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Numerical Investigations of Single Mode Gyrotron Equation*

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Abstract. A stationary problem with the integral boundary condition arising in the mathematical modelling of a gyrotron is numerically investigated. The Chebyshev's polynomials of the second kind are used as the tool of calculations. The main result with physical meaning is the possibility to determine the maximal value of electrons efficiency.

Key words: finite-difference schemes, mathematical modelling of gyrotron, Chebyshev's polynomials of the second kind.

1 Introduction

Gyrotrons are microwave sources whose operation is based on the stimulated cyclotron radiation of electrons oscillating in a static magnetic field. The main application of powerful gyrotrons is electron cyclotron resonance plasma heating in tokamaks and stellarators and the noninductive current drive in tokamaks. Extensive literature exists on various aspects of these devices, (see, [4, 12, 13, 14]). In the previous papers (see, [5, 9, 10, 11]) we numerically investigated problems arising for a gyrotron operated at the fundamental cyclotron harmonic. The present work continues these investigations in the case of the stationary problem.

Competition between the normalized amplitude f of the high frequency field in the resonator of gyrotron in the single mode case and the dimensionless complex transverse orbital momentum p of electron can be described by the

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following system of two complex differential equations:

$$\begin{cases} \dfrac{\partial p}{\partial x} + i\left(\Delta + |p|^2 - 1\right) p = i f(x,t), \\ \dfrac{\partial^2 f}{\partial x^2} - i\dfrac{\partial f}{\partial t} + \delta f = I < p > . \end{cases} \tag{1.1}$$

Here i is imaginary unit, $x \in [0, L]$ and $t \geq 0$ are the normalized axial and time coordinates, L is the length of the interaction space, Δ is the cyclotron resonance mismatch (real constant), $\delta(x)$ describes the frequency mismatch, I is the dimensionless current, $\theta_0 \in [0, 2\pi]$ is the flight phase parameter and $< p >= \frac{1}{2\pi}\int_0^{2\pi} p\, d\theta_0$ is an averaged value of p.

The system (1.1) is supplemented by the standard initial conditions

$$p(0, t, \theta_0) = \exp(i\theta_0), \quad f(x, 0) = f_0(x), \tag{1.2}$$

where $f_0(x)$ is the given complex function. The boundary conditions are given for the field at the entrance and at the exit to the interaction space in the cavity of gyrotron

$$\begin{aligned} & f(0,t) = 0, \\ & f(L,t) + \frac{1}{\sqrt{\pi i}} \int_0^t \frac{1}{\sqrt{t-\zeta}} \frac{\partial f(L,\zeta)}{\partial x}\, d\zeta = 0. \end{aligned} \tag{1.3}$$

The second boundary condition is a well known impedance boundary condition which is equivalent to nonlocal transparent boundary condition. Such a Dirichlet-to-Neumann map enables us to reduce the original half space problem to a problem in a finite domain. Analysis of transparent boundary conditions for general Schrodinger type equations and their numerical approximations were considered in many papers, see e.g. [2, 8, 15].

Let us note, that

$$\begin{aligned} & \int_0^t \frac{1}{\sqrt{t-\zeta}} \frac{\partial f(L,\zeta)}{\partial x}\, d\zeta \\ & \quad = \frac{\partial f(L,t)}{\partial x} \int_0^t \frac{1}{\sqrt{t-\zeta}}\, d\zeta + \int_0^t \frac{1}{\sqrt{t-\zeta}} \Big(\frac{\partial f(L,\zeta)}{\partial x} - \frac{\partial f(L,t)}{\partial x} \Big)\, d\zeta \\ & \quad = 2\sqrt{t} \frac{\partial f(L,t)}{\partial x} - \int_0^t \frac{1}{\sqrt{t-\zeta}} \Big(\frac{\partial (f(L,t) - f(L,\zeta))}{\partial x} \Big)\, d\zeta. \end{aligned}$$

So we get the following boundary condition

$$\frac{1}{\sqrt{\pi i}} \left(2\sqrt{t} \frac{\partial f(L,t)}{\partial x} - \int_0^t \frac{1}{\sqrt{t-\zeta}} \left(\frac{\partial (f(L,t) - f(L,\zeta))}{\partial x} \right)\, d\zeta \right) + f(L,t) = 0. \tag{1.4}$$

In [6] we have considered boundary conditions

$$f(0,t) = 0, \quad \frac{\partial f(L,t)}{\partial x} = -i\gamma f(L,t), \tag{1.5}$$

where γ is a positive parameter which describes the wave number at resonator exit.The second of the boundary conditions (1.5) is a simplified version of the transparent boundary conditions, it describes a perfect absorbation of a simple wave.

