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Qualitative Properties for a Sixth–Order Thin Film Equation*

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Abstract. In this article, the author studies the qualitative properties of weak solutions for a sixth-order thin film equation, which arises in the industrial application of the isolation oxidation of silicon. Based on the Schauder type estimates, we establish the global existence of classical solutions for regularized problems. After establishing some necessary uniform estimates on the approximate solutions, we prove the existence of weak solutions. The nonnegativity and the expansion of the support of solutions are also discussed.

Keywords: Sixth-order thin film equation, degenerate, existence, nonnegativity.

AMS Subject Classification: 35D05; 35K55; 35K65; 76A20.

1 Introduction

In this article, we investigate the sixth-order thin film equation

$$\frac{\partial u}{\partial t} = \partial_x\left[m(u)\left(\partial_x^5 u + \partial_x(|u|^{p-1}u)\right)\right], \quad \text{in } Q_T,\ p > 2, \tag{1.1}$$

where $Q_T = I \times (0,T)$, $I = (0,1)$ and $m(u) = |u|^n, n > 0$. On the basis of physical consideration, as usual the equation (1.1) is supplemented with the zero-contact-angle, zero-shearing force and zero-flux conditions

$$\partial_x u\big|_{x=0,1} = \partial_x^3 u\big|_{x=0,1} = \partial_x^5 u\big|_{x=0,1} = 0, \quad t > 0, \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x). \tag{1.3}$$

The equation (1.1) is a typical higher order equation, which has a sharp physical background and a rich theoretical connotation. It arises in the industrial application of the isolation oxidation of silicon [8, 10].

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During the past years, only a few works have been devoted to the sixth-order thin film equation [5, 11, 16, 17]. Bernis and Friedman [5] have studied the initial boundary value problems to the thin film equation

$$\frac{\partial u}{\partial t} + (-1)^{m-1}\partial_x\big(f(u)\partial_x^{2m+1}u\big) = 0,$$

where $f(u) = |u|^n f_0(u)$, $f_0(u) > 0$, $n \geq 1$ and proved the existence of weak solutions preserving nonnegativity. Barrett, Langdon and Nuernberg [1] considered the above equation with $m = 2$. A finite element method is presented which proves to be well posed and convergent. Numerical experiments illustrate the theory.

Recently, Evans, Galaktionov and King [8, 9] considered the sixth-order thin film equation containing an unstable (backward parabolic) second-order term

$$\frac{\partial u}{\partial t} = \mathrm{div}\big[|u|^n\nabla\Delta^2 u\big] - \Delta(|u|^{p-1}u), \quad n > 0, p > 1. \tag{1.4}$$

By a formal matched expansion technique, they show that, for the first critical exponent $p = p_0 = n + 1 + \frac{4}{N}$ for $n \in (0, \frac{5}{4})$, where N is the space dimension, the free-boundary problem with zero-height, zero-contact-angle, zero-moment, and zero-flux conditions at the interface admits a countable set of continuous branches of radially symmetric self-similar blow-up solutions

$$u_k(x,t) = (T-t)^{-(N/(nN+6))}f_k(y), \quad y = \frac{x}{(T-t)^{1/(nN+6)}},$$

where $T > 0$ is the blow-up time. Some other results can be found in [15].

Remark 1. In [8, 9], the authors using a combination of formal asymptotic and numerical methods, from the point of view of numerical analysis show that the solutions of problem (1.4) blow up at a finite time when the second-order term is $-\Delta(|u|^{p-1}u)$. Our result from the point of view of theoretical analysis shows that the problem (1.1) has global solutions for the second-order term with the opposite sign.

We also refer the following relevant equation

$$\frac{\partial u}{\partial t} = -\partial_x\left(u^n\partial_x^3 u\right), \tag{1.5}$$

which has been extensively studied. Bernis and Friedman [5] have studied the initial boundary value problems to the thin film equation $n > 0$ and proved the existence of weak solutions preserving nonnegativity (see also [2, 13, 18, 20, 22]). They proved that if $n \geq 2$ the support of the solutions $u(\cdot, t)$ is nondecreasing with respect to t. Some references to unstable fourth order equations can be found in [21].

Remark 2. In [19], the Lyapunov functional might not exist for the convective Cahn-Hilliard equation. The author based on uniform Schauder type estimates via the framework of Campanato spaces proved the global existence of classical solutions for regularized problems. In this paper, in order to prove the existence, we construct a new Lyapunov functional. On the other hand, the Bernis estimates can not be applied, so we introduce a suitable integral inequalities which are then used to prove the expansion of the support.

In this paper, we study the problem (1.1)–(1.3). Because of the degeneracy, the problem does not admit classical solutions in general. So, we introduce the weak solutions in the following sense.

