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Identification of Microstructured Materials by Phase and Group Velocities*

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Abstract. An inverse problem to determine parameters of microstructured solids by means of group and phase velocities of wave packets is studied. It is proved that in the case of normal dispersion the physical solution is unique and in the case of anomalous dispersion two physical solutions occur. Numerical tests are presented.

Key words: inverse problem, microstructure, dispersion, phase and group velocity.

1 Introduction

Microstructured materials like alloys, crystallites, ceramics, functionally graded materials, etc. have gained a wide utilization [5, 7, 10]. In many cases the nondestructive evaluation of physical properties of such materials is necessary. The simplest method to gather data for nondestructive evaluation is to generate waves and measure their characteristics, such as velocities, amplitudes, frequencies, etc. [1, 2, 3, 12, 13].

The wave propagation in materials with microstructure is influenced by various scales of microstructure and involves dispersive effects [5, 6, 7, 11]. In case of presence of dispersion the group and phase velocities are different and expected to contain information about the physical properties.

In this paper we will study an inverse problem to reconstruct physical parameters of microstructured materials from measured phase and group velocities of wave packets. This is a continuation of the research of the paper [9], where these parameters were recovered from spectra of linear waves. Similar results for a simplified approximate model were obtained in [8].

Our method is as follows. Firstly, we determine coefficients of a dispersion equation (see eq. (3.1) given below) by means of the phase and group velocities

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and then we compute the physical parameters by means of these coefficients. We expect that this quite general method can be adjusted for nondestructive evaluation of other dispersive media, too.

2 Model Description

We follow the Mindlin model of microstructure [11], where the microelement is taken as a deformable cell (e.g. a molecule of a polymer, a crystallite of a polycrystal or a grain of a granular material). 1D system of governing equations in Mindlin model was derived in [6]:

$$\begin{cases} \rho u_{tt} \ = \ a u_{xx} + D\psi_x, \\ I\psi_{tt} \ = \ C\psi_{xx} - Du_x - B\psi. \end{cases} \tag{2.1}$$

Here u is the macrodisplacement, ψ is the microdeformation and ρ, I, a, B, C are physical parameters of the material satisfying the inequalities

$$\rho,\, I,\, a,\, B,\, C > 0. \tag{2.2}$$

We rewrite (2.1) in dimensionless variables $X = \frac{x}{L}$, $T = \frac{t}{T_0}$, $U = \frac{u}{U_0}$ where L, T_0, U_0 are certain constant values. Introducing in addition the geometric parameters

$$\delta = \frac{l^2}{L^2}\,, \quad \epsilon = \frac{U_0}{L}\,, \quad \vartheta = \frac{T_0^2}{L^2}, \tag{2.3}$$

where l is the scale of the microstructure, system (2.1) can be written as

$$\begin{cases} \rho U_{TT} = a\vartheta U_{XX} + \dfrac{D\vartheta}{\epsilon}\psi_X, \\ \delta \dfrac{I}{\vartheta l^2}\psi_{TT} = \delta \dfrac{C}{l^2}\psi_{XX} - D\epsilon U_X - B\psi. \end{cases} \tag{2.4}$$

For the macrodeformation v we have the relation $v = u_x = \epsilon U_X$. Let us replace the first equation in (2.4) by the corresponding equation for v, namely

$$\rho v_{TT} \ = \ a\vartheta v_{XX} + \frac{N\vartheta}{2}(v^2)_{XX} + D\vartheta\psi_{XX},$$

and rewrite the resulting system by means of the lower-case letters x and t:

$$\begin{cases} \rho v_{tt} \ = \ a\vartheta v_{xx} + D\vartheta\psi_{xx}, \\ \delta \dfrac{I}{\vartheta l^2}\psi_{tt} \ = \ \delta \dfrac{C}{l^2}\psi_{xx} - Dv - B\psi. \end{cases} \tag{2.5}$$

This system has in total 7 coefficients: ρ, $a\vartheta$, $D\vartheta$, $\frac{I}{\vartheta l^2}$, $\frac{C}{l^2}$, D and B related to the physical properties of the material. We suppose that these coefficients are unknown. The geometric parameter δ is assumed to be given.

Evidently, it is not possible to recover all coefficients of the homogeneous system (2.5). Indeed, any vector of coefficients that fits to this system can be multiplied by an arbitrary constant to get another vector of coefficients that also fits to this system. The determination of all coefficients could be possible only in the case of non-homogeneous system containing mass-forces. Therefore, we divide equations (2.5) by ρ and $\frac{I}{\vartheta l^2}$, respectively, to obtain the system

$$\begin{cases} v_{tt} \,=\, a_0 v_{xx} + \alpha \psi_{xx} \,, \\ \psi_{tt} \,=\, a_1 \psi_{xx} - \dfrac{\beta}{\delta} v - \dfrac{\gamma}{\delta} \psi, \end{cases} \tag{2.6}$$

that contains 5 unknown coefficients:

$$a_0 = \frac{a\vartheta}{\rho}, \quad \alpha = \frac{D\vartheta}{\rho}, \quad a_1 = \frac{C\vartheta}{I}, \quad \beta = \frac{D\vartheta l^2}{I}, \quad \gamma = \frac{B\vartheta l^2}{I} \,. \tag{2.7}$$

It seems that a realistic problem could be to determine five coefficients a_0, a_1, γ, α and β. However, it was shown in [9] that it is not possible to separate α and β from their product $\alpha\beta$ by means of information gathered from linear waves in macro-level. Therefore, the inverse problem we will pose and study consists in determination of four parameters a_0, a_1, γ and $\alpha\beta$.

