zero, the parabolic boundary layer with the typical width ε appears in a neighbourhood of the strip boundary. A similar problem first was considered in [14], where the condensing grid method and classical difference approximations of the boundary value problem were applied for the construction of ε-uniformly convergent difference schemes. In this paper, using the integro-interpolational method, a conservative nonlinear finite difference scheme is constructed that converges ε-uniformly at the rate $\mathcal{O}\big(N_1^{-2} \ln^2 N_1 + N_2^{-2} + N_0^{-1}\big)$, where $N = \min_s N_s$, while $N_1 + 1$ and $N_0 + 1$ are the numbers of mesh points on the x_1-axis and the t-axis, respectively, and $N_2 + 1$ is the minimal number of nodes in the x_2-axis on per unit length. A conservative linearized ε-uniformly convergent difference scheme is also considered whose solution components on the current temporal level are found from the disjoined system of linear equations. Conservative ε-uniformly convergent difference schemes for semilinear systems of singularly perturbed partial differential equations have never been studied.

The formulation of the initial-boundary value problem and the aim of the research are given in Section 2; the spatial derivatives in the differential equations are written in divergent form. Compatibility conditions that ensure the required smoothness of the solution are discussed in Section 3. *A priori* estimates for the problem solutions and their regular and singular components that are needed for the construction and study of difference schemes are exposed in

Section 4. To derive *a priori* estimates and justify convergence of special finite difference schemes, a technique is applied that had been developed for a system of linear singularly perturbed elliptic [13, 16] and parabolic [15, 17] equations on a rectangle. A *nonlinear conservative difference scheme* on the rectangular grid with an arbitrary distribution of nodes (in particular, on uniform grid) is constructed in Section 5. *Special nonlinear and linearized conservative difference schemes* that converge to the solution of the boundary value problem ε-uniformly are constructed in Section 6. Generalizations and remarks are discussed in Section 7. Conclusions are exposed in Section 8.

2 Problem Formulation. The Aim of Research

Let G be the domain $D \times (0, T]$ with the boundary $S = \overline{G} \setminus G$, where $\overline{D}$ is the vertical strip[1]

$$\overline{D} = D \bigcup \Gamma, \quad D = D_{(2.1)} = \{x: \quad 0 < x_1 < d, \quad |\, x_2 \,| < \infty\}, \tag{2.1}$$

$$S = S^L \bigcup S_0, \quad S^L = \Gamma \times (0, T], \quad S_0 = \overline{S}_0,$$

where S^L and S_0 are the lateral and lower parts of the boundary S. On the strip $\overline{D}$ we consider the Dirichlet problem for the system of two semilinear singularly perturbed parabolic reaction-diffusion equations

$$L\,\mathbf{u}(x,t) = \mathbf{g}\big(x,t,\ \mathbf{u}(x,t)\big), \quad (x,t) \in G, \tag{2.2a}$$

$$\mathbf{u}(x,t) = \boldsymbol{\varphi}(x,t), \quad (x,t) \in S. \tag{2.2b}$$

Here $L\,\mathbf{u}(x,t) = L(\varepsilon)\,\mathbf{u}(x,t) \equiv \left\{\varepsilon^2 L_2 - C(x,t) - P(x,t)\,\frac{\partial}{\partial t}\right\} \mathbf{u}(x,t)$,

$$L_2 = \begin{pmatrix} L_2^1 & 0 \\ 0 & L_2^2 \end{pmatrix},$$

$$L_2^1 = \sum_{s=1,2} \frac{\partial}{\partial x_s}\left(a_s^1(x,t)\,\frac{\partial}{\partial x_s}\right), \qquad L_2^2 = \sum_{s=1,2} \frac{\partial}{\partial x_s}\left(a_s^2(x,t)\,\frac{\partial}{\partial x_s}\right),$$

$$C(x,t) = \begin{pmatrix} c^{11}(x,t) & c^{12}(x,t) \\ c^{21}(x,t) & c^{22}(x,t) \end{pmatrix}, \quad P(x,t) = \begin{pmatrix} p^1(x,t) & 0 \\ 0 & p^2(x,t) \end{pmatrix},$$

$\mathbf{u}(x,t)$, $\mathbf{g}\big(x,t,\ \mathbf{u}\big)$ and $\boldsymbol{\varphi}(x,t)$ are vector-functions;

$$\mathbf{u}(x,t) = (u^1(x,t),\, u^2(x,t))^T, \quad (x,t) \in \overline{G}.$$

Note that the operator L_2 has divergent form [8].

We shall use both the vector form of the boundary value problem and the scalar form

$$\begin{aligned} L^i\,\mathbf{u}(x,t) &= g^i\big(x,t,\ \mathbf{u}(x,t)\big), \quad (x,t) \in G, \\ u^i(x,t) &= \varphi^i(x,t), \quad (x,t) \in S, \quad i = 1,2; \end{aligned} \tag{2.2c}$$

[1] The notation $L_{(j.k)}$ ($\overline{G}_{(j.k)}, M_{(j.k)}$) means that these operators (domains, constants) are introduced in formula $(j.k)$.

the operator $L^i = L^i_{(2.2)}$ is defined by the relation

$$L^i \, \mathbf{u}(x,t) = \varepsilon^2 \, L^i_2 \, u^i(x,t) - \sum_{j=1,2} c^{ij}(x,t) \, u^j(x,t) - p^i(x,t) \, \frac{\partial}{\partial t} u^i(x,t).$$

The functions $a^i_s(x,t)$, $p^i(x,t)$, $c^{ij}(x,t)$, $g^i\big(x,t,\ \mathbf{u}\big)$, and also $\varphi^i(x,t)$ are assumed to be sufficiently smooth on the set $\overline{G}$, $\overline{Q} \equiv \overline{G} \times R^2$ and the boundary S, respectively. For simplicity, we assume also that the following conditions hold: [2]

$$a_0 \le a^i_s(x,t) \le a^0, \ \ p_0 \le p^i(x,t) \le p^0, \ \ (x,t) \in \overline{G}, \ \ s = 1,2, \ \ a_0, \, p_0 > 0; \qquad (2.3a)$$

$$^{\vee}c^{ii}(x,t) \ge c_0, \quad m \, {}^{\vee}c^{ii}(x,t) \ge {}^{\wedge}c^{ij}(x,t), \quad (x,t) \in \overline{G}, \qquad (2.3b)$$

$$i,j = 1,2, \quad i \ne j, \quad c_0 > 0, \quad m = m_{(2.3)} < 1.$$

Here

$$^{\vee}c^{ii}(x,t) = c^{ii}(x,t) - g^i_i(x,t), \quad {}^{\wedge}c^{ij}(x,t) = |c^{ij}(x,t)| + g^i_j(x,t), \quad (x,t) \in \overline{G},$$

where

$$\left| \frac{\partial}{\partial u^i} \, g^i\big(x,t,\, \mathbf{u}\big) \right| \le g^i_i(x,t), \quad \left| \frac{\partial}{\partial u^j} \, g^i\big(x,t,\, \mathbf{u}\big) \right| \le g^i_j(x,t), \quad \big(x,t,\, \mathbf{u}\big) \in \overline{Q}.$$

The parameter ε takes arbitrary values in the open-closed interval $(0,1]$.

By a solution of the problem (2.2), we mean a function $\mathbf{u} \in C^{2,1}(G)$ that is continuous on $\overline{G}$ and satisfies the differential equation (2.2a) on G and the boundary condition (2.2b) on S.