In the case of the boundary condition (1.5) instead (1.3), we note that the solution of the given problem highly depends on the parameter γ. It was shown in [6] that the value of η is the increasing function of γ and its maximal value is obtained for $\gamma = \infty$, which is not possible to realize in the real physical models. So, as it is difficult to obtain separate wave numbers at the resonator exit, it is more convenient to use the integral boundary condition (1.3).

Using the method of stationarity for the system (1.1) we approximate the derivative $\partial f/\partial t$ by the discrete difference $(f^{(s+1)} - f^{(s)})/\tau$, where $s = 0, 1, \ldots, S$ is the parameter of iterations, $f^{(0)} = f_0(x)$ is the given complex function. We write simply $f^{(s+1)} = f$, $p^{(s+1)} = p$, $p^{(0)} = \exp(i\theta_0)$ when no confusion can arise. The number of iterations S is determined from the condition $|f^{(s+1)}(L) - f^{(s)}(L)| \le \varepsilon$, where $\varepsilon > 0$ is a desirable precision. In this case for every iteration s we can rewrite the system of equations (1.1) in following form

$$\begin{cases} \dfrac{\partial p}{\partial x} + i\left(\Delta + |p^{(s)}|^2 - 1\right) p = i f^{(s)} \\ \dfrac{\partial^2 f}{\partial x^2} - \dfrac{i}{\tau}(f - f^{(s)}) + \delta f = I(\omega < p > + (1-\omega) < p^{(s)} >), \end{cases}$$

where ω is the under relaxation parameter, i.e. $\omega < 1$.

The numerical simulation of problem (1.1)–(1.3) allows us to calculate the optimal values of parameters Δ, I and δ in order to attain the maximal value of the electron efficiency coefficient

$$\eta = 1 - \frac{1}{2\pi} \int_0^{2\pi} |p(L, t, \theta_0)|^2 \, d\theta_0, \tag{1.6}$$

which features the extraction of the electron orbital momentum from the beam.

Section 2 describes the finite-difference method with nonuniform grid for this numerical simulation based on the nonsaturated approximation taking into account the endpoints of the interval $[0, L]$, furthermore, Section 3 demonstrates some of obtained numerical results. The main conclusions are summarised in the Section 4.

2 The Numerical Method

We consider the finite-difference method based on the nonsaturated approximation. The concept of nonsaturatedness for approximations of functions at first was defined in [3]. There was shown that error of approximation decreases if smoothness of the function increases. In [7] is proved that nonsaturated approximations of functions is possible to realize by means of Lagrange interpolations, which are the simplest global approximations of functions and simultaneously give simple expressions also for approximation of the derivatives. Particularly, if

derivative $f^{(m)}$ satisfies Holder condition in the interval $[-1,1]$ with $\gamma \in (0,1)$, then the remainder $R_{N-1}(t) = f(t) - P_{N-1}(t)$ of Lagrange interpolation with the roots of Chebyshev polynomials of second kind

$$U_{N-1}(t) = \frac{\sin(N\alpha)}{\sin(\alpha)}, \quad \alpha = \arccos t,$$

yields the asymptotic estimations

$$\|R_{N-1}\| = \max_{|t|\le 1} |R_{N-1}(t)| = O(N^{-m-\gamma} \ln N),$$

where $P_{N-1}(t)$ are Lagrange interpolation polynomials. For analytic functions estimations are $\|R_{N-1}\| = O(\exp(-\nu N))$, where ν is the fixed constant.