We note that the coefficients to be determined satisfy the following a priori inequalities

$$a_0, \;\; a_1, \;\; \gamma, \;\; \alpha\beta > 0, \tag{2.8}$$

that easily follow from (2.7) in view of the physical inequalities (2.2). Moreover, in the case when the scale of the microstructure is zero, i.e. $\delta = 0$, from (2.6) we get $v = -\gamma\psi/\beta$. Plugging this relation into (2.6) we reach the equation for the macrodeformation:

$$v_{tt} \,=\, \left(a_0 - \frac{\alpha\beta}{\gamma} \right) v_{xx}.$$

From this equation we infer the following necessary hyperbolicity condition for the coefficients:

$$a_0 \gamma - \alpha\beta > 0 \,. \tag{2.9}$$

3 Dispersion Relation. Wave Packets

Harmonic wave solutions of system (2.6) represent synchronous sinus – oscillations in the macro- and microlevel [9]:

$$v(x,t) \,=\, A e^{i(kx - \omega t)} \,, \quad \psi(x,t) \,=\, \frac{A(\omega^2 - a_0 k^2)}{\alpha k^2} e^{i(kx-\omega t)} \,.$$

Here $A > 0$, $k \in \mathbb{R}$ and $\omega \in \mathbb{R}$ are the amplitude of the macrodeformation, the wavenumber and the frequency, respectively. The wavenumber and frequency satisfy the following quartic dispersion equation [9]:

$$\omega^4 + \varkappa_1 \omega^2 k^2 + \varkappa_2 k^4 + \varkappa_3 \omega^2 + \varkappa_4 k^2 \,=\, 0, \tag{3.1}$$

where

$$\varkappa_1 = -(a_0 + a_1), \quad \varkappa_2 = a_0 a_1, \quad \varkappa_3 = -\frac{\gamma}{\delta}, \quad \varkappa_4 = \frac{a_0\gamma - \alpha\beta}{\delta}. \tag{3.2}$$

For given ω equation (3.1) has 4 roots:

$$k = \pm\omega\sqrt{\frac{1}{2a_0a_1}\left[a_0 + a_1 - \frac{a_0\gamma - \alpha\beta}{\delta\omega^2} \pm R_1(\omega)\right]}$$

where

$$R_1(\omega) = \sqrt{\left(a_0 - a_1 - \frac{a_0\gamma - \alpha\beta}{\delta\omega^2}\right)^2 + \frac{4a_1\alpha\beta}{\delta\omega^2}}.$$

We will limit our analysis to acoustic branches of the dispersion relation that are continuous real-valued functions satisfying the condition $w = 0 \Leftrightarrow k = 0$. There exist only two solutions that meet these conditions: $k = \pm k(\omega)$ where

$$k(\omega) = \omega R_2(\omega)\,, \quad R_2(\omega) = \sqrt{\frac{1}{2a_0a_1}\left[a_0 + a_1 - \frac{a_0\gamma - \alpha\beta}{\delta\omega^2} + R_1(\omega)\right]}. \tag{3.3}$$

Physically, $k = k(\omega)$ and $k = -k(\omega)$ correspond to the waves propagating to the right and left, respectively. The phase velocity of a harmonic wave with the number $k(\omega)$ is given by $c_{ph}(\omega) = \omega/k(\omega)$.

Then, a superposition of harmonic waves has the form (we write only the macro-component v)

$$v(x,t) = \int_{-\infty}^{\infty} A_+(\omega)e^{i(k(\omega)x - \omega t)}\,d\omega + \int_{-\infty}^{\infty} A_-(\omega)e^{i(-k(\omega)x - \omega t)}\,d\omega.$$

Here A_+ and A_- are some functions (may be singular distributions, too).

Given a packet of harmonic waves propagating to the right with a central frequency ω, the group velocity of this packet is equal to $c_g(\omega) = \frac{1}{k'(\omega)}$. Examples of such wave packets are harmonic oscillations with amplitudes modulated by Gaussian curves (see e.g. [4, 8]).