The problem as formulated arises, for example, in modeling a diffusion process in combination with chemical reactions. The parameter multiplying the highest-order derivatives characterizes the diffusion coefficient of the agents and the functions $c^{ij}(x,t)$ determine the rates of the direct and inverse chemical reactions (see, e.g., [1]).

We assume that the solution of the problem is sufficiently smooth for fixed values of the parameter ε. When ε tends to zero, a parabolic boundary layer appears in a neighbourhood of the set S^L.

Our aim for the boundary value problem (2.2), (2.1) is to construct a finite difference scheme that converges ε-uniformly.

3 Compatibility Conditions for Problem (2.2), (2.1)

We give conditions imposed on the data of the problem (2.2), (2.1) that guarantee the required smoothness of the solution.

We introduce some notation. We denote by $\varGamma_j$ with $\varGamma = \bigcup \varGamma_j$ for $j = 1,2$, different sides of the strip D, where the side $\varGamma_1$ passes through the point $(0,0)$. Set

$$S_j = \varGamma_j \times (0,\, T], \quad j = 1,2. \qquad (3.1a)$$

[2] Here and below M, M_i (or m) denote sufficiently large (small) positive constants which do not depend on ε and on the discretization parameters.

We denote by S^c the set of "edges"

$$S^c = \overline{S}^L \bigcap S_0. \tag{3.1b}$$

3.1. In the case when the data of the problem (2.2), (2.1) satisfy the conditions

$$A^1,\, A^2 \in C^{l^{(1)}+1+\alpha,\,(l^{(1)}+\alpha)/2}(\overline{G}), \quad C_{(2.2)},\, P \in H^{l^{(1)}+\alpha}(\overline{G}), \tag{3.2a}$$
$$\mathbf{g} \in C^{l^{(1)}+\alpha,\,(l^{(1)}+\alpha)/2,\,l^{(1)}+\alpha}(\overline{Q}), \quad l^{(1)} \geq 0,$$

$$\boldsymbol{\varphi} \in H^{l^{(2)}+\alpha}(\overline{S}_j), \;\; \boldsymbol{\varphi} \in H^{l^{(2)}+\alpha}(S_0), \;\; \boldsymbol{\varphi} \in C(S), \quad j = 1, 2, \tag{3.2b}$$
$$l^{(2)} \geq 2, \quad \alpha \in (0,1),$$

where $A^i(x,t) = \begin{pmatrix} a_1^i(x,t) & 0 \\ 0 & a_2^i(x,t) \end{pmatrix}$, $(x,t) \in \overline{G}$, $i = 1, 2$, then for the solution of this problem one has $\mathbf{u} \in H^{l^{(3)}+\alpha_1}(G)$, $\mathbf{u} \in H^{\alpha_1}(\overline{G})$, where $l^{(3)} = \min\,(l^{(1)} + 2,\, l^{(2)})$, $\alpha_1 \in (0,1)$ (see [2, 5]).

Let

$$\textit{the data of the problem (2.2), (2.1) on the set } S^c_{(3.1)} \tag{3.2c}$$
$$\textit{satisfy compatibility conditions up to order } [l^{(4)}/2], \;\; l^{(4)} \leq l^{(3)},$$

where $[l/2] = [l/2]_{(3.2)}$ is the integer part of the number $l/2$. For the description of compatibility conditions (for the derivatives in t of the solution to the boundary value problem) on the set $S^c_{(3.1)}$ see [5]. Then the solution of the problem (2.2), (2.1) satisfies the inclusion $\mathbf{u} \in H^{l^{(4)}+\alpha_1}(\overline{G})$ (see [2, 5]).

3.2. In the case when the data of the problem (2.2), (2.1) satisfy the condition (3.2), where

$$l^{(1)} = l^{(2)} = l^{(4)} = l + 2, \tag{3.3}$$

then the solution satisfies the inclusion $\mathbf{u} \in H^{l+2+\alpha}(\overline{G})$ (see [2, 5]).

We shall assume that the following condition (we call it the condition (3.4)) holds:

The data of the problem (2.2), (2.1) *satisfy the conditions* (3.2), (3.3) *that guarantee the smoothness of the solution of the boundary value problem on* $\overline{G}$. *When constructing a priori estimates for the regular and singular components of the solution in the representations* (4.3), (4.6), (4.10) (*from Section* 4) *, the following condition is assumed to be fulfilled in addition to the conditions* (3.2), (3.3):

$$A^1,\, A^2 \in C^{l_1+1+\alpha,\,(l_1+\alpha)/2}(\overline{G}), \qquad C_{(2.2)},\, P \in H^{l_1+\alpha}(\overline{G}),$$
$$\mathbf{g} \in C^{l_1+\alpha,\,(l_1+\alpha)/2,\,l_1+\alpha}(\overline{Q}), \tag{3.4}$$
$$\boldsymbol{\varphi} \in H^{l_1+\alpha}(\overline{S}_j), \quad \boldsymbol{\varphi} \in H^{l_1+\alpha}(S_0), \quad \boldsymbol{\varphi} \in C(S); \quad j = 1, 2, \quad l_1 \geq l,$$

that guarantee the smoothness of the regular and singular components of the solution.

The actual values of l and l_1 are specified where it is required. The fulfilment of other conditions in addition to (3.2), (3.3), (3.4) is not assumed.

Note that the condition (3.4) belongs to sufficient conditions that are required for the construction of *a priori* estimates, and at the same time, this condition is sufficiently simple.

4 *A Priori* Estimates for Solutions

When constructing and studying convergence of classical and special difference schemes, we need estimates of the solutions and their derivatives.

4.1. Introducing the new variables $\tilde{x}_i = \varepsilon^{-1} x_i$, for $i = 1, 2$, we bring the problem (2.2), (2.1) to a form in which the coefficients at the high-order derivatives are equal to one. In that case derivatives of the function $\widetilde{\mathbf{u}}(\tilde{x}, t) = \mathbf{u}\big(x(\tilde{x}), t\big)$ in the new variables become of order one [2, 5]. Returning to the original variables, in the case of condition (3.4), where

$$l \geq K - 2, \tag{4.1}$$

we obtain the estimates

$$|\mathbf{u}(x,t)| \leq M, \qquad \Big| \frac{\partial^{k+k_0}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_0}} \, \mathbf{u}(x,t) \Big| \leq M \, \varepsilon^{-k}, \tag{4.2}$$

$$(x,t) \in \overline{G}, \quad k + 2\,k_0 \leq K, \quad k = k_1 + k_2,$$

where

$$|\mathbf{u}(x,t)| = \max_{\overline{G}} \, |\mathbf{u}(x,t)| = \max_{\overline{G},i} \, |u^i(x,t)|.$$

Theorem 1. *Let the data of the boundary value problem* (2.2), (2.1) *satisfy the conditions* (3.4), (4.1), *where* $K \geq 2$. *Then the solution of the problem satisfies the estimates* (4.2).