For given boundary conditions using the roots of polynomials $U_{N-1}(t), t \in [-1,1]$ we have the grid points

$$t_k = -\cos\frac{(k-1)\pi}{N}, \quad k = 1, \dots, N+1$$

and it is possible to take as t_k the roots of polynomials of $N+1$ -order

$$\omega_{N+1}(t) = \prod_{k=1}^{N+1} (t - t_k) = 2^{1-N} U_{N-1}(t)(t^2 - 1).$$

It is also easy to approximate the derivatives of the function f. The $N+1$-order Lagrange interpolation polynomials are in the following form $P_{N+1}(t) = \sum_{k=1}^{N+1} l_k(t) f(t_k)$, where

$$l_k(t) = \frac{\omega_{N+1}(t)}{(t - t_k)\omega'_{N+1}(t_k)}$$

are the elementary Lagrange multipliers. The elements of $N+1$-order interpolation matrix of the derivatives $\tilde{D}$ ($P' = \tilde{D}P$, where P, P' are the vector-columns of the corresponding values of $P_{N+1}(t_k), P'_{N+1}(t_k), k = 1, \dots, N+1$) have following expressions

$$d_{j,k} = \frac{\omega'_{N+1}(t_j)}{(t_j - t_k)\omega'_{N+1}(t_k)}, j \ne k, \quad d_{k,k} = \frac{\omega''_{N+1}(t_k)}{2\omega'_{N+1}(t_k)}.$$

For calculations we use following representations

$$d_{j,k} = \frac{(-1)^{j+k} c_j}{c_k(t_j - t_k)}, j \ne k, \quad d_{k,k} = -\sum_{j\ne k}^{N+1} d_{j,k},$$

where $c_1 = c_{N+1} = 2$, otherwise $c_j = 1$.

Using the transformation $x = 0.5L(1+t)$, $t \in [-1,1]$ we obtain that the grid points of nonuniform grid are

$$x_k = 0.5L\left(1 - \cos\frac{\pi(k-1)}{N}\right), \tag{2.1}$$

and the matrix of first order derivative in the interval $[0, L]$ is $D = \frac{2}{L}\tilde{D}$.

Remark 1. We note that in [16] the optimal approximation on the uniform grid is used for numerical solution of similar problems, but the approach of this paper is more useful in the case of our investigation.

Using the grid (2.1) we approximate the derivatives $\partial/\partial x$, $\partial^2/\partial x^2$ in the equations (1.1) with matrixes D, D^2 as

$$f'_h = Df_h, \quad f''_h = D^2 f_h, \quad p'_h = Dp_u,$$

where $f_h = (f_1, f_2, \ldots, f_{N+1})$, $f'_h = (f'_1, f'_2, \ldots, f'_{N+1})$ are the vector-columns of the corresponding values of the grid functions,

$$f_j \approx f(x_j), \quad p_j \approx p(x_j, \theta_0), \quad f'_j \approx \frac{\partial f(x_j)}{\partial x}, \quad f''_j \approx \frac{\partial^2 f(x_j)}{\partial x^2}.$$

From the Lagrange interpolation follows, that elements $d_{j,k}$ of matrix D are given in the form

$$d_{j,k} = \frac{dl_k(x_j)}{dx}, \quad j,k = 1,2,\ldots,N+1,$$

where $l_k(x)$ are the elementary Lagrange multipliers.

The approximation of the first equation (1.1) on the grid x_j, $j = 2,\ldots,N+1$ is defined in following form:

$$\sum_{k=1}^{N+1} d_{j,k}p_k + i(\Delta + |p_j^{(s)}|^2 - 1)p_j = if_j. \tag{2.2}$$

Using the initial condition (1.2) we rewrite equation (2.2) in the matrix form

$$B^{(1)}p_h = g_h^{(1)}, \tag{2.3}$$

where $B^{(1)} = D^{(1)} + i(\Delta - 1)E + iP(p)$ is the matrix of order N, $g_h^{(1)}$ is the vector-column of order N with elements

$$g_j^{(1)} = if_j - d_{j,1}\exp(i\theta_0), \quad j = 2,3,\ldots,N+1,$$

E is the unit matrix of order N. Matrix $P(p)$ is a diagonal matrix with nonlinear elements $|p_j^{(s)}|^2$.

Similarly, the approximation of the second equation (1.1) in the inner grid points x_j, $j = 2,3,\ldots,N$ has the form

$$\sum_{k=1}^{N+1} \tilde{d}_{j,k}f_k - if_j/\tau + \delta_j f_j = Ip_j^{av} - if_j^{(s)}/\tau, \tag{2.4}$$

where $\tilde{d}_{j,k}$ are elements of matrix D^2, δ_j and p_j^{av} are the values of grid functions (vectors-columns) δ and $p^{av} =< p >$.