For our analysis we have to establish the type of dispersion that depends on the sign of $c_{ph}(\omega) - c_g(\omega)$. Differentiating the formula (3.3) we immediately obtain the relation

$$k'(\omega) = \frac{k(\omega)}{\omega} + \frac{Q_1(\omega) + Q_2(\omega)}{a_0a_1R_1(\omega)R_2(\omega)} \quad \Rightarrow \quad \frac{c_{ph}(\omega) - c_g(\omega)}{c_{ph}(\omega)c_g(\omega)} = \frac{Q_1(\omega) + Q_2(\omega)}{a_0a_1R_1(\omega)R_2(\omega)},$$

where

$$Q_1(\omega) = \frac{a_0\gamma - \alpha\beta}{\delta\omega^2}R_1(\omega),$$

$$Q_2(\omega) = \left(a_0 - a_1 - \frac{a_0\gamma - \alpha\beta}{\delta\omega^2}\right)\frac{a_0\gamma - \alpha\beta}{\delta\omega^2} - \frac{2a_1\alpha\beta}{\delta\omega^2}.$$

In further computations we follow the basic physical inequalities (2.8), (2.9) and the relations $R_1(\omega), R_2(\omega), c_{ph}(\omega), c_g(\omega) > 0$. Firstly, we get

$$\operatorname{sign}(c_{ph}(\omega) - c_g(\omega)) = \operatorname{sign}(Q_1(\omega) + Q_2(\omega)).$$

Further, using elementary calculations, one can check that

$$\operatorname{sign} Q_2(\omega) = \operatorname{sign}\left[\frac{a_0(a_0\gamma - a_1\gamma - \alpha\beta) - a_1\alpha\beta}{(a_0\gamma - \alpha\beta)^2} - \frac{1}{\delta\omega^2}\right].$$

Next let us compare the quantities $Q_1(\omega)$ and $Q_2(\omega)$. Squaring these quantities, subtracting, simplifying and observing that $Q_1(\omega) > 0$, we deduce the relation

$$\operatorname{sign}(Q_1(\omega) - |Q_2(\omega)|) = \operatorname{sign}(a_0\gamma - a_1\gamma - \alpha\beta).$$

Finally, we come to the following conclusions.

1. In case $a_0\gamma - a_1\gamma - \alpha\beta > 0$ the quantity $Q_1(\omega) + Q_2(\omega)$ is positive, hence $c_{ph}(\omega) > c_g(\omega)$ for any $\omega \in \mathbb{R}$. The model possesses *normal dispersion*.
2. In case $a_0\gamma - a_1\gamma - \alpha\beta < 0$ the quantities $Q_1(\omega) - |Q_2(\omega)|$ and $Q_2(\omega)$ are negative for any $\omega \in \mathbb{R}$. This implies that $Q_1(\omega) + Q_2(\omega)$ is negative, hence $c_{ph}(\omega) < c_g(\omega)$ for any $\omega \in \mathbb{R}$. The model possesses *anomalous dispersion*.
3. In case $a_0\gamma - a_1\gamma - \alpha\beta = 0$ there holds $Q_1(\omega) - |Q_2(\omega)| = 0$ and $Q_2(\omega) < 0$ for any $\omega \in \mathbb{R}$. This implies that $Q_1(\omega) + Q_2(\omega) = 0$, hence $c_{ph}(\omega) = c_g(\omega)$ for any $\omega \in \mathbb{R}$. The model has *no dispersion*.

4 Inverse Problem for Phase and Group Velocities

Let us assume that we have the group and phase velocities $c_{g,j}$ and $c_{ph,j}$ of two wave packets with central frequencies ω_j, $j = 1, 2$, such that $\omega_1^2 \neq \omega_2^2$. We pose the *inverse problem* to determine the coefficients a_0, a_1, γ and the product $\alpha\beta$.

In the sequel we will see that the properties of the inverse problem depend on the type of the dispersion. The latter one can be immediately inferred from the data, namely the sign of $c_{ph,j} - c_{g,j}$ which is the same for $j = 1$ and $j = 2$. Indeed, by virtue of the previous section we have the relation

$$\operatorname{sign}(c_{ph,j} - c_{g,j}) = \operatorname{sign}(a_0\gamma - a_1\gamma - \alpha\beta),$$

which implies that the case of different signs of $c_{ph,1} - c_{g,1}$ and $c_{ph,2} - c_{g,2}$ is contradictive.

In the sequel we focus ourselves to the solution algorithm and the uniqueness issues. From (3.1) we easily deduce the relation for the derivative $k'(\omega)$:

$$\varkappa_1\big(\omega k(\omega)^2 + \omega^2 k(\omega)k'(\omega)\big) + 2\varkappa_2 k(\omega)^3 k'(\omega) + \varkappa_3\omega + \varkappa_4 k(\omega)k'(\omega) = -2\omega^3. \quad (4.1)$$