Definition 1. A function u is said to be a weak solution of (1.1)–(1.3), if the following conditions are satisfied:

1. $u, \partial_x u \in C^\alpha(\overline{Q}_T)$, $u \in L^\infty(0,T;H^2(0,1))$, $|u|^{n/2}\partial_x^5 u \in L^2(P)$.
2. For $\varphi \in C^1(\overline{Q}_T)$ and $Q_T = \Omega \times (0,T)$,

$$
\begin{aligned}
&-\int_0^1 u(x,T)\varphi(x,T)\,dx + \int_0^1 u_0(x)\varphi(x,0)\,dx + \iint_{Q_T} u\frac{\partial\varphi}{\partial t}\,dx\,dt \\
&= \iint_P |u|^n(\partial_x^5 u + \partial_x(|u|^{p-1}u))\partial_x\varphi\,dx\,dt,
\end{aligned}
$$

where $P = \overline{Q}_T \backslash (\{u(x,t)=0\} \cup \{t=0\})$.

We investigate the existence of weak solutions. The main difficulties for treating the problem (1.1)–(1.3) are caused by the nonlinearity of the principal part and the lack of maximum principle. Because of the degeneracy, we first consider the regularized problem. To prove the existence of classical solutions for the regularized problem, the basic a priori estimates are the L^2 norm estimates on u and $\partial_x u$. Our method is based on uniform Schauder type estimates for local in time solutions. Based on the uniform estimates for the approximate solutions, we obtain the existence. Owing to the background, we are much interested in the nonnegativity of the weak solutions and the solutions with the expansion of the support. As it is well known, one of the important properties of solutions of the porous medium equation is the expansion of the support. So from the point of view of physical background, it seems to be natural to investigate this property for thin film equation. On the other hand, the mathematical description of this property is that if $\operatorname{supp} u_0$ is bounded, then for any $t > 0$, $\operatorname{supp} u(\cdot,t)$ is also bounded. So from the point of view of mathematics, this problem seems to be quite interesting. The expansion of the support is completely open for pure sixth order thin film equation. Here we face a substantial difficulty, which is caused by the nonlinearity of the second-order term. Comparing the equations (1.1) with (1.5). Bernis and Friedman [5] replaced u^n by $m_\sigma(u)$ in (1.5), where $m_\sigma(s) = |s|^{n+4}/(\sigma|s|^n + |s|^4)$. Then the approximating problem of equation (1.5) has a unique positive solution, hence Bernis' inequality [4] holds. However, for the problem (1.1)–(1.3) the Bernis estimates can not be applied. This means that we should find a new approach to establish the required estimates. This goal would in principle justify introducing a different approximating scheme in order to obtain a-priori, suitable integral inequalities which are then used to prove the expansion of the support.

This paper is arranged as follows. We first study the regularized problem in Section 2, and then establish the existence and the nonnegativity of weak solutions in Section 3. Subsequently, we discuss the expansion of the support in Section 4.

2 Regularized Problems

To discuss the existence, we adopt the method of parabolic regularization, namely, the desired solution will be obtained as the limit of some subsequence of solutions of the following regularized problem

$$\frac{\partial u_\varepsilon}{\partial t} = \partial_x \Big[m_\varepsilon(u_\varepsilon)\Big(\partial_x^5 u_\varepsilon + \partial_x(|u_\varepsilon|^{p-1}u_\varepsilon)\Big)\Big], \quad (x,t) \in Q_T, \tag{2.1}$$

$$\partial_x u_\varepsilon\Big|_{x=0,1} = \partial_x^3 u_\varepsilon\Big|_{x=0,1} = \partial_x^5 u_\varepsilon\Big|_{x=0,1} = 0, \quad t > 0, \tag{2.2}$$

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x), \tag{2.3}$$

where $m_\varepsilon(u_\varepsilon) = (|u_\varepsilon|^2 + \varepsilon)^{\frac{n}{2}}$.

Theorem 1. *For each fixed $\varepsilon > 0$, $p > 2$ and*

$$u_{0\varepsilon} \in C^{6+\alpha}, \quad \partial_x^i u_{0\varepsilon}(0) = \partial_x^i u_{0\varepsilon}(1) = 0 \ (i = 1, 3, 5),$$

then (2.1)–(2.3) *admits a unique classical solution $u_\varepsilon \in C^{6+\alpha, 1+(\alpha/6)}(\overline{Q}_T)$, for some $\alpha \in (0,1)$.*

From the classical approach [6, 12], it is not difficult to conclude that the problem (2.1)–(2.3) admits a unique classical solution local in time. So, it is sufficient to make a priori estimates. As an important step, we give the Hölder norm estimate on the local in time solutions.