Let $t = s\tau$. Using the mean rectangle formula by calculating the boundary integral (1.4) we get

$$f^s + \frac{1}{\sqrt{\pi i}}\Big(2F^s\sqrt{n\tau} - \sum_{j=0}^{s-1}\frac{F^s - F^j}{\sqrt{s-j}}\sqrt{\tau}\Big) = 0,$$

where $f^s = f(t_s, L)$, $F^s = \dfrac{\partial f(t_s, L)}{\partial x}$, $F^j = \dfrac{\partial f(t_j, L)}{\partial x}$, $t_j = j\tau$. Then

$$\frac{\sqrt{\pi i}}{\sqrt{\tau}} f^s_{N+1} + \Big(2\sqrt{s} - \sum_{j=0}^{s-1} \frac{1}{\sqrt{s-j}}\Big) F^s_{N+1} + \sum_{j=0}^{s-1} \frac{F^j_{N+1}}{\sqrt{s-j}} = 0,$$

where $f^s_{N+1} = f(t_s, x_{N+1})$, $F^s_{N+1} = \dfrac{\partial f(t_s, x_{N+1})}{\partial x}$, $F^j_{N+1} = \dfrac{\partial f(t_j, x_{N+1})}{\partial x}$ and $x_{N+1} = L$.

Let us denote by

$$S_{1s} = 2\sqrt{s} - \sum_{j=0}^{s-1} \frac{1}{\sqrt{s-j}}, \quad S_{2s} = \sum_{j=0}^{s-1} \frac{F^j_{N+1}}{\sqrt{s-j}}.$$

Then

$$F^s_{N+1} = d_{N+1,N+1} f^s_{N+1} + \sum_{k=2}^{N} d_{N+1,k} f^s_k$$

yields

$$\begin{aligned} f^s_{N+1} &= -\frac{1}{S_{1s} d_{N+1,N+1} + \sqrt{\pi i/\tau}} \Big(S_{2s} + S_{1s} \sum_{k=2}^{N} d_{N+1,k} f^s_k\Big) \\ &= -\frac{1}{d_{N+1,N+1} + S^{-1}_{1s}\sqrt{\pi i/\tau}} \sum_{k=2}^{N} d_{N+1,k} f^s_k - \frac{S_{2s}}{S_{1s} d_{N+1,N+1} + \sqrt{\pi i/\tau}}. \quad (2.5) \end{aligned}$$

From (2.4) it follows that

$$\begin{aligned} \sum_{k=1}^{N+1} \tilde{d}_{j,k} f^s_k &= \sum_{k=2}^{N} \tilde{d}_{j,k} f^s_k + \tilde{d}_{j,N+1} f^s_{N+1} \\ &= \sum_{k=2}^{N} \tilde{d}_{j,k} f^s_k - \frac{\tilde{d}_{j,N+1}}{S_{s1} d_{N+1,N+1} + \sqrt{\pi i/\tau}} \Big(S_{2s} + S_{1s} \sum_{k=2}^{N} d_{N+1,k} f^s_k\Big) \\ &= \sum_{k=2}^{N} b_{j,k} f^s_k - \frac{S_{2s}}{S_{1s} d_{N+1,N+1} + \sqrt{\pi i/\tau}} \tilde{d}_{j,N+1}, \end{aligned}$$

where

$$b_{j,k} = \tilde{d}_{j,k} - \frac{d_{N+1,k} \tilde{d}_{j,N+1}}{d_{N+1,N+1} + S^{-1}_{1s}\sqrt{\pi i/\tau}}, \quad j = 2, 3, \ldots, N.$$

Let B be the matrix with elements $b_{j,k}$. We can rewrite (2.4) in the matrix form

$$B^{(2)} f_h = g^{(2)}_h, \quad (2.6)$$

where $B^{(2)} = B + D_0$ is the matrix of order $N-1$ (matrix D_0 is a diagonal matrix with elements $\delta_j - i/\tau$), $g^{(2)}_h$ is the vector-column with components

$$g^{(2)}_j = I p^{av}_j - i f^*_j/\tau + \frac{S_{2s}}{S_{1s} d_{N+1,N+1} + \sqrt{\pi i/\tau}} \tilde{d}_{j,N+1}, \quad j = 2, 3, \ldots, N.$$