Remark 1. In the case of condition (2.3), the solution of the boundary value problem (2.2), (2.1) satisfies the estimate

$$|\mathbf{u}(x,t)| \leq 2\,(1 - m^2)^{-1} \, \max \big[c_0^{-1} \, \max_{\overline{G}} \, |\mathbf{g}(x, t, \, \mathbf{0})|, \; \max_{S} \, |\boldsymbol{\varphi}(x,t)| \big], \quad (x,t) \in \overline{G},$$

where $m = m_{(2.3)}$. For the component $u^i(x,t)$ we have the estimate

$$|u^i(x,t)| \leq m \, \max_{\overline{G}} \, |u^{3-i}(x,t)| + c_0^{-1} \, \max_{\overline{G}} \, |g^i(x,t, \, \mathbf{0})| + \max_{S} \, |\varphi^i(x,t)|,$$

$$(x,t) \in \overline{G}, \quad i = 1, 2.$$

4.2. We now give estimates that are obtained using the main terms of an asymptotic expansion of the solution (see, e.g., [6, 9, 16] in the case of linear equations). First, we write the solution of the problem as the sum of functions

$$\mathbf{u}(x,t) = \mathbf{U}(x,t) + \mathbf{V}(x,t), \quad (x,t) \in \overline{G}, \tag{4.3}$$

where $\mathbf{U}(x,t)$ and $\mathbf{V}(x,t)$ are the regular and singular terms of the solution decomposition. The function $\mathbf{U}(x,t)$, $(x,t) \in \overline{G}$, is the restriction to $\overline{G}$ of the function $\mathbf{U}^e(x,t)$, $(x,t) \in \overline{G}^{\,e}$, where the set $\overline{G}^{\,e}$, i.e., the extension of $\overline{G}$ beyond the boundary $\overline{S}^L$, includes $\overline{G}$ along with its m_0-neighbourhood; $\overline{G}^{\,e} = \overline{D}^{\,e} \times [0,\,T]$. The function $\mathbf{U}^e(x,t)$ is the solution of the problem

$$\begin{aligned} &L^e\,\mathbf{U}^e(x,t) = \mathbf{g}^e\big(x,t,\,\mathbf{U}^e(x,t)\big), && (x,t) \in G^e, \\ &\mathbf{U}^e(x,t) = \boldsymbol{\varphi}^e(x,t), && (x,t) \in S^e. \end{aligned} \tag{4.4}$$

Here L^e and $\mathbf{g}^e(x,t,\,\mathbf{u})$, $(x,t) \in \overline{Q}$ are smooth continuations of the operator $L_{(2.2)}$ and the function $\mathbf{g}(x,t,\,\mathbf{u})$ (that preserve the properties of (2.3)) the function $\boldsymbol{\varphi}^e(x,t)$, $(x,t) \in S^e$ is chosen sufficiently smooth, $\boldsymbol{\varphi}^e(x,t) = \boldsymbol{\varphi}(x,t)$, $(x,t) \in S_0$. Assume that the functions $\mathbf{g}^e(x,t,\,\mathbf{u})$ and $\boldsymbol{\varphi}^e(x,t)$ are equal to zero outside a nearest m_1-neighbourhood of the set $\overline{G}$, where $m_1 < m_0$. The function $\mathbf{V}(x,t)$ is the solution of the problem

$$\begin{aligned} &L_{(2.2)}\,\mathbf{V}(x,t) = \mathbf{g}\big(x,t,\,\mathbf{U}(x,t)+\mathbf{V}(x,t)\big) - \mathbf{g}\big(x,t,\,\mathbf{U}(x,t)\big), \quad (x,t) \in G, \\ &\mathbf{V}(x,t) = \boldsymbol{\varphi}(x,t) - \mathbf{U}(x,t) \equiv \boldsymbol{\varphi}_{\mathbf{V}}(x,t), \quad (x,t) \in S. \end{aligned} \tag{4.5}$$

4.2.1. Now we estimate the regular component of the problem solution in the representation (4.3). Let us write the function $\mathbf{U}(x,t)$ as the sum of functions

$$\mathbf{U}(x,t) = \sum_{k=0}^{n} \varepsilon^{2k}\mathbf{U}_k(x,t) + \mathbf{v}^n_{\mathbf{U}}(x,t) \equiv \mathbf{U}^n(x,t) + \mathbf{v}^n_{\mathbf{U}}(x,t),\ (x,t) \in \overline{G}, \tag{4.6}$$

that corresponds to the representation of the function $\mathbf{U}^e(x,t)$, $(x,t) \in \overline{G}^{\,e}$, which is the solution of problem (4.4):

$$\mathbf{U}^e(x,t) = \sum_{k=0}^{n} \varepsilon^{2k}\mathbf{U}^e_k(x,t) + \mathbf{v}^{en}_{\mathbf{U}}(x,t), \quad (x,t) \in \overline{G}^{\,e}.$$

The functions $\mathbf{U}^e_k(x,t)$, $(x,t) \in \overline{G}^{\,e}$, i.e., components in the expansion of the regular part of the solution, are solutions of the problems

$$\begin{aligned} &L_{(4.7)}\,\mathbf{U}^e_0(x,t) = \mathbf{g}^e\big(x,t,\,\mathbf{U}^e_0(x,t)\big), && (x,t) \in \overline{G}^{\,e} \setminus S^e_0, \\ &\mathbf{U}^e_0(x,t) = \boldsymbol{\varphi}^e(x,t), && (x,t) \in S^e_0; \\ &L_{(4.7)}\,\mathbf{U}^e_k(x,t) = \varepsilon^{-2}\left\{L_{(4.7)} - L^e_{(4.4)}\right\}\mathbf{U}^e_{k-1}(x,t) \\ &\quad + \varepsilon^{-2k}\left\{\mathbf{g}^e\Big(x,t,\,\sum_{k_1=0}^{k}\varepsilon^{2k_1}\,\mathbf{U}^e_{k_1}(x,t)\Big) - \mathbf{g}^e\Big(x,t,\,\sum_{k_1=0}^{k-1}\varepsilon^{2k_1}\,\mathbf{U}^e_{k_1}(x,t)\Big)\right\}, \\ & && (x,t) \in \overline{G}^{\,e} \setminus S^e_0, \\ &\mathbf{U}^e_k(x,t) = \mathbf{0}, && (x,t) \in S^e_0, \quad k > 0, \end{aligned} \tag{4.7}$$

where

$$L_{(4.7)} = L^e_{(4.4)}\big|_{\varepsilon=0} = -C^e(x,t) - P^e(x,t)\,\frac{\partial}{\partial t}.$$

For the function $\mathbf{v}_{\mathbf{U}}^{en}(x,t)$ we have the following estimate (see, e.g., [4]):

$$|\mathbf{v}_{\mathbf{U}}^{en}(x,t)| \le M\varepsilon^{2n+2}, \quad (x,t) \in \overline{G}.$$

In the case of condition (3.4), where

$$l \ge K-2, \quad l_1 \ge K+2\,n, \tag{4.8a}$$

for

$$n = [(K+1)/2]_{(3.2)} - 2, \quad K \ge 4 \tag{4.8b}$$

one has $\mathbf{U}^e \in H^{K+\alpha}(\overline{G}^{\,e})$. For the function $\mathbf{U}(x,t)$ we obtain the estimate

$$\Big|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}\,\mathbf{U}(x,t)\Big| \le M\,\left[1+\varepsilon^{K-k-2}\right],\ (x,t)\in\overline{G},\ k+2\,k_0 \le K. \tag{4.9}$$

Moreover, for the components $\mathbf{U}^n(x,t)$ and $\mathbf{v}_{\mathbf{U}}^n(x,t)$ we have the estimates

$$\Big|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}\,\mathbf{U}^n(x,t)\Big| \ \le\ M,$$

$$\Big|\frac{\partial^{k}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}\,\mathbf{v}_{\mathbf{U}}^n(x,t)\Big| \le\ M\,\varepsilon^{K-k-2}, \quad (x,t)\in\overline{G}, \quad k+2\,k_0 \le K.$$