Thus, denoting $k_j := k(\omega_j) = \omega_j / c_{ph,j}$ and $k'_j := k'(\omega_j) = 1/c_{g,j}$, from (3.1) and (4.1) we get the following equations for $\varkappa_1, \ldots, \varkappa_4$:

$$\begin{aligned} &k_j^2\omega_j^2\varkappa_1 + k_j^4\varkappa_2 + \omega_j^2\varkappa_3 + k_j^2\varkappa_4 = -\omega_j^4, \quad j = 1, 2, \\ &\big(\omega_j k_j^2 + \omega_j^2 k_j k'_j\big)\varkappa_1 + 2k_j^3 k'_j\varkappa_2 + \omega_j\varkappa_3 + k_j k'_j\varkappa_4 = -2\omega_j^3, \quad j = 1, 2. \end{aligned} \quad (4.2)$$

We propose the following solution procedure that consists of two steps:

1. Solve the 4×4 linear system (4.2) for the quantities $\varkappa_1, \ldots, \varkappa_4$;

2. Plug $\varkappa_1, \ldots, \varkappa_4$ into equations (3.2) and solve them for a_0, a_1, γ, $\alpha\beta$.

The solution of (4.2) is unique in the case of presence of the dispersion, i.e. when the parameters satisfy $a_0\gamma - a_1\gamma - \alpha\beta \neq 0$. This follows from the next theorem, whose proof is shifted to Section 6.

Theorem 1. *Assume that $a_0, a_1, \gamma, \alpha, \beta$ satisfy the inequality $a_0\gamma - a_1\gamma - \alpha\beta \neq 0$ and $\varkappa_1, \ldots, \varkappa_4$ are given by (3.2) in terms $a_0, a_1, \gamma, \alpha, \beta$. Furthermore, let $\omega_j \in \mathbb{R}$, $\omega_j \neq 0$, $j = 1, 2$ be such that $\omega_1^2 \neq \omega_2^2$. Let us define $k_j = k(\omega_j)$, $k_j' = k'(\omega_j)$, $j = 1, 2$, where $k(\omega)$ is given by (3.3) in terms of $a_0, a_1, \gamma, \alpha, \beta$. By these definitions, $\varkappa_1, \ldots, \varkappa_4$ solve (4.2) with the data ω_j, k_j, k_j', $j = 1, 2$. In addition, if $\widetilde{\varkappa}_1, \ldots, \widetilde{\varkappa}_4$ solve (4.2) with the same data, then $\widetilde{\varkappa}_j = \varkappa_j$, $j = 1, \ldots, 4$.*

It is easy to check that the second subproblem, i.e. the system (3.2), has two solutions:

$$a_0 = a_0^+, \;\; a_1 = a_1^-, \;\; \gamma = -\delta\varkappa_3, \;\; \alpha\beta = (\alpha\beta)_1 := a_0^+\gamma - \delta\varkappa_4, \tag{4.3}$$

$$a_0 = a_0^-, \;\; a_1 = a_1^+, \;\; \gamma = -\delta\varkappa_3, \;\; \alpha\beta = (\alpha\beta)_2 := a_0^-\gamma - \delta\varkappa_4, \tag{4.4}$$

where

$$a_0^\pm = a_1^\pm = \frac{-\varkappa_1 \pm \sqrt{\varkappa_1^2 - 4\varkappa_2}}{2}. \tag{4.5}$$

From the physical point of view, the solution must satisfy the inequalities (2.8). The first three inequalities $a_0 > 0$, $a_1 > 0$ and $\gamma > 0$ in (2.8) can be rewritten in the form of necessary conditions for the data $\varkappa_1, \varkappa_2, \varkappa_3$ of the second subproblem:

$$\varkappa_1 + \sqrt{\varkappa_1^2 - 4\varkappa_2} < 0\,, \quad \varkappa_3 < 0.$$

The fourth inequality $\alpha\beta > 0$ in (2.8) enables to extract the physical solutions from the set of mathematical solutions. Note that in view of the definitions of $(\alpha\beta)_1$ and $(\alpha\beta)_2$, the relation

$$(\alpha\beta)_2 = a_0^-\gamma - a_0^+\gamma + (\alpha\beta)_1 \tag{4.6}$$

is valid. Further, since $a_0^- \leq a_0^+$ (cf. (4.5)) and $\gamma > 0$, we have from (4.6) that $(\alpha\beta)_1 \geq (\alpha\beta)_2$. Thus, two different cases may occur:

1. $(\alpha\beta)_1 > 0, \quad (\alpha\beta)_2 \leq 0$;

2. $(\alpha\beta)_1 > 0, \quad (\alpha\beta)_2 > 0$.

We exclude the third case $(\alpha\beta)_1 \le 0$, $(\alpha\beta)_2 \le 0$ because then the problem has no physical solution at all.