We use discrete values of the parameter θ_0 in the form $\theta_m = m\frac{2\pi}{M}$, $m = 1, 2, \ldots, M$ where M is the number of angular grid points. Let us denote $p_j(\theta_m) = p(x_j, \theta_m)$. Using the trapezoid rule of the quadrature formula for calculation of the integral $g_j^{(2)}$ in (2.6), we obtain the following expression

$$p_j^{av} = \frac{1}{M}\sum_{m=1}^{M} p_j(\theta_m), \quad j = 1, 2, \ldots, N. \tag{2.7}$$

The approximation error is defined as $r_j = \dfrac{\pi^2}{3M^3}\dfrac{d^2 p_j(\xi)}{d\theta_0^2}$, where $\xi \in (0, 2\pi)$.

Similarly, from (1.6) we can calculate

$$\eta = 1 - \frac{1}{M}\sum_{m=1}^{M} \left|p_{N+1}(\theta_m)\right|^2.$$

Therefore from (2.3) for a fixed value θ_m we can calculate the vector-column $p_h(\theta_m)$, $m = 1, 2, \ldots, M$ and obtain a rectangular matrix W with $N-$columns and $M-$rows (in the m-row is the transposed vector $p_h(\theta_m)$).

From (2.6) it follows that $f_h = (B^{(2)})^{-1} g_h^{(2)}$, where the vector $g_h^{(2)}$ contains components of (2.7). Adding to this vector component f_{N+1} defined by (2.5) we obtain the vector f_h of the order N for calculation of vector $p_h(\theta_m)$. Using (2.3) we write it in the following form $p_h(\theta_m) = (B^{(1)})^{-1} g_h^{(1)}$.

In the iteration process for the matrix $P(p)$ the diagonal elements are calculated by using $|p_j^{(s)}|^2$ from previous iterations and the new value is obtained by using the expression

$$p_h^{(s+1)}(\theta_m) = \omega p_h(\theta_m) + (1-\omega) p_h^{(s)}(\theta_m).$$

3 Numerical Results

In [1] the maximum efficiency $\eta = 0.75$ is obtained for $\Delta = 0.6$, $\delta = 0$, $I = 0.01$ in the case of the so-called reflectionless boundary conditions in the integral form. In the present work we consider the boundary condition (1.3) for $\Delta \in [0, 0.7]$ depending on $\delta = \delta(x)$.

We consider two different initial conditions $f_0(x) = 0.1 \sin\left(\frac{\pi x}{L}\right)$ and $f_0(x) = 0.1\exp(-(x-\beta)^2)$, where $\beta \in [0.7, 11]$. The computations are performed by using "MATLAB" tool for $L = 15$, $M = 36$, $N = 40$, $S \in [1000, 5000]$, $\varepsilon = 0.001$, $\omega = 0.5$, $\tau \in [0.05, 0.1]$, $I \in [0.01, 0.1]$.

For the function δ we investigated three cases: 1) $\delta = 0$, 2) $\delta = \sinh(2x - L)/\sinh(L)$, 3) $\delta = \tanh(x - L/2)$. Using $M = 40$, $N = 50$, $S = 5000$ we proved, that the value of the electron efficiency η is accurate in average up to 4^{th} decimal sign.

The results of calculations with $\delta = 0$, $I = 0.01$, $\tau = 0.05$, $S = 2000$ for the analysis of the efficiency coefficient η and the high frequency field amplitude at the resonator ($|f(L)|$ for different values of Δ are summarized in Tab. 1. If $\Delta = 0.6$ and $S = 4000$, then $\eta = 0.7474$, but for $S = 5000$ we get $\eta = 0.7486$.

Table 1. The values of η and $|f(L)|$ for $\delta = 0$, $I = 0.01$.

No.	Δ	η	$\lvert f(L)\rvert$
1	0.5	0.4858	0.0959
2	0.6	0.7411	0.1605
3	0.7	0.0281	0.0688

Fig. 1 shows distributions of the averaged value of p and the values $p(L)$ in the complex plane for $M = 360$, and the values of the function $f(x)$ (real, imaginary parts and modulus with respect to x).