Remark 2. According to the decomposition (4.6), the function $\boldsymbol{\varphi}_{\mathbf{V}(4.5)}(x,t)$ has the representation

$$\boldsymbol{\varphi}_{\mathbf{V}}(x,t) = \sum_{k=0}^{n} \varepsilon^{2k}\boldsymbol{\varphi}_{k\mathbf{V}}(x,t) + \boldsymbol{\varphi}_{\mathbf{v}}^n(x,t) \equiv \boldsymbol{\varphi}_{\mathbf{V}}^n(x,t) + \boldsymbol{\varphi}_{\mathbf{v}}^n(x,t), \quad (x,t)\in S,$$

where

$$\boldsymbol{\varphi}_{0\mathbf{V}}(x,t) = \boldsymbol{\varphi}(x,t) - \mathbf{U}_0(x,t), \quad \boldsymbol{\varphi}_{k\mathbf{V}}(x,t) = -\mathbf{U}_k(x,t), \quad k\ge 1,$$

$$\boldsymbol{\varphi}_{\mathbf{v}}^n(x,t) = -\mathbf{v}_{\mathbf{U}}^n(x,t), \quad (x,t)\in S.$$

4.2.2. Let us consider the decomposition of the singular part of the solution to the boundary value problem.

We construct the function $\mathbf{V}(x,t)$ as the sum of the functions

$$\mathbf{V}(x,t) = \sum_{k=0}^{n}\varepsilon^{2k}\,\mathbf{V}_k(x,t) + \mathbf{v}_{\mathbf{V}}^n(x,t) \equiv \mathbf{V}^n(x,t) + \mathbf{v}_{\mathbf{V}}^n(x,t),\ (x,t)\in\overline{G}. \tag{4.10}$$

The functions $\mathbf{V}_k(x,t)$, $(x,t)\in\overline{G}$, i.e., components of the singular part of the problem solution, are solutions of the problems

$$\begin{aligned}
L_{(4.11)}\,\mathbf{V}_0(x,t) &= \mathbf{g}\big(x,t,\,\mathbf{U}_0(x,t)+\mathbf{V}_0(x,t)\big)-\mathbf{g}\big(x,t,\,\mathbf{U}_0(x,t)\big), \quad (x,t)\in G,\\
\mathbf{V}_0(x,t) &= \boldsymbol{\varphi}_{0\mathbf{V}}(x,t), \qquad (x,t)\in S;\\
L_{(4.11)}\,\mathbf{V}_k(x,t) &= \varepsilon^{-2}\left\{L_{(4.11))}-L_{(4.4)}\right\}\mathbf{V}_{k-1}(x,t) \qquad (4.11)\\
&+\varepsilon^{-2k}\Big\{\mathbf{g}\Big(x,t,\,\sum_{k_1=0}^{k}\varepsilon^{2k_1}\left[\mathbf{U}_{k_1}(x,t)+\mathbf{V}_{k_1}(x,t)\right]\Big)\\
&-\mathbf{g}^e\Big(x,t,\,\sum_{k_1=0}^{k}\varepsilon^{2k_1}\,\mathbf{U}_{k_1}(x,t)+\sum_{k_1=0}^{k-1}\varepsilon^{2k_1}\,\mathbf{V}_{k_1}(x,t)\Big)\Big\}, \quad (x,t)\in G,\\
\mathbf{V}_k(x,t) &= \boldsymbol{\varphi}_{k\mathbf{V}}(x,t), \qquad (x,t)\in S, \quad k>0,
\end{aligned}$$

where

$$L_{(4.11)}\equiv\varepsilon^2\begin{pmatrix}\dfrac{\partial}{\partial x_1}\left(a_1^1(x,t)\dfrac{\partial}{\partial x_1}\right) & 0\\ 0 & \dfrac{\partial}{\partial x_1}\left(a_1^2(x,t)\dfrac{\partial}{\partial x_1}\right)\end{pmatrix}-C(x,t)-P(x,t)\frac{\partial}{\partial t}.$$

Under the conditions (3.4), (4.8), applying a technique similar to one given in [16], we obtain the estimate

$$\left|\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_0}}\,\mathbf{V}(x,t)\right| \le M\,\left(\varepsilon^{-k_1}+\varepsilon^{K-k-2}\right)\exp\left(-m\,\varepsilon^{-1}\,r(x,\Gamma)\right),$$
$$(x,t)\in\overline{G}, \quad k+2\,k_0\le K. \qquad (4.12)$$

Here $r(x,\Gamma)$ is the distance from the point x to the boundary Γ, and m is an arbitrary constant from the interval $(0,m_0)$, where $m_0=c_0^{1/2}(1-m_{(2.3)})^{1/2}$, for $c_0=c_{0(2.3)}$.

Theorem 2. *Let the data of the boundary value problem* (2.2), (2.1) *satisfy the conditions* (3.4), (4.8), *where* $K\ge 4$. *Then the solution components* $\mathbf{U}(x,t)$ *and* $\mathbf{V}(x,t)$ *in the decomposition* (4.3) *satisfy the estimates* (4.9) *and* (4.12).

Remark 3. In the case when the condition (2.3b) is violated, we pass in the problem (2.2), (2.1) from the function $\mathbf{u}(x,t)$ to the function $\mathbf{u}^*(x,t)$, $\mathbf{u}(x,t)=\mathbf{u}^*(x,t)\,\exp(\alpha\,t)$. We choose the value α sufficiently large so as to satisfy the condition

$$\begin{aligned}
&\Psi^i(x,t;\,\alpha)\equiv\alpha\,p_0+c^{ii}(x,t)-g_i^i(x,t)\ge c_0,\\
&m\,\Psi^i(x,t;\,\alpha)\ge|c^{ij}(x,t)|+g_j^i(x,t), \quad (x,t)\in\overline{G}, \quad i,\,j=1,2, \quad i\ne j,
\end{aligned}$$

where $c_0>0$, m is an arbitrary constant that satisfies the condition $m<1$ and $p_0=p_{0(2.3)}$. Estimating the function $\mathbf{u}^*(x,t)$ and its components, we return to the function $\mathbf{u}(x,t)$. It is not difficult to verify that constants m and M in an estimate of type (4.12), which is obtained for the function $V(x,t)$ in that case, depend on α. Moreover, the constant $m = m(\alpha)$ can be chosen arbitrary sufficiently small, and the constant $M = M(\alpha)$ grows as $\alpha \to \infty$. Thus, the statement of Theorem 2 is preserved also in the case when the condition (2.3b) is violated.