In the case $(\alpha\beta)_1 > 0$, $(\alpha\beta)_2 \le 0$ only the solution (4.3) is physical. Then in view of (4.6) and the relation $a_0^- = a_1^-$ we have

$$0 \ge (\alpha\beta)_2 = a_0^-\gamma - a_0^+\gamma + (\alpha\beta)_1 = a_1^-\gamma - a_0^+\gamma + (\alpha\beta)_1.$$

This implies the inequality $a_0^+\gamma - a_1^-\gamma - (\alpha\beta)_1 \ge 0$ for the solution (4.3). Consequently, either the material has the normal dispersion or dispersion is absent. In terms of $\varkappa$-s the relations $(\alpha\beta)_1 > 0$, $(\alpha\beta)_2 \le 0$ can be written as

$$\varkappa_3\big(\varkappa_1 + \sqrt{\varkappa_1^2 - 4\varkappa_2}\big) \le 2\varkappa_4 \; < \; \varkappa_3\big(\varkappa_1 - \sqrt{\varkappa_1^2 - 4\varkappa_2}\big).$$

In the case $(\alpha\beta)_1 > 0$, $(\alpha\beta)_2 > 0$ both solutions (4.3) and (4.4) are physical. Again, in view of (4.6) and the relations $a_0^\pm = a_1^\pm$ we get

$$\begin{aligned} 0 &< (\alpha\beta)_2 = a_0^-\gamma - a_0^+\gamma + (\alpha\beta)_1 = a_1^-\gamma - a_0^+\gamma + (\alpha\beta)_1, \\ 0 &< (\alpha\beta)_1 = a_0^+\gamma - a_0^-\gamma + (\alpha\beta)_2 = a_1^+\gamma - a_0^-\gamma + (\alpha\beta)_2. \end{aligned}$$

This yields $a_0^+\gamma - a_1^-\gamma - (\alpha\beta)_1 < 0$ and $a_0^-\gamma - a_1^+\gamma - (\alpha\beta)_2 < 0$ for the solutions (4.3) and (4.4), respectively. Consequently, the material has the anomalous dispersion. In terms of $\varkappa$-s the relations $(\alpha\beta)_1 > 0$, $(\alpha\beta)_2 > 0$ can be written as

$$\varkappa_3\big(\varkappa_1 + \sqrt{\varkappa_1^2 - 4\varkappa_2}\big) > \; 2\varkappa_4.$$

Taking into account Theorem 1 and the discussion concerning the system (3.2) we come to the following conclusions.

1. In the case of normal dispersion the physical solution of the inverse problem is unique. It has the form (4.3), where a_0^+, a_1^- are given by (4.5) and $\varkappa_1, \ldots, \varkappa_4$ solve (4.2).
2. In the case of anomalous dispersion the inverse problem has two physical solutions. They have the forms (4.3) and (4.4), where $a_0^\pm$, $a_1^\pm$ are given by (4.5) and $\varkappa_1, \ldots, \varkappa_4$ solve (4.2).

In non-dispersive media, i.e. when $a_0\gamma - a_1\gamma - \alpha\beta = 0$, Theorem 1 doesn't apply. It is easy to see that in this case the inverse problem has infinitely many solutions. Indeed, then the function $k(\omega)$ has the degenerate form $k(\omega) = \omega/\sqrt{a_1}$ and this means that the data of the inverse problem contain information about the coefficient a_1 only, i.e.:

$$a_1 = \frac{\omega_j^2}{k_j^2} = c_{ph,j}^2 = c_{g,j}^2, \quad j = 1, 2.$$

The remaining coefficients a_0, γ and $\alpha\beta$ may be arbitrary numbers satisfying the equation $a_0\gamma - a_1\gamma - \alpha\beta = 0$.

Although the non-dispersive case is rather theoretical, one should take into account that when the weak dispersion occurs, i.e. $a_0\gamma - a_1\gamma - \alpha\beta \approx 0$, then the matrix of (4.2) is ill-conditioned, i.e. close to singular. This may cause large computational errors in the solution.

5 Numerical Tests

We have tested the sensitivity of the solution with respect to errors of the phase and group velocities. The computations were performed using Mathematica 5.1. In all examples we chose $\delta = 10^{-4}$ and the central frequencies $\omega_1 = 1$, $\omega_2 = 2$. The synthetic data for chosen parameters a_0, a_1, γ, α and β were computed by the formulas $c_{ph,j} = \omega_j/k(\omega_j)$, $c_{q,j} = 1/k'(\omega_j)$, $j = 1, 2$, where $k(\omega)$ is given by (3.3), and is disturbed in the following manner:

$$c^{\epsilon}_{ph,j} = c_{ph,j} + R^j_{ph}\epsilon\,, \quad c^{\epsilon}_{g,j} = c_{g,j} + R^j_g \epsilon,$$

where ϵ is a given noise level and R^j_{ph}, R^j_g are uniformly distributed random numbers in the interval $[-1, 1]$. The vector $(a^{\epsilon}_0, a^{\epsilon}_1, \gamma^{\epsilon}, (\alpha\beta)^{\epsilon})$ stands for the solution corresponding to the synthetic noisy data $c^{\epsilon}_{ph,j}$, $c^{\epsilon}_{g,j}$, $j = 1, 2$.

For every noise level we made 50 computations with different random factors and picked up the largest relative errors of solution components.