Proposition 1. *Assume that u_ε is a smooth solution of the problem* (2.1)–(2.3)*. Then there exists a constant C depending only on the known quantities, such that for any $(x_1,t_1), (x_2,t_2) \in Q_T$ and some $0 < \alpha < 1$,*

$$|u_\varepsilon(x_1,t_1) - u_\varepsilon(x_2,t_2)| \le C(|t_1 - t_2|^{\alpha/6} + |x_1 - x_2|^{\alpha}), \tag{2.4}$$
$$|\partial_x u_\varepsilon(x_1,t_1) - \partial_x u_\varepsilon(x_2,t_2)| \le C(|t_1 - t_2|^{1/12} + |x_1 - x_2|^{1/2}).$$

Proof. Now, we set

$$F_\varepsilon(t) = \int_0^1 \Big[\frac{1}{2}(\partial_x^2 u_\varepsilon)^2 + H(u_\varepsilon)\Big] dx,$$

where $H(s) = \frac{1}{p+1}|s|^{p+1}$. Integrating by parts and using the equation (2.1) itself and boundary value condition (2.2), we see that

$$\begin{aligned}
\frac{dF_\varepsilon(t)}{dt} &= \int_0^1 \Big[\partial_x^2 u_\varepsilon \partial_x^2 u_{\varepsilon t} + |u_\varepsilon|^{p-1}u_\varepsilon \frac{\partial u_\varepsilon}{\partial t}\Big] dx = \int_0^1 \big[\partial_x^4 u_\varepsilon + |u_\varepsilon|^{p-1}u_\varepsilon\big] \frac{\partial u_\varepsilon}{\partial t} dx \\
&= \int_0^1 \big[\partial_x^4 u_\varepsilon + |u_\varepsilon|^{p-1}u_\varepsilon\big] \partial_x \big[m_\varepsilon(u_\varepsilon)(\partial_x^5 u_\varepsilon + \partial_x(|u_\varepsilon|^{p-1}u_\varepsilon))\big] dx \\
&= -\int_0^1 \big[\partial_x^5 u_\varepsilon + \partial_x(|u_\varepsilon|^{p-1}u_\varepsilon)\big] \big[m_\varepsilon(u_\varepsilon)(\partial_x^5 u_\varepsilon + \partial_x(|u_\varepsilon|^{p-1}u_\varepsilon))\big] dx \le 0,
\end{aligned}$$

which implies that

$$\int_0^1 |u_\varepsilon|^{p+1} dx \le C, \quad \int_0^1 (\partial_x^2 u_\varepsilon)^2 dx \le C. \tag{2.5}$$

On the other hand, integrating the equation (2.1) on $Q_t = (0,1) \times (0,t)$, we have

$$\int_0^1 u_\varepsilon(x,t)\, dx = \int_0^1 u_{0\varepsilon}(x)\, dx.$$

Applying Poincaré's inequality and Friedrichs' inequality [7], we conclude

$$\int_0^1 (u_\varepsilon)^2\, dx \le C, \quad \int_0^1 (\partial_x u_\varepsilon)^2 dx \le C. \tag{2.6}$$

By the Sobolev imbedding theorem,

$$\sup_{Q_T} |u_\varepsilon| \le C, \quad \sup_{Q_T} |\partial_x u_\varepsilon| \le C. \tag{2.7}$$

Multiplying both sides of the equation (2.1) by $\partial_x^4 u_\varepsilon$ and then integrating the resulting relation with respect to x over $(0,1)$, we get

$$\int_0^1 \frac{\partial u_\varepsilon}{\partial t} \partial_x^4 u_\varepsilon\, dx = \int_0^1 \partial_x \Big[m_\varepsilon(u_\varepsilon) \Big(\partial_x^5 u_\varepsilon + \partial_x(|u_\varepsilon|^{p-1} u_\varepsilon) \Big) \Big] \partial_x^4 u_\varepsilon\, dx.$$

After integration by parts, and used the boundary value conditions, the above equality becomes

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\partial_x^2 u_\varepsilon)^2 dx + \int_0^1 m_\varepsilon(u_\varepsilon)|\partial_x^5 u_\varepsilon|^2 dx = -\int_0^1 m_\varepsilon(u_\varepsilon)\partial_x(|u_\varepsilon|^{p-1}u_\varepsilon)\partial_x^5 u_\varepsilon\, dx.$$

Hölder's inequality and (2.7) give the following result

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\partial_x^2 u_\varepsilon)^2 dx + \int_0^1 m_\varepsilon(u_\varepsilon)|\partial_x^5 u_\varepsilon|^2 dx \le \frac{1}{2}\int_0^1 m_\varepsilon(u_\varepsilon)(\partial_x^5 u_\varepsilon)^2 dx + C.$$

Hence

$$\iint_{Q_T} m_\varepsilon(u_\varepsilon)(\partial_x^5 u_\varepsilon)^2 dx\, dt \le C. \tag{2.8}$$

By (2.6) and (2.7), we have

$$\big|u_\varepsilon(x_1,t) - u_\varepsilon(x_2,t)\big| \le C\big|x_1 - x_2\big|^\alpha, \quad 0 < \alpha < 1.$$