5 Finite Difference Scheme for Problem (2.2), (2.1)

5.1. When constructing a finite difference scheme for the problem (2.2), (2.1), we use the *integro-interpolational method* (see, e.g., [8]). On the set $\overline{G}$ we introduce the grid

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0, \quad \overline{D}_h = \overline{\omega}_1 \times \omega_2. \tag{5.1}$$

Here $\overline{\omega}_1$ and ω_2 are meshes on the interval $[0,\, d]$ and on the x_2-axis; $\overline{\omega}_0$ is a mesh on the interval $[0,T]$; all meshes are, in general, arbitrary nonuniform. Set $h_s^i = x_s^{i+1} - x_s^i$ with $x_1^i,\, x_1^{i+1} \in \overline{\omega}_1$ and $x_2^i,\, x_2^{i+1} \in \omega_2$; $h_s = \max_i h_s^i$, $h = \max_s h_s$, $s = 1,2$; $h_t^k = t^{k+1} - t^k$ with $t^k,\, t^{k+1} \in \overline{\omega}_0$, $h_t = \max_k h_t^k$. Assume that the conditions $h \le MN^{-1}$ and $h_t \le MN_0^{-1}$ are satisfied, where $N = \min_s N_s$ for $s = 1,2$, N_1+1 and N_0+1 are the numbers of nodes in the meshes $\overline{\omega}_1$ and $\overline{\omega}_0$, respectively; N_2+1 is the minimal number of nodes in the mesh ω_2 in the unit interval.

On the grid $\overline{G}_h$ for the solution of the problem, we use the nonlinear difference scheme

$$\begin{aligned} &\Lambda\, \mathbf{z}(x,t) = \mathbf{g}\big(x,t,\ \mathbf{z}(x,t)\big), \quad (x,t) \in G_h, \\ &\mathbf{z}(x,t) = \boldsymbol{\varphi}(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{5.2a}$$

Here

$$\begin{aligned} &G_h = G \bigcap \overline{G}_h, \quad S_h = S \bigcap \overline{G}_h, \\ &\Lambda\, \mathbf{z}(x,t) \equiv \varepsilon^2\, \Lambda_2\, \mathbf{z}(x,t) - C(x,t)\, \mathbf{z}(x,t) - P(x,t)\, \delta_{\overline{t}}\, \mathbf{z}(x,t), \\ &\Lambda_2 = \begin{pmatrix} \Lambda_2^1 & 0 \\ 0 & \Lambda_2^2 \end{pmatrix}, \quad \Lambda_2^k = \delta_{\widehat{x1}}\big(a_{1,i-1/2}^k(x,t)\, \delta_{\overline{x1}}\big) + \delta_{\widehat{x2}}\big(a_{2,r-1/2}^k(x,t)\, \delta_{\overline{x2}}\big), \\ &\qquad k = 1,\, 2, \qquad \mathbf{z}(x,t) = \big(z^1(x,t),\, z^2(x,t)\big)^T, \quad (x,t) \in \overline{G}_h, \end{aligned}$$

where

$$\begin{aligned} &\delta_{\widehat{x1}}\big(a_{1,i-1/2}^k(x,t)\, \delta_{\overline{x1}}\, \mathbf{z}(x,t)\big), \quad \delta_{\widehat{x2}}\big(a_{2,r-1/2}^k(x,t)\, \delta_{\overline{x2}}\, \mathbf{z}(x,t)\big), \\ &\delta_{\overline{xs}}\, \mathbf{z}(x,t), \quad \delta_{xs}\, \mathbf{z}(x,t), \quad s = 1,2, \quad \delta_{\overline{t}}\, \mathbf{z}(x,t) \end{aligned}$$

are difference derivatives defined as [8]:

$$\delta_{x1}\,\mathbf{z}(x,t) = \frac{\mathbf{z}(x_1^{i+1},\,x_2,\,t) - \mathbf{z}(x,t)}{h_1^i}, \quad \delta_{\overline{x1}}\,\mathbf{z}(x,t) = \frac{\mathbf{z}(x,t) - \mathbf{z}(x_1^{i-1},\,x_2,\,t)}{h_1^{i-1}},$$

$$\delta_{\widehat{x1}}\big(a^k_{1,i-1/2}(x,t)\,\delta_{\overline{x1}}\,\mathbf{z}(x,t)\big) = 2\,\frac{a_1^k\big(x_{1,i+1/2},x_2,t\big)\,\delta_{x1}\,\mathbf{z}(x,t) - a_1^k\big(x_{1,i-1/2},x_2,t\big)\,\delta_{\overline{x1}}\,\mathbf{z}(x,t)}{h_1^i + h_1^{i-1}},$$

$$\delta_{\widehat{x2}}\big(a^k_{2,r-1/2}(x,t)\,\delta_{\overline{x2}}\,\mathbf{z}(x,t)\big) = 2\,\frac{a_2^k\big(x_1,x_{2,r+1/2},t\big)\,\delta_{x2}\,\mathbf{z}(x,t) - a_2^k\big(x_1,x_{2,r-1/2},t\big)\,\delta_{\overline{x2}}\,\mathbf{z}(x,t)}{h_2^i + h_2^{i-1}},$$

$$\delta_{\overline{t}}\,\mathbf{z}(x,t) = (h_t^{j-1})^{-1}\,\big[\mathbf{z}(x,t) - \mathbf{z}(x,\,t^{j-1})\big], \quad (x,t) = (x_1^i,\,x_2^r,\,t^j) \in G_h,$$

$$x_{1,i+1/2} = 2^{-1}\,\big(x_1^i + x_1^{i+1}\big), \quad x_{1,i-1/2} = 2^{-1}\,\big(x_1^i + x_1^{i-1}\big),$$

$$x_{2,r+1/2} = 2^{-1}\,\big(x_2^r + x_2^{r+1}\big), \quad x_{2,r-1/2} = 2^{-1}\,\big(x_2^r + x_2^{r-1}\big), \quad k = 1,\,2.$$

In a scalar form, the finite difference scheme takes the form

$$\begin{aligned} &\Lambda^i\,\mathbf{z}(x,t) = g^i\big(x,t,\ \mathbf{z}(x,t)\big), && (x,t) \in G_h, \\ &z^i(x,t) = \varphi^i(x,t), && (x,t) \in S_h, \quad i = 1,2. \end{aligned} \tag{5.2b}$$

Here the operators Λ^i, $i = 1,2$ are defined by the following relations:

$$\Lambda^i\,\mathbf{z}(x,t) \equiv \varepsilon^2\,\Lambda_2^i\,z^i(x,t) - \sum_{j=1,2} c^{ij}(x,t)\,z^j(x,t) - p^i(x,t)\,\delta_{\overline{t}} z^i(x,t). \tag{5.2c}$$

Note that the discrete function

$$w_{1h}^k\big(x_{1,i+1/2},x_2,t\big) = -\varepsilon^2\,a_1^k\big(x_{1,i+1/2},x_2,t\big)\,\delta_{x1}\,z^k(x,t),$$

$$(x,t) = (x_1^i,x_2,\,t) \in G_h, \quad x_1^i,\,x_1^{i+1} \in \overline{\omega}_1, \quad k = 1,\,2$$

corresponds to a diffusion "flux" of the k-th substance along the x_1-axis at the point $\big(x_{1,i+1/2},x_2,t\big)$, which is the middle point between the nodes $\big(x_1^i,x_2,t\big)$ and $\big(x_1^{i+1},x_2,t\big)$ in the set $\overline{G}_h$. Analogously, the discrete function

$$w_{2h}^k\big(x_2,x_{2,r+1/2},t\big) = -\varepsilon^2\,a_2^k\big(x_1,x_{2,r+1/2},t\big)\,\delta_{x2}\,z^k(x,t),$$

$$(x,t) = (x_1,x_{2,r+1/2},\,t) \in G_h, \quad x_2^r,\,x_2^{r+1} \in \overline{\omega}_2, \quad k = 1,\,2$$

corresponds to a diffusion "flux" of the k-th substance along the x_2-axis in the point$\big(x_1,x_{2,r+1/2},\,t\big)$, which is the middle point between the nodes $\big(x_1,x_2^r,t\big)$ and $\big(x_1,x_2^{r+1},t\big)$ in the set $\overline{G}_h$.