In the first example we took $a_0 = 100$, $a_1 = 1$, $\gamma = 10^{-4}$, $\alpha = 0.1$ and $\beta = 10^{-4}$ (the parameters γ and β contain the small quantity l^2 (formulas (2.3),(2.7)), hence it is realistic to choose them small). Then the quantity related to the dispersion equals $a_0\gamma - a_1\gamma - \alpha\beta \approx 10^{-2}$. The values of $\varkappa$-s and velocities are given as

$$\varkappa_1 = -101\,, \quad \varkappa_2 = 100, \quad \varkappa_3 = -1, \quad \varkappa_4 = 99.9,$$
$$c_{ph,1} \approx 9.53463\,, \quad c_{ph,2} \approx 1.15451\,, \quad c_{g,1} \approx 1.03936\,, \quad c_{g,2} \approx 0.86617.$$

The results of experiments are presented in Table 1.

Table 1. Relative errors in case $a_0 = 100$, $a_1 = 1$, $\gamma = 10^{-4}$, $\alpha\beta = 10^{-5}$ (strong dispersion).

ϵ	$\left\lvert\frac{a^{\epsilon}_0 - a_0}{a_0}\right\rvert$	$\left\lvert\frac{a^{\epsilon}_1 - a_1}{a_1}\right\rvert$	$\left\lvert\frac{\gamma^{\epsilon} - \gamma}{\gamma}\right\rvert$	$\left\lvert\frac{(\alpha\beta)^{\epsilon} - \alpha\beta}{\alpha\beta}\right\rvert$
10^{-4}	0.084%	0.008%	0.058%	0.46%
10^{-3}	0.81%	0.034%	0.19%	5.3%
10^{-2}	7.3%	0.86%	1.2%	73%

For comparison, the following relative errors were obtained in the method of spectral decomposition for the same data in case $\epsilon = 10^{-2}$ (see, [9]):

$$\left|\frac{a^{\epsilon}_0 - a_0}{a_0}\right| = 3.6\%, \quad \left|\frac{a^{\epsilon}_1 - a_1}{a_1}\right| = 2.2\%, \quad \left|\frac{\gamma^{\epsilon} - \gamma}{\gamma}\right| = 1.7\%, \quad \left|\frac{(\alpha\beta)^{\epsilon} - \alpha\beta}{\alpha\beta}\right| = 35\%.$$

In the second example we took $a_0 = 2.1$, $a_1 = 1$, $\gamma = 10^{-4}$, $\alpha = 1$ and $\beta = 10^{-4}$. Then the dispersion is weaker:

$$a_0\gamma - a_1\gamma - \alpha\beta = 10^{-5}.$$

The values of $\varkappa$-s and velocities are given by

$$\varkappa_1 = -3.1, \quad \varkappa_2 = 2.1, \quad \varkappa_3 = -1\,, \quad \varkappa_4 = 1.1\,,$$
$$c_{ph,1} \approx 1.02470, \quad c_{ph,2} \approx 1.00949, \quad c_{g,1} \approx 0.99970, \quad c_{g,2} \approx 0.99398.$$

Table 2. Relative errors in case $a_0 = 2.1$, $a_1 = 1$, $\gamma = 10^{-4}$, $\alpha\beta = 10^{-4}$ (weak dispersion).

ϵ	$\left\lvert\frac{a_0^\epsilon - a_0}{a_0}\right\rvert$	$\left\lvert\frac{a_1^\epsilon - a_1}{a_1}\right\rvert$	$\left\lvert\frac{\gamma^\epsilon - \gamma}{\gamma}\right\rvert$	$\left\lvert\frac{(\alpha\beta)^\epsilon - \alpha\beta}{\alpha\beta}\right\rvert$
10^{-4}	15%	0.041%	3.6%	51%
10^{-3}	577%	0.32%	27%	820%
10^{-2}	–	2%	106%	–

Due to the much weaker dispersion, the system (4.2) is ill-conditioned. This is clearly seen from the results presented in Table 2.

Summing up, in the case of strong dispersion the sensitivity of parameters with respect to the noise is moderate. But in the case of weak dispersion all parameters except a_1 are very sensitive. This is in accordance with the theoretical statement that in non-dispersive medium only a_1 is identifiable. In addition, we see that $\alpha\beta$ is much more sensitive than other parameters even in the case of strong dispersion. This fact is also observed in the method of spectral decomposition [9]. Probably the reason is that linear waves in macro-scale contain scarce information about $\alpha\beta$.

6 Proof of Theorem 1

For any $\omega \in \mathbb{C}$ we define the following set of maximally four elements:

$$K(\omega) = \{k \in \mathbb{C} : k \text{ solves (3.1) for given } \omega\}.$$

Evidently, $K(\omega)$ depends on the coefficients $\varkappa_1, \ldots, \varkappa_4$ of equation (3.1), and in turn on $a_0, a_1, \gamma, \alpha, \beta$. We will need the following lemma proved in [9].