Integrating the equation (2.1) with respect to (x,t) over $(y, y+(\Delta t)^{1/6}) \times (t_1,t_2)$, where $0 < t_1 < t_2 < T$, $\Delta t = t_2 - t_1$, we see that

$$\int_y^{y+(\Delta t)^{1/6}} \big[u_\varepsilon(z,t_2) - u_\varepsilon(z,t_1)\big] dz = \int_{t_1}^{t_2} \Big[m_\varepsilon\big(u_\varepsilon(y',s)\big)\big(\partial_x^5 u_\varepsilon(y',s)$$
$$+\partial_x(|u_\varepsilon|^{p-1}u_\varepsilon)(y',s)\big) - m_\varepsilon\big(u_\varepsilon(y,s)\big)\big(\partial_x^5 u_\varepsilon(y,s) + \partial_x(|u_\varepsilon|^{p-1}u_\varepsilon)(y,s)\big)\Big] ds. \tag{2.9}$$

For simplicity, set

$$
\begin{aligned}
N(s,y) =& m_\varepsilon\big(u_\varepsilon(y',s)\big)\big(\partial_x^5 u_\varepsilon(y',s)+\partial_x(|u_\varepsilon|^{p-1}u_\varepsilon)(y',s)\big)\\
&- m_\varepsilon\big(u_\varepsilon(y,s)\big)\big(\partial_x^5 u_\varepsilon(y,s)+\partial_x(|u_\varepsilon|^{p-1}u_\varepsilon)(y,s)\big),
\end{aligned}
$$

where $y' = y + (\Delta t)^{1/6}$. Then (2.9) is converted into

$$
(\Delta t)^{1/6}\int_0^1 \Big[u_\varepsilon(y+\theta(\Delta t)^{1/6},t_2) - u_\varepsilon(y+\theta(\Delta t)^{1/6},t_1)\Big]d\theta = \int_{t_1}^{t_2} N(s,y)ds.
$$

Integrating the above equality with respect to y over $(x, x+(\Delta t)^{1/6})$, we get

$$
(\Delta t)^{1/3}\big(u_\varepsilon(x^*,t_2) - u_\varepsilon(x^*,t_1)\big) = \int_{t_1}^{t_2}\int_x^{x+(\Delta t)^{1/6}} N(s,y)\,dy\,ds.
$$

Here, we have used the mean value theorem, where $x^* = y^* + \theta^*(\Delta t)^{1/6}$, $y^* \in (x, x+(\Delta t)^{1/6})$, $\theta^* \in (0,1)$. Therefore, by Hölder's inequality and (2.7), (2.8), we end up with

$$
\big|u_\varepsilon(x^*,t_2) - u_\varepsilon(x^*,t_1)\big| \le C(\Delta t)^{\alpha/6}, \quad 0<\alpha<1.
$$

Similar to the discussion above, we have

$$
\big|\partial_x u_\varepsilon(x_1,t_1) - \partial_x u_\varepsilon(x_2,t_2)\big| \le C\big(\big|x_1-x_2\big|^{1/2} + \big|t_1-t_2\big|^{1/12}\big). \tag{2.10}
$$

The proof is complete. □

Proof. [Proof of Theorem 1] The conclusion follows immediately from the classical theory, since we can transform the equation (2.1) into the form

$$
\frac{\partial u_\varepsilon}{\partial t} + a_1(x,t)\partial_x^6 u_\varepsilon + b_1(x,t)\partial_x^5 u_\varepsilon + a_2(x,t)\partial_x^2 u_\varepsilon + b_2(x,t)\partial_x u_\varepsilon = 0,
$$

where the Hölder norms on

$$
\begin{aligned}
&a_1(x,t) = -m_\varepsilon(u_\varepsilon(x,t)), \quad b_1(x,t) = -m_\varepsilon'(u_\varepsilon(x,t))\partial_x u_\varepsilon(x,t),\\
&a_2(x,t) = -pm_\varepsilon(u_\varepsilon(x,t))|u_\varepsilon(x,t)|^{p-1},\\
&b_2(x,t) = -[pm_\varepsilon'|u_\varepsilon|^{p-1} + p(p-1)m_\varepsilon|u_\varepsilon|^{p-3}u_\varepsilon]\partial_x u_\varepsilon(x,t)
\end{aligned}
$$

have been estimated in the above discussion. The proof is complete. □

3 Existence

After the discussion of the regularized problem, we can now turn to the investigation of the existence of weak solutions of the problem (1.1)–(1.3). The main existence result is the following

Theorem 2. *Assume that $u_0 \in H_0^2(I)$, then the problem* (1.1)–(1.3) *admits at least one weak solution.*

Proof. Let u_ε be the approximate solution of the problem (2.1)–(2.3) constructed in the previous section. Using the estimates (2.4), (2.5) and (2.10), we can extract a subsequence from $\{u_\varepsilon\}$, denoted also by $\{u_\varepsilon\}$, such that

$$\begin{aligned} &u_\varepsilon(x,t) \to u(x,t), && \text{uniformly in } \overline{Q}_T, \\ &\partial_x u_\varepsilon(x,t) \to \partial_x u(x,t), && \text{uniformly in } \overline{Q}_T, \end{aligned}$$

and the limiting function $u, \partial_x u \in C^{1/2,1/12}(\overline{Q}_T)$. By (2.5), we also have $u \in L^\infty(0,T;H^2(I))$.