In the difference scheme (5.2), (5.1), the first-order difference derivatives in x_s of the component $z^k(x,t)$ are used. They approximate the first-order differential derivatives $(\partial/\partial x_s)u^k(x,t)$ in the "middle" points of the mesh $\overline{\omega}_s$ with the second order of accuracy with respect to the mesh interval, which includes this "middle" point.

5.2. We write the grid equation from (5.2) in the node $(x,t) = (x_1^{i_1},\, x_2^{i_2},\, t) \in G_h$ in the following form:

$$\Big\{\big[-w^k_{1h}\big(x_{1,i_1+1/2}, x_2^{i_2}, t\big) + w^k_{1h}\big(x_{1,i_1-1/2}, x_2^{i_2}, t\big)\big]\,\triangle x_2 \tag{5.3a}$$
$$+\big[-w^k_{2h}\big(x_1^{i_1}, x_{2,i_2+1/2}, t\big) + w^k_{2h}\big(x_1^{i_1}, x_{2,i_2-1/2}, t\big)\big]\,\triangle x_1\Big\}\,\triangle t$$
$$+\Big\{-\sum_{j=1,2} c^{kj}(x,t)\, z^j(x,t) - g^k\big(x,t,\,\mathbf{z}(x,t)\big)\Big\}\,\triangle F(x)\,\triangle t$$
$$+\Big\{-p(x,t)\,\big[z^k(x,t) - z^k(x,\,\breve{t})\big]\Big\}\,\triangle F(x) = 0, \quad k = 1,\,2.$$

Here

$$\begin{aligned} w^k_{1h}\big(x_{1,i_1+1/2}, x_2^{i_2}, t\big) &= -\varepsilon^2\, a^k_1\big(x_{1,i_1+1/2}, x_2^{i_2}, t\big)\,\delta_{x1}\, z^k(x,t),\\ w^k_{2h}\big(x_1^{i_1}, x_{2,i_2+1/2}, t\big) &= -\varepsilon^2\, a^k_2\big(x_1^{i_1}, x_{2,i_2+1/2}, t\big)\,\delta_{x2}\, z^k(x,t). \end{aligned} \tag{5.3b}$$

In the equations (5.3a), (5.3b):

$$\triangle x_1 = x_{1,i_1+1/2} - x_{1,i_1-1/2}, \quad \triangle x_2 = x_{2,i_2+1/2} - x_{2,i_2-1/2},$$
$$\triangle F(x) = \triangle x_1\,\triangle x_2, \quad \triangle t = t - \breve{t} = t^j - t^{j-1}, \quad t^j, t^{j-1} \in \overline{\omega}_0,$$

$w^k_{1h}\big(x_{1,i_1+1/2}, x_2^{i_2}, t\big)\triangle x_2 \triangle t$ is the diffusion "flux" of the component z^k along the x_1-axis through the right side of the rectangle

$$\overline{D}^1 = \big[x_{1,i_1-1/2},\, x_{1,i_1+1/2}\big]\,\big[x_{2,i_2-1/2},\, x_{2,i_2+1/2}\big] \tag{5.4}$$

for the time interval $\triangle t = t^{j-1} - t^j$. Analogously, $w^k_{2h}\big(x_1^{i_1}, x_{2,i_2+1/2}, t\big)\triangle x_1 \triangle t$ is the diffusion "flux" z^k along the x_2-axis through the lower side of the rectangle (5.4).

The equation (5.3a) corresponds to a *conservation law* with respect to the spatial variables (see [8]) for the equation (2.2c) for an elementary volume in $\overline{G}$ generated by the flux grid [8] in x_1, x_2

$$\overline{G}^a_h = \omega^a_1 \times \omega^a_2 \times \overline{\omega}_0, \tag{5.5}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(5.1)}$, ω^a_1 is the flux mesh on the interval $[0,\, d]$ on the x_1-axis with the nodes

$$x_{1,\,1/2}, \;\; x_{1,\,1+1/2}, \;\; \dots, \;\; x_{1,\,i_1+1/2}, \;\; \dots, \;\; x_{1,\,N-1/2},$$

and ω^a_2 is the flux mesh on the x_2-axis with the nodes

$$\dots,\, x_{2,\,i_2-1/2},\, x_{2,\,i_2+1/2},\, x_{2,\,i_2+3/2},\, \dots.$$

Here

$$\begin{aligned} x_{1,\,i_1+1/2} &= 2^{-1}\left(x_1^{i_1} + x_1^{i_1+1}\right), \quad x_1^{i_1},\, x_1^{i_1+1} \in \overline{\omega}_1,\\ x_{2,\,i_2+1/2} &= 2^{-1}\left(x_2^{i_2} + x_2^{i_2+1}\right), \quad x_2^{i_2},\, x_2^{i_2+1} \in \overline{\omega}_2. \end{aligned}$$

The fluxes $w^k_{1h}(x,t)$ and $w^k_{2h}(x,t)$ are defined, respectively, on the flux grids $\overline{G}^a_{1h}$ in x_1 and $\overline{G}^a_{2h}$ in x_2, where

$$\overline{G}^a_{1h} = \omega^a_1 \times \omega_2 \times \overline{\omega}_0 \quad \text{and} \quad \overline{G}^a_{2h} = \overline{\omega}_1 \times \omega^a_2 \times \overline{\omega}_0. \tag{5.6}$$

By summing up equations (5.3a) corresponding to nodes from G_h on the rectangle

$$\overline{D}^2 = \left[x_{1,i_1-1/2},\, x_{1,i_1+k_1+1/2}\right] \left[x_{2,i_2-1/2},\, x_{2,i_2+k_2+1/2}\right] \tag{5.7}$$

for $t = t^j$, where

$$x_{1,i_1-1/2},\, \ldots,\, x_{1,i_1+k_1+1/2} \in \omega^a_1 \quad \text{and} \quad x_{2,i_2-1/2},\, \ldots,\, x_{2,i_2+k_2+1/2} \in \omega^a_2$$

with $k_1,\, k_2 \ge 1$, we obtain a relation between the fluxes $w_{1h}(x,t)$ and $w_{2h}(x,t)$ on the sides of the rectangle $\overline{D}^2_{(5.7)}$ for $t = t^j$ and the effective sources acting in nodes of the grid D_h inside the rectangle $\overline{D}^2$. The effective sources include the sources $\mathbf{g}\big(x, t,\, \mathbf{z}(x,t)\big)$ and also the terms $C(x,t)\; \mathbf{z}(x,t)$ and

$$P(x,t)\; (\triangle t)^{-1}[\mathbf{z}(x,t) - \mathbf{z}(x,\; t^{j-1})],$$

where $x \in D^2 \bigcap \overline{D}_h$.

By summing up equations (5.3a) with respect to the nodes of the parallelepiped $\overline{G}^2 = \overline{D}^2 \times T^2$, where $\overline{D}^2 = \overline{D}^2_{(5.7)}$ and $T^2 = [t^j,\, t^{j+l}]$ with $l \ge 1$, under the condition

$$P(x,t) = P(x), \quad (x,t) \in \overline{G}, \tag{5.8}$$

we obtain a relation between the fluxes $w_{1h}(x,t)$ and $w_{2h}(x,t)$ on the lateral faces of the parallelepiped $\overline{G}^2$, the values $\mathbf{z}(x,t)$ on the upper and lower faces of the parallelepiped $\overline{G}^2$ and effective sources involving $\mathbf{f}(x,t)$ and $C(x,t)\,\mathbf{z}(x,t)$ in nodes of the grid G_h inside the parallelepiped $\overline{G}^2$.