Lemma 1. *Assume that $a_0, a_1, \gamma, \alpha, \beta$ satisfy $a_0\gamma - a_1\gamma - \alpha\beta \neq 0$ and $\varkappa_1, \ldots, \varkappa_4$ are given by* (3.1) *in terms of $a_0, a_1, \gamma, \alpha, \beta$. Moreover, let $\omega_1, \omega_2 \in \mathbb{C}$, $\omega_1, \omega_2 \neq 0$, and $k_j \in K(\omega_j)$, $j = 1, 2$. If $\omega_1^2 \neq \omega_2^2$ then the quantities $s_j = \frac{k_j}{\omega_j}$ satisfy $s_1^2 \neq s_2^2$.*

Further, consider the following equation for a quantity k' obtained form (3.1) by differentiation with respect to ω:

$$2\omega^3 + \varkappa_1\left(\omega k^2 + \omega^2 k k'\right) + 2\varkappa_2 k^3 k' + \varkappa_3\omega + \varkappa_4 k k' = 0. \tag{6.1}$$

This can be easily solved for k':

$$k' = -\frac{\omega(2\omega^2 + \varkappa_1 k^2 + \varkappa_3)}{k(\varkappa_1\omega^2 + 2\varkappa_3 k^2 + \varkappa_4)}. \tag{6.2}$$

Note that in case $k \in K(\omega)$, the quantity k' depends on the coefficients $\varkappa_1, \ldots, \varkappa_4$ of the equation (6.1).

Instead of Theorem 1 we prove the following more general theorem.

Theorem 2. *Let the assumptions of Theorem* 1 *be valid for $a_0, a_1, \gamma, \alpha, \beta$, $\varkappa_1, \ldots, \varkappa_4$ and $\omega_j \in \mathbb{C}$, $\omega_j \neq 0$, $j = 1, 2$ be such that $\omega_1^2 \neq \omega_2^2$. Let us choose some $k_j \in K(\omega_j)$, $j = 1, 2$ and denote by k_j' the solution of* (6.2) *corresponding* *to* $\omega = \omega_j$, $k = k_j$. *By this construction,* $\varkappa_1, \ldots, \varkappa_4$ *solve* (4.2) *with the data* ω_j, k_j, k_j', $j = 1, 2$. *If* $\widetilde{\varkappa}_1, \ldots, \widetilde{\varkappa}_4$ *solve* (4.2) *with the same data, then* $\widetilde{\varkappa}_j = \varkappa_j$, $j = 1, \ldots, 4$.

Proof. Dividing the first equations in the system (4.2) by ω_j^4, second equations by ω_j^3 and denoting $s_j = k_j/\omega_j$ we obtain the following system

$$\begin{cases} 1 + \varkappa_1 s_j^2 + \varkappa_2 s_j^4 + \frac{1}{\omega_j^2}\left(\varkappa_3 + \varkappa_4 s_j^2\right) = 0\,, \\ 1 + \widetilde{\varkappa}_1 s_j^2 + \widetilde{\varkappa}_2 s_j^4 + \frac{1}{\omega_j^2}\left(\widetilde{\varkappa}_3 + \widetilde{\varkappa}_4 s_j^2\right) = 0\,, \\ 2 + \varkappa_1 s_j^2 + \frac{\varkappa_3}{\omega_j^2} + \left(\varkappa_1 + 2\varkappa_2 s_j^2 + \frac{\varkappa_4}{\omega_j^2}\right) s_j k_j' = 0\,, \\ 2 + \widetilde{\varkappa}_1 s_j^2 + \frac{\widetilde{\varkappa}_3}{\omega_j^2} + \left(\widetilde{\varkappa}_1 + 2\widetilde{\varkappa}_2 s_j^2 + \frac{\widetilde{\varkappa}_4}{\omega_j^2}\right) s_j k_j' = 0\,, \end{cases} \tag{6.3}$$

where $j = 1, 2$. We plan to eliminate the quantities ω_j and k_j' from (6.3). One possibility of elimination is the multiplication of the first equation by $-\widetilde{\varkappa}_3 - \widetilde{\varkappa}_4 s_j^2$, the second equation by $\varkappa_3 + \varkappa_4 s_j^2$ and addition. This results in the relations

$$(\varkappa_4\widetilde{\varkappa}_2 - \widetilde{\varkappa}_4\varkappa_2)s_j^6 + (\varkappa_3\widetilde{\varkappa}_2 - \widetilde{\varkappa}_3\varkappa_2 + \varkappa_4\widetilde{\varkappa}_1 - \widetilde{\varkappa}_4\varkappa_1)s_j^4 + (\varkappa_4 - \widetilde{\varkappa}_4 \\ + \varkappa_3\widetilde{\varkappa}_1 - \widetilde{\varkappa}_3\varkappa_1)s_j^2 + \varkappa_3 - \widetilde{\varkappa}_3 = 0, \quad j = 1, 2.$$