Now, let $\delta > 0$ be fixed and set $P_\delta = \{(x,t);\ |u|^n(x,t) > \delta\}$. We choose $\varepsilon_0(\delta) > 0$, such that

$$\begin{aligned} &\big(|u_\varepsilon|^2(x,t) + \varepsilon\big)^{\frac{n}{2}} \ge \delta/2, \quad (x,t) \in P_\delta,\ \ 0 < \varepsilon < \varepsilon_0(\delta), \\ &|u_\varepsilon|^n \le 2\delta, \quad (x,t) \in Q_T \backslash P_\delta,\ \ 0 < \varepsilon < \varepsilon_0(\delta). \end{aligned} \tag{3.1}$$

Then from (2.8)

$$\iint_{P_\delta} \big(\partial_x^5 u_\varepsilon\big)^2\, dx\, dt \le \frac{C}{\delta},$$

where the constant C is independent of ε and δ. By employing a diagonal selection, we obtain a subsequence from $\{u_\varepsilon\}$, denoted also by $\{u_\varepsilon\}$, such that

$$\partial_x^5 u_\varepsilon(x,t) \to \partial_x^5 u(x,t), \text{ weakly in } L^2(P_\delta).$$

Noting that

$$\begin{aligned} \iint_{P_\delta} |u|^n (\partial_x^5 u)^2 dx\, dt &\le \iint_{P_\delta} |u|^n \partial_x^5 u(\partial_x^5 u - \partial_x^5 u_\varepsilon) dx\, dt + \iint_{P_\delta} |u|^n \partial_x^5 u \partial_x^5 u_\varepsilon dx\, dt \\ &\le \iint_{P_\delta} |u|^n \partial_x^5 u(\partial_x^5 u - \partial_x^5 u_\varepsilon) dx\, dt + \frac{1}{2}\iint_{P_\delta} |u|^n (\partial_x^5 u)^2 dx\, dt + \frac{1}{2}\iint_{P_\delta} |u|^n (\partial_x^5 u_\varepsilon)^2 dx\, dt, \end{aligned}$$

hence

$$\iint_{P_\delta} |u|^n (\partial_x^5 u)^2 dx\, dt \le 2\Big|\iint_{P_\delta} |u|^n \partial_x^5 u(\partial_x^5 u - \partial_x^5 u_\varepsilon) dx\, dt\Big| + \iint_{P_\delta} |u|^n (\partial_x^5 u_\varepsilon)^2 dx\, dt.$$

This and the fact that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \iint_{P_\delta} |u|^n \partial_x^5 u(\partial_x^5 u - \partial_x^5 u_\varepsilon) dx\, dt = 0, \\ &\lim_{\varepsilon\to 0} \iint_{P_\delta} \Big|(|u_\varepsilon|^2 + \varepsilon)^{\frac{n}{2}} - |u|^n\Big| (\partial_x^5 u_\varepsilon)^2 dx\, dt = 0, \end{aligned}$$

yield

$$\iint_{P_\delta} |u|^n (\partial_x^5 u)^2 dx\, dt \le \underline{\lim}_{\varepsilon\to 0} \iint_{P_\delta} \Big(|u_\varepsilon|^2 + \varepsilon\Big)^{\frac{n}{2}} (\partial_x^5 u_\varepsilon)^2 dx\, dt \le C.$$

To prove the integral equality in the definition of solutions, it suffices to pass the limit as $\varepsilon \to 0$ in

$$
\begin{aligned}
&-\int_0^1 u_\varepsilon(x,T)\varphi(x,T)dx + \int_0^1 u_{0\varepsilon}\varphi(x,0)dx + \iint_{Q_T} u_\varepsilon \frac{\partial \varphi}{\partial t} dx\,dt \\
&= \iint_{Q_T} (|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}} \partial_x^5 u_\varepsilon \partial_x \varphi dx\,dt + \iint_{Q_T} (|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}} \partial_x(|u_\varepsilon|^{p-1}u_\varepsilon)\partial_x\varphi dx\,dt.
\end{aligned}
$$

The limits

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^1 u_\varepsilon(x,T)\varphi(x,T)dx = \int_0^1 u(x,T)\varphi(x,T)dx, \\
&\lim_{\varepsilon\to 0}\int_0^1 u_{0\varepsilon}(x)\varphi(x,0)dx = \int_0^1 u_0(x)\varphi(x,0)dx, \\
&\lim_{\varepsilon\to 0}\iint_{Q_T} u_\varepsilon \frac{\partial\varphi}{\partial t}dx\,dt = \iint_{Q_T} u\frac{\partial\varphi}{\partial t}dx\,dt, \\
&\lim_{\varepsilon\to 0}\iint_{Q_T} (|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}}\partial_x(|u_\varepsilon|^{p-1}u_\varepsilon)\partial_x\varphi dx\,dt = \iint_{Q_T} |u|^n\partial_x(|u|^{p-1}u)\partial_x\varphi dx\,dt
\end{aligned}
$$

are obvious. It remains to show

$$
\lim_{\varepsilon\to 0}\iint_{Q_T} (|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}}\partial_x^5 u_\varepsilon\partial_x\varphi dx\,dt = \iint_P |u|^n\partial_x^5 u\partial_x\varphi dx\,dt. \tag{3.2}
$$