Thus, taking into account the corresponding source terms, the difference scheme (5.2), (5.1) is *conservative* [8] for subsets from G generated by rectangular cells-parallelepipeds that have nodes from the set $G^a_{h(5.5)}$ as vertexes. The scheme belongs to conservative in spatial variables on the temporal levels $t = t^j$ on G, and under the condition (5.8) it belongs to conservative in spatial and temporal variables on the whole set G. Note that in the case of the scheme (5.2), (5.1) the grid $G^a_{h(5.5)}$ is only an auxiliary grid with respect to which the conservation property is revealed.

We call all discrete equations (5.3) connecting the functions $\mathbf{z}(x,t)$, $\mathbf{w}_{1h}(x,t)$ and $\mathbf{w}_{2h}(x,t)$ that are defined, respectively, on the grids $\overline{G}_{h(5.1)}$, $\overline{G}^a_{1h(5.6)}$ and $\overline{G}^a_{2h(5.6)}$ the conservative flux difference scheme (5.3), (5.1), (5.6), and we call the function $\mathbf{z}(x,t)$, $(x,t) \in \overline{G}_h$ the solution of this scheme.

5.3. When investigating convergence of the difference scheme (5.2), (5.1), a maximum principle is used [8]. We assume that the solution of the boundary value problem (2.2), (2.1) satisfies the estimates of Theorem 1. Note that the operators

$$\Lambda^i_{(5.9)} \equiv \varepsilon^2\, \Lambda^i_2 - c^{ii}(x,t) - p^i(x,t)\,\delta_{\overline{t}}, \quad (x,t) \in G_h, \quad i = 1, 2. \tag{5.9}$$

in (5.2c) are monotone [8]. Taking into account the estimate

$$\left|z^i(x,t)\right| \le m \max_{\overline{G}_h} \left|z^{3-i}(x,t)\right| + M \left[\max_{\overline{G}_h} \left|g^i(x,t,\ \mathbf{0})\right| + \max_{S_h} \left|\varphi^i(x,t)\right|\right],$$

$$(x,t) \in \overline{G}_h, \quad i = 1, 2,$$

where $m < 1$ by virtue of (2.3), we obtain the estimate

$$|\mathbf{z}(x,t)| \le M \left[\max_{\overline{G}_h} |\mathbf{g}(x,t,\ \mathbf{0})| + \max_{S_h} |\boldsymbol{\varphi}(x,t)|\right], \quad (x,t) \in \overline{G}_h.$$

Taking into account the *a priori* estimates of the solution to the problem (2.2), (2.1) and using the monotonicity of the operator $\Lambda^i_{(5.9)}$, for the solution of the difference scheme (5.2), (5.1) we establish the estimate

$$|\,\mathbf{u}\,(x,t) - \mathbf{z}(x,t)\,| \ \le\ M\left[(\varepsilon + N^{-1})^{-1}\, N^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h. \quad (5.10)$$

On the uniform grid

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 \quad (5.11)$$

we obtain the estimate

$$|\,\mathbf{u}\,(x,t) - \mathbf{z}(x,t)\,| \ \le\ M\left[(\varepsilon + N^{-1})^{-2}\, N^{-2} + N_0^{-1}\right], \quad (x,t) \in \overline{G}^u_h. \quad (5.12)$$

Theorem 3. *Assume that the solution of the boundary value problem* (2.2), (2.1) *satisfies the estimates of Theorem* 1*, where* $K = 4$*. Then the difference scheme* (5.2), (5.1) *converges under the condition* $N^{-1} = o(\varepsilon)$*. The solution of the scheme* (5.2), (5.1) (*of the scheme* (5.2), (5.11)) *satisfies the estimate* (5.10) (*the estimate* (5.12)).

6 Special Finite Difference Schemes

6.1. From the estimates of Theorem 2 it follows that the derivatives of the solution in a neighbourhood of the boundary S^L increase without bound as the parameter ε tends to zero. In the case of the boundary value problem (2.2), (2.1), the boundary layer is sufficiently simple. To solve the boundary value problem, we apply a piecewise-uniform grid that condenses in a neighbourhood of the boundary.

Let us construct a special finite difference scheme for the problem (2.2), (2.1). On the set $\overline{G}$ we introduce the grid

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0, \quad \overline{D}_h = \overline{D}^S_h = \overline{\omega}^S_1 \times \omega_2, \quad (6.1)$$

where $\overline{\omega}_0 = \overline{\omega}_{0(5.11)}$, $\omega_2 = \omega_{2(5.11)}$, $\overline{\omega}^S_1 = \overline{\omega}^S_1(\sigma)$ is a piecewise-uniform mesh on the interval $[0, d]$. The step-size in the mesh $\overline{\omega}^S_1$ equals $h^{(1)}_1 = 4\sigma N_1^{-1}$ on the sets $[0, \sigma]$, $[d - \sigma, d]$ and $h^{(2)}_1 = 2(d - 2\sigma)N_1^{-1}$ on the set $[\sigma, d - \sigma]$. The value σ is defined by

$$\sigma = \sigma(\varepsilon, N_1) = \min\left[\,4^{-1}\, d,\ M\, \varepsilon\, \ln N_1\right],$$

where $M = 2m^{-1}_{(4.12)}$.

To solve the problem (2.2), (2.1) we use the conservative difference scheme

$$\begin{aligned} \Lambda\,\mathbf{z}(x,t) &= \mathbf{g}\big(x,t,\ \mathbf{z}(x,t)\big), \quad (x,t)\in G_h, \\ \mathbf{z}(x,t) &= \boldsymbol{\varphi}(x,t), \qquad\qquad (x,t)\in S_h, \end{aligned} \tag{6.2}$$

where $\Lambda = \Lambda_{(5.2)}$, $\overline{G}_h = \overline{G}_{h(6.1)}$. Taking into account the estimates of Theorem 2 and the monotonicity of the operator $\Lambda^i_{(5.9)}$, we establish the ε-uniform convergence of the difference scheme (6.2), (6.1)

$$|\,\mathbf{u}(x,t) - \mathbf{z}(x,t)\,| \ \le\ M\left[N^{-2}\ln^2 N + N_0^{-1}\right], \quad (x,t)\in\overline{G}_h. \tag{6.3}$$

Theorem 4. *Let the components in the decomposition* (4.3) *of the solution to the boundary value problem* (2.2), (2.1) *satisfy the estimates of Theorem* 2 *for* $K = 4$. *Then the solution of the difference scheme* (6.2), (6.1) *converges to the solution of the boundary value problem ε-uniformly. The discrete solution satisfies the estimate* (6.3).