These relations show that $\sigma = s_j^2$, $j = 1, 2$, solve the cubic equation:

$$f(\sigma) := (\varkappa_4\widetilde{\varkappa}_2 - \widetilde{\varkappa}_4\varkappa_2)\sigma^3 + (\varkappa_3\widetilde{\varkappa}_2 - \widetilde{\varkappa}_3\varkappa_2 + \varkappa_4\widetilde{\varkappa}_1 - \widetilde{\varkappa}_4\varkappa_1)\sigma^2 \\ + (\varkappa_4 - \widetilde{\varkappa}_4 + \varkappa_3\widetilde{\varkappa}_1 - \widetilde{\varkappa}_3\varkappa_1)\sigma + \varkappa_3 - \widetilde{\varkappa}_3 = 0.$$

Another possibility of elimination contains the following steps. We multiply in (6.3) the first equation by $\big(2\omega_j^2\big(\widetilde{\varkappa}_1 + 2\widetilde{\varkappa}_2 s_j^2 + \frac{\widetilde{\varkappa}_4}{\omega_j^2}\big) - \widetilde{\varkappa}_4\big)$, the second equation by $\big(-2\omega_j^2\big(\varkappa_1 + 2\varkappa_2 s_j^2 + \frac{\varkappa_4}{\omega_j^2}\big) + \varkappa_4\big)$, the third equation by $\big(-\omega_j^2\big(\widetilde{\varkappa}_1 + 2\widetilde{\varkappa}_2 s_j^2 + \frac{\widetilde{\varkappa}_4}{\omega_j^2}\big)\big)$, the fourth equation by $\omega_j^2\big(\varkappa_1 + 2\varkappa_2 s_j^2 + \frac{\varkappa_4}{\omega_j^2}\big)$ and add the obtained equations. After simplification we get the relation

$$3(\varkappa_4\widetilde{\varkappa}_2 - \widetilde{\varkappa}_4\varkappa_2)s_j^4 + 2(\varkappa_3\widetilde{\varkappa}_2 - \widetilde{\varkappa}_3\varkappa_2 + \varkappa_4\widetilde{\varkappa}_1 - \widetilde{\varkappa}_4\varkappa_1)s_j^2 \\ + \varkappa_4 - \widetilde{\varkappa}_4 + \varkappa_3\widetilde{\varkappa}_1 - \widetilde{\varkappa}_3\varkappa_1 = 0, \quad j = 1, 2.$$

From these relations we see that $\sigma = s_j^2$, $j = 1, 2$, solve the equation $f'(\sigma) = 0$. Consequently, s_j^2, $j = 1, 2$ are double roots of the cubic function $f(\sigma)$. Since ω_j^2, $j = 1, 2$ are different, Lemma 1 implies that s_j^2, $j = 1, 2$ are also different. This means that the cubic function $f(\sigma)$ has two different double roots, hence it is trivial. Setting the coefficients of $f(\sigma)$ equal to zero, after some transformations we get the following 4×4 linear system for the determination of vector $(\widetilde{\varkappa}_1 - \varkappa_1, \widetilde{\varkappa}_2 - \varkappa_2, \widetilde{\varkappa}_3 - \varkappa_3, \widetilde{\varkappa}_4 - \varkappa_4)$:

$$\begin{cases} \widetilde{\varkappa}_3 - \varkappa_3 = 0, \\ \varkappa_3(\widetilde{\varkappa}_1 - \varkappa_1) - \varkappa_1(\widetilde{\varkappa}_3 - \varkappa_3) - (\widetilde{\varkappa}_4 - \varkappa_4) = 0, \\ \varkappa_4(\widetilde{\varkappa}_1 - \varkappa_1) + \varkappa_3(\widetilde{\varkappa}_2 - \varkappa_2) - \varkappa_2(\widetilde{\varkappa}_3 - \varkappa_3) - \varkappa_1(\widetilde{\varkappa}_4 - \varkappa_4) = 0, \\ \varkappa_4(\widetilde{\varkappa}_2 - \varkappa_2) - \varkappa_2(\widetilde{\varkappa}_4 - \varkappa_4) = 0. \end{cases}$$

The determinant of this system is

$$\Delta = -\varkappa_2\varkappa_3^2 - \varkappa_4^2 + \varkappa_1\varkappa_3\varkappa_4 = \frac{(a_0\gamma - a_1\gamma - \alpha\beta)\alpha\beta}{\delta^2} \neq 0\,,$$

because $\alpha\beta > 0$ and $a_0\gamma - a_1\gamma - \alpha\beta \neq 0$. This implies that the system has only the trivial solution. Consequently, $\widetilde{\varkappa}_1 = \varkappa_1$, $\widetilde{\varkappa}_2 = \varkappa_2$, $\widetilde{\varkappa}_3 = \varkappa_3$, $\widetilde{\varkappa}_4 = \varkappa_4$. This proves Theorem 2. $\square$

Theorem 1 follows from Theorem 2 because $k(\omega_j) \in K(\omega_j)$, $j = 1, 2$, and $k'_j = k'(\omega_j)$ solves (6.2).

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