In fact, for any fixed $\delta > 0$,

$$
\begin{aligned}
&\Big|\iint_{Q_T} (|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}}\partial_x^5 u_\varepsilon\partial_x\varphi dx\,dt - \iint_P |u|^n\partial_x^5 u\partial_x\varphi dx\,dt\Big| \\
&\le \Big|\iint_{P_\delta} (|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}}\partial_x^5 u_\varepsilon\partial_x\varphi dx\,dt - \iint_{P_\delta} |u|^n\partial_x^5 u\partial_x\varphi dx\,dt\Big| \\
&\quad + \Big|\iint_{Q_T\setminus P_\delta} (|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}}\partial_x^5 u_\varepsilon\partial_x\varphi dx\,dt\Big| + \Big|\iint_{P\setminus P_\delta} |u|^n\partial_x^5 u\partial_x\varphi dx\,dt\Big|.
\end{aligned}
$$

Using Hölder's inequality and the estimates (2.8), (3.1), we have

$$
\begin{aligned}
&\Big|\iint_{Q_T\setminus P_\delta} (|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}}\partial_x^5 u_\varepsilon\partial_x\varphi dx\,dt\Big| \le \Big(\iint_{Q_T\setminus P_\delta} (|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}}\Big(\partial_x^5 u_\varepsilon\Big)^2 dx\,dt\Big)^{\frac{1}{2}} \times \\
&\Big(\iint_{Q_T\setminus P_\delta} (|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}} dx\,dt\Big)^{\frac{1}{2}} \sup|\partial_x\varphi| \le C(\delta^2+\varepsilon)^{\frac{n}{4}}\sup|\partial_x\varphi|, \quad 0<\varepsilon<\varepsilon_0(\delta).
\end{aligned}
$$

Similarly, we obtain

$$
\Big|\iint_{P\setminus P_\delta} |u|^n\partial_x^5 u\partial_x\varphi dx\,dt\Big| \le C\sqrt{\delta}\sup|\partial_x\varphi|.
$$

On the other hand, we see that

$$\begin{aligned}&\Big|\iint_{P_\delta}(|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}}\partial_x^5 u_\varepsilon \partial_x\varphi dx\,dt-\iint_{P_\delta}|u|^n\partial_x^5 u\partial_x\varphi dx\,dt\Big|\\ &\le\iint_{P_\delta}\Big|(|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}}-|u|^n\Big||\partial_x^5 u_\varepsilon||\partial_x\varphi|dx\,dt+\Big|\iint_{P_\delta}|u|^n\Big(\partial_x^5 u_\varepsilon-\partial_x^5 u\Big)\partial_x\varphi dx\,dt\Big|\\ &\le\sup|(|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}}-|u|^n||\partial_x\varphi|\frac{C}{\sqrt{\delta}}+\Big|\iint_{P_\delta}|u|^n(\partial_x^5 u_\varepsilon-\partial_x^5 u)\partial_x\varphi dx\,dt\Big|.\end{aligned}$$

Hence

$$\overline{\lim_{\varepsilon\to 0}}\Big|\iint_{Q_T}(|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}}\partial_x^5 u_\varepsilon\partial_x\varphi dx\,dt-\iint_P|u|^n\partial_x^5 u\partial_x\varphi dx\,dt\Big|\le C\sqrt{\delta}\sup|\partial_x\varphi|.$$

By the arbitrariness of δ, we see that the limit (3.2) holds. The proof is complete. □

Theorem 3. *The weak solution u satisfies $u(x,t)\ge 0$, if $u_0(x)\ge 0$.*

Proof. Suppose the contrary, that is, the set

$$E=\{(x,t)\in\overline{Q}_T;\ u(x,t)<0\}$$

is nonempty. For any fixed $\delta>0$, choose a C^∞ function $H_\delta(s)$ such that $H_\delta(s)=-\delta$ for $s\ge-\delta$, $H_\delta(s)=-1$, for $s\le-2\delta$ and that $H_\delta(s)$ is nondecreasing for $-2\delta<s<-\delta$. Also, we extend the function $u(x,t)$ to be defined in the whole plane $\mathbb{R}^2$ such that the extension $\bar u(x,t)=0$ for $t\ge T+1$ and $t\le-1$. Let $\alpha(s)$ be the kernel of mollifier in one dimension, that is, $\alpha(s)\in C^\infty(\mathbb{R})$, $\operatorname{supp}\alpha=[-1,1]$, $\alpha(s)>0$ in $(-1,1)$, and $\int_{-1}^1\alpha(s)ds=1$. For any fixed $k>0,\delta>0$, define

$$u^h(x,t)=\int_{\mathbb{R}}\bar u(s,x)\alpha_h(t-s)\,ds,\quad \beta_\delta(t)=\int_t^{+\infty}\alpha\left(\frac{s-T/2}{T/2-\delta}\right)\frac{1}{T/2-\delta}\,ds,$$

where $\alpha_h(s)=\frac{1}{h}\alpha(s/h)$. The function $\varphi_\delta^h(x,t)\equiv\big[\beta_\delta(t)H_\delta(u^h)\big]^h$ is clearly an admissible test function, that is, the following integral equality holds

$$\begin{aligned}&-\int_0^1 u(x,T)\varphi_\delta^h(T,x)dx+\int_0^1 u_0(x)\varphi_\delta^h(x,0)dx+\iint_{Q_T}u\frac{\partial\varphi_\delta^h}{\partial t}dx\,dt\\ &=\iint_P|u|^n\big(\partial_x^5 u+\partial_x(|u|^{p-1}u)\big)\partial_x\varphi_\delta^h\,dx\,dt.\end{aligned}$$

To proceed further, we analyze the properties of the test function $\varphi_\delta^h(x,t)$. The remaining part of the proof can be done in the same way as that in the proof of Theorem 3.1 in [22] (or [19]). □

4 Expansion of the support

Let us observe again the physical phenomenon described by the thin film. Suppose that at the initial time, the oil film occupies the domain Ω_0. Then as the time evolves, due to the effect of gravity, a touching domain Ω_t will expand. So from the point of view of physical background, this problem seems to be quite interesting. On the other hand, the mathematical description of this property is that the set $\operatorname{supp} u(\cdot,t)$ increases with t. Therefore, in this section, we study the expansion of the support.

Theorem 4. *Assume* $0 < n < 1$, $u_0 \in H_0^2(I)$, $u_0 \geq 0$, $\operatorname{supp} u_0 \subset [x_1, x_2]$, $0 < x_1 < x_2 < 1$, *and* u *is the weak solution of the problem* (1.1)–(1.3), *then for any fixed* $t > 0$, *we have*

$$\operatorname{supp} u(x,\cdot) \subset [x_1(t), x_2(t)] \cap [0,1],$$

where $x_1(t)$, $x_2(t)$ *can be expressed by* $x_1(t) = x_1 - C_1 t^\gamma$, $x_2(t) = x_2 + C_2 t^\gamma$, *with positive constants* C_1, C_2, γ *depending only on* p *and* u_0.

We need a series of uniform estimates on such approximate solutions u_ε.

Lemma 1. *Let* u *be the weak solution of the problem* (1.1)–(1.3). *If* $0 < n < 1$, *then the following integral inequality holds*

$$\int_0^1 u^{2-n} dx + (1-n)(2-n) \iint_{Q_t} (\partial_x^3 u)^2 dx\, ds \leq \int_0^1 u_0^{2-n} dx.$$

Proof. Let u_ε be the solution of the problem (2.1)–(2.3). Denote

$$g_\varepsilon(u) = \int_0^u \frac{dr}{(|r|^2+\varepsilon)^{n/2}}, \quad G_\varepsilon(u) = \int_0^u g_\varepsilon(r) dr.$$

Multiplying both sides of the equation (2.1) by $g_\varepsilon(u_\varepsilon)$, and then integrating over Q_t, we obtain

$$\int_0^1 G_\varepsilon(u_\varepsilon)(x,t)dx + \iint_{Q_t} (\partial_x^3 u_\varepsilon)^2 dx\, ds + p \iint_{Q_t} |u_\varepsilon|^{p-1} (\partial_x u_\varepsilon)^2 dx ds = \int_0^1 G_\varepsilon(u_{0\varepsilon}(x)) dx.$$

Letting $\varepsilon \to 0$ and using the fact that $G_\varepsilon(u_\varepsilon) \to u^{2-n}/(1-n)(2-n)$ and $u_\varepsilon \to u$ pointwise and the lower semi-continuity of the integrals, we immediately get the conclusion of the lemma. The proof is complete. □

Lemma 2. *Let* u *be the weak solution of the problem* (1.1)–(1.3). *If* $0 < n < 1$, *then for any* $\alpha > 4$ *and* $y \in \mathbb{R}^+$, *the following integral inequality holds*

$$\int_0^1 (x-y)_+^\alpha u^{2-n} dx + \iint_{Q_t} (x-y)_+^\alpha (\partial_x^3 u)^2 dx\, ds \leq C \iint_{Q_t} (x-y)_+^{\alpha-4} (\partial_x u)^2 dx\, ds$$
$$+ C \left(\int_y^1 |u_0|^{2-n} dx \right)^{\frac{2-n}{4}} + C \iint_{Q_t} (x-y)_+^{\alpha-2} (\partial_x^2 u)^2 dx ds,$$

where C *depends only on* n, u_0 *and* $(x-y)_+$ *denotes the positive part of* $x - y$.

Proof. Let $g_\varepsilon(u)$ and $G_\varepsilon(u)$ be defined as in the proof of Lemma 1. Let u_ε be the approximate solutions derived from the problem (2.1)–(2.3). Then, using the equation (2.1) and integrating by parts, we get

$$\int_0^1 (x-y)_+^\alpha G_\varepsilon(u_\varepsilon)dx - \int_0^1 (x-y)_+^\alpha G_\varepsilon(u_0)dx$$
$$= -\iint_{Q_t} \big(|u_\varepsilon|^2+\varepsilon\big)^{\frac{n}{2}} \big(\partial_x^5 u_\varepsilon + \partial_x(|u_\varepsilon|^{p-1}u_\varepsilon)\big)\partial_x\big[(x-y)_+^\alpha g_\varepsilon(u_\varepsilon)\big]dx\,ds$$
$$= -\iint_{Q_t} \big(\partial_x^5 u_\varepsilon + \partial_x(|u_\varepsilon|^{p-1}u_\varepsilon)\big)(x-y)_+^\alpha \partial_x u_\varepsilon\,dx\,ds$$
$$- \iint_{Q_t} \big(|u_\varepsilon|^2+\varepsilon\big)^{\frac{n}{2}} \big(\partial_x^5 u_\varepsilon + \partial_x(|u_\varepsilon|^{p-1}u_\varepsilon)\big)\big[\alpha(x-y)_+^{\alpha-1} g_\varepsilon(u_\varepsilon)\big]dx\,ds$$
$$\equiv I_1 + I_2.$$

As for I_1, integrating by parts, we have

$$I_1 = -\iint_{Q_t} \big[\partial_x^5 u_\varepsilon + p|u_\varepsilon|^{p-1}\partial_x u_\varepsilon\big](x-y)_+^\alpha \partial_x u_\varepsilon dx\,ds$$
$$= \iint_{Q_t} \partial_x^4 u_\varepsilon \partial_x\big[(x-y)_+^\alpha \partial_x u_\varepsilon\big]dx\,ds - \iint_{Q_t} p|u_\varepsilon|^{p-1}(x-y)_+^\alpha(\partial_x u_\varepsilon)^2 dx\,ds$$
$$= \iint_{Q_t} \partial_x^4 u_\varepsilon (x-y)_+^\alpha \partial_x^2 u_\varepsilon dx\,ds + \iint_{Q_t} \partial_x^4 u_\varepsilon \partial_x u_\varepsilon \alpha (x-y)_+^{\alpha-1} dx\,ds$$
$$- \iint_{Q_t} p|u_\varepsilon|^{p-1}(x-y)_+^\alpha(\partial_x u_\varepsilon)^2 dx\,ds = -\iint_{Q_t} \partial_x^3 u_\varepsilon \partial_x[(x-y)_+^\alpha \partial_x^2 u_\varepsilon]dx\,ds$$
$$- \iint_{Q_t} \partial_x^3 u_\varepsilon \partial_x[\partial_x u_\varepsilon \alpha(x-y)_+^{\alpha-1}]dx\,ds - \iint_{Q_t} p|u_\varepsilon|^{p-1}(x-y)_+^\alpha(\partial_x u_\varepsilon)^2 dx\,ds$$
$$= -\iint_{Q_t} (x-y)_+^\alpha(\partial_x^3 u_\varepsilon)^2 dx\,ds - \iint_{Q_t} \alpha(x-y)_+^{\alpha-1}\partial_x^3 u_\varepsilon \partial_x^2 u_\varepsilon dx\,ds$$
$$- \iint_{Q_t} \partial_x^3 u_\varepsilon \partial_x^2 u_\varepsilon \alpha(x-y)_+^{\alpha-1} dx\,ds - \iint_{Q_t} p|u_\varepsilon|^{p-1}(x-y)_+^\alpha(\partial_x u_\varepsilon)^2 dx\,ds$$
$$- \iint_{Q_t} \partial_x^3 u_\varepsilon \partial_x u_\varepsilon \alpha(\alpha-1)(x-y)_+^{\alpha-2} dx\,ds.$$

In addition, I_2 yields, by integrating by parts,

$$I_2 = -\iint_{Q_t} \big(|u_\varepsilon|^2+\varepsilon\big)^{\frac{n}{2}} \big[\partial_x^5 u_\varepsilon + p|u_\varepsilon|^{p-1}\partial_x u_\varepsilon\big]\big[\alpha(x-y)_+^{\alpha-1} g_\varepsilon(u_\varepsilon)\big]dx\,ds$$
$$= -\iint_{Q_t} \big(|u_\varepsilon|^2+\varepsilon\big)^{\frac{n}{2}} D^5 u_\varepsilon g_\varepsilon(u_\varepsilon)\alpha(x-y)_+^{\alpha-1} dx\,ds$$
$$- \iint