6.2. In order to solve the boundary value problem (2.2), (2.1), it is convenient to use a linearized difference scheme where each component $z^1(x,t)$ and $z^2(x,t)$ at the temporal level $t\in\omega_0$ is found from the disjoined system of difference equations. We approximate the boundary value problem by the finite difference scheme

$$\begin{aligned} \Lambda\,\mathbf{z}(x,t) &= \mathbf{F}\left(\breve{\mathbf{z}}(x,t),\, \mathbf{g}\big(x,t,\ \breve{\mathbf{z}}(x,t)\big)\right), \quad (x,t)\in G_h, \\ \mathbf{z}(x,t) &= \boldsymbol{\varphi}(x,t), \quad (x,t)\in S_h. \end{aligned} \tag{6.4}$$

Here

$$\breve{\mathbf{z}}(x,t) = \mathbf{z}(x,\ t^{k-1}), \quad (x,t) = (x,\ t^k)\in\overline{G}_h, \quad t^k\in\omega_0;$$

$$\Lambda = \Lambda_{(6.4)}(\varepsilon) \equiv \varepsilon_2\,\Lambda_2 - C_1(x,t) - P(x,t)\,\delta_{\overline{t}}, \quad \Lambda_2 = \Lambda_{2(5.2)},$$

$$\mathbf{F}\left(\breve{\mathbf{z}}(x,t),\, \mathbf{g}\big(x,t,\ \breve{\mathbf{z}}(x,t)\big)\right) \equiv C_2(x,t)\,\breve{\mathbf{z}}(x,t) + \mathbf{g}\big(x,t,\ \breve{\mathbf{z}}(x,t)\big),$$

$$C_1(x,t) = \begin{pmatrix} c^{11}(x,t) & 0 \\ 0 & c^{22}(x,t) \end{pmatrix}, \quad C_2(x,t) = \begin{pmatrix} 0 & c^{12}(x,t) \\ c^{21}(x,t) & 0 \end{pmatrix}.$$

The components $z^i(x,t)$, $(x,t)\in\overline{G}_h$, $i = 1,2$, are found from the disjoined system of linear difference equations.

In scalar form, we have the difference scheme

$$\Lambda^1 z^1(x,t) \equiv \Big\{\varepsilon^2 \sum_{s=1,2} \delta_{\widehat{xs}}\big(a^1_{s,i-1/2}(x,t)\delta_{\overline{xs}}\big) - c^{11}(x,t) - p^1(x,t)\delta_{\overline{t}}\Big\} z^1(x,t)$$

$$= c^{12}(x,t)\,z^2(x,\ t^{k-1}) - g^1\big(x,t,\ \mathbf{z}(x,\ t^{k-1})\big) \equiv F^1\left(\breve{z}^2(x,t),\ g^1\big(x,t,\ \breve{\mathbf{z}}(x,t)\big)\right),$$

$$\Lambda^2 z^2(x,t) \equiv \Big\{\varepsilon^2 \sum_{s=1,2} \delta_{\widehat{xs}}\big(a^2_{s,i-1/2}(x,t)\delta_{\overline{xs}}\big) - c^{22}(x,t) - p^2(x,t)\delta_{\overline{t}}\Big\} z^2(x,t) =$$

$$= c^{21}(x,t)\,z^1(x,\ t^{k-1}) - g^2\big(x,t,\ \mathbf{z}(x,\ t^{k-1})\big) \equiv F^2\left(\breve{z}^1(x,t),\ g^2\big(x,t,\ \breve{\mathbf{z}}(x,t)\big)\right),$$

$$(x,t)\in\overline{G}_h, \quad t = t^k\in\omega_0.$$

The linearized difference scheme (6.4), (6.1) is conservative.

Taking into account the estimates of Theorem 2 and the monotonicity of the operator $\Lambda^i_{(6.4)}$, one can derive the estimate

$$|\,\mathbf{u}(x,t)-\mathbf{z}(x,t)\,| \;\le\; M\,[N^{-2}\ln^2 N+N_0^{-1}], \quad (x,t)\in\overline{G}_h. \qquad (6.5)$$

Thus, we have the following theorem.

Theorem 5. *Let the hypotheses of Theorem* 4 *be satisfied. Then the solution* $\mathbf{z}$ *of the conservative linearized difference scheme* (6.4)*,* (6.1) *converges to the solution* $\mathbf{u}$ *of the boundary value problem* (2.2)*,* (2.1) *ε-uniformly. The discrete solution satisfies the estimate* (6.5)*.*

7 Generalizations and Remarks

7.1. It is possible *to weaken the condition* (2.3b), changing it by a simpler condition, e.g., such as

$$c^{ii}(x,t)\ge c_0, \quad mc^{ii}(x,t)\ge |c^{ij}(x,t)|+\sum_{s=1,2}\max_{\mathbf{u}\in R^2}\left|\frac{\partial}{\partial u_s}g^i(x,t,\,\mathbf{u})\right|,$$

$$(x,t)\in\overline{G}, \quad i,j=1,2, \quad i\ne j, \quad m=m_{(7.1)}<1. \qquad (7.1)$$

In this case, *all constructions and justifications* in the paper *are preserved.*

7.2. By virtue of Remark 3 the *statements of Theorems* 3, 4 *and* 5 *are preserved* also in the case when the *condition* (2.3b) *is omitted.*

7.3. We write the equation (2.2) in the form

$$L\mathbf{u}(x,t)\equiv\left\{\varepsilon^2L_2-C^*(x,t)\right\}\mathbf{u}(x,t)-\frac{\partial}{\partial t}\left(P(x,t)\mathbf{u}(x,t)\right)=\mathbf{g}\big(x,t,\,\mathbf{u}(x,t)\big), \qquad (7.2)$$

for $(x,t)\in G$. Here $C^*(x,t)=C(x,t)+\dfrac{\partial}{\partial t}P(x,t)$. We construct a finite difference scheme for problem (7.2), (2.1) similarly to the scheme (5.2), (6.1). Such a scheme is *conservative already on the whole set* G and it converges ε-uniformly (see discussions in 5.2. (Section 5) in the case of the scheme (5.2), (6.1) under the condition (5.8)).

7.4. In the case of the *conservative finite difference scheme* (5.3), (5.1), (5.6) on *flux grids*, the technique from [12] allows us to establish ε-uniform convergence of the *solutions* $\mathbf{z}(x,t)$, $(x,t)\in\overline{G}_h$, *and the fluxes* $\mathbf{w}_{sh}(x,t)$, $(x,t)\in\overline{G}^a_{sh}$, $s=1,\,2$.

7.5. The exposed technique for the construction and investigation of ε-uniformly convergent difference schemes for the problem (2.2), (2.1) allows us to construct conservative ε-uniformly convergent difference schemes for a system of p equations, where $p>2$.

8 Conclusions

8.1. For an initial-boundary value problem (on a vertical strip) for a system of two semilinear singularly perturbed parabolic reaction-diffusion equations, *nonlinear and linearized conservative finite difference schemes* are constructed that converge in the maximum norm ε-uniformly with the second (up to a logarithmic factor) accuracy order in the spatial variable and with the first order in the temporal variable.

8.2. In the research, the following approach and methods were applied:

- an approach for the construction of conservative finite difference schemes based on the *integro-interpolational method* [8];
- the method of special grids *condensing* in a neighbourhoods of boundary layers [16];
- the method for the construction of *a priori estimates* based on a *special decomposition of the solution* into the regular and singular components [16].

8.3. The approach for the construction of conservative difference schemes developing in the paper for a system of semilinear singularly perturbed parabolic reaction-diffusion equations can be applied to construct and study ε-uniformly convergent conservative difference schemes for wide classes of singularly perturbed problems. The use of conservative difference schemes gives an opportunity to obtain numerical solutions for complicated problems, e.g., for problems with *large Reynolds numbers* and also in computations of *long-term nonstationary processes* [7].

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