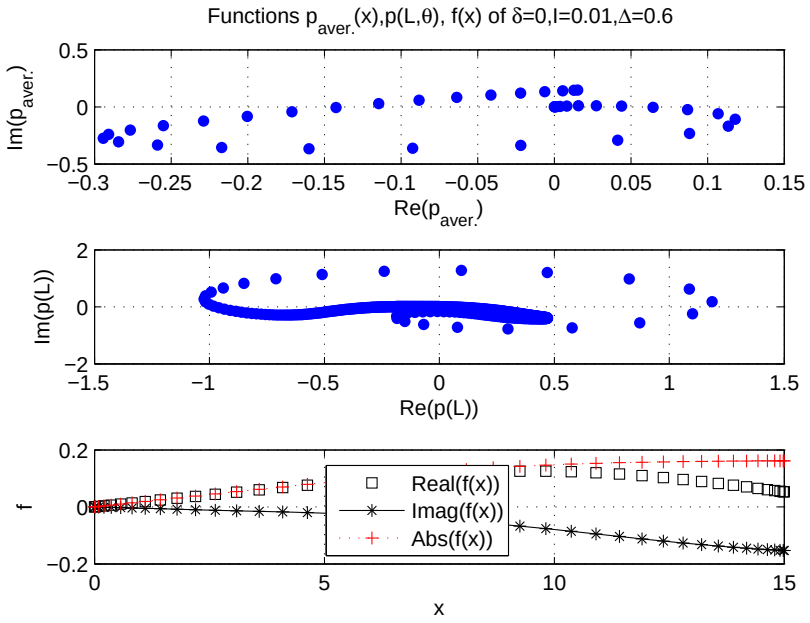

Figure 1. Distributions of functions of p and f for $I = 0.01$, $\Delta = 0.6$ $\delta = 0$.

For the varying $\delta = \delta(x)$, if $\Delta = 0.6, I = 0.01, \tau = 0.05$ we get the following values of the electron efficiency coefficient η:

1. If $\delta(x) = \sinh(2x - L)/\sinh(L)$, then $\eta = 0.5748$,
2. If $\delta(x) = \tanh(x - L/2)$, then $\eta = 0.3788$, but for $I = 0.1$ we get $\eta = 0.1870$ ($\delta = 0$).

In the case $f_0(x) = 0.1\exp(-(x - \beta)^2)$ for different values of β we obtain the following results, presented in Tab. 2.

1. If $\delta(x) = \tanh(x - L/2)$, then $\eta = 0.4429$ and $|f(L)| = 0.0623$.
2. If $\delta(x) = \sinh(2x - L)'\sinh(L)$, then $\eta = 0.5141$ and $|f(L)| = 0.0785$.

Table 2. The values of η and $|f(L)|$ for $\Delta = 0.6$, $\delta = 0$, $I = 0.01$.

No.	β	η	$\lvert f(L)\rvert$
1	0.75	0.4671	0.0584
2	5.5	0.5998	0.1774
3	8.5	0.7312	0.1641
4	9.0	0.7339	0.1632
5	10.5	**0.7449**	0.1591
6	11.0	0.7275	0.1568

The optimal case with $f_0(x) = 0.1 \sin \frac{\pi x}{2}$ was achieved with number of iterations $S = 7000$.

4 Conclusions

Numerical experiments show that the electron efficiency coefficient η depends on the complex valued function of high-frequency field in resonator.

For fixed values of this function it is possible to obtain different values of the electron efficiency coefficient η, where the maximal value depends on parameters of gyrotron. This maximum is increasing for high frequency field in resonator and depends on the frequency mismatch δ. The maximal value $\eta = 0.7486$ is obtained for $I = 0.01$, $\delta = 0$, $\Delta = 0.6$ and corresponds to the results of [1].

On the other hand these results differs from the results which were obtained for local boundary conditions with parameter γ [6]. As far as the electron efficiency coefficient η strongly depends on the parameter γ in the boundary condition (1.3), the admissible values of this parameter must be obtained using physical considerations. These values, of course, depend on the geometry of gyrotron and other parameters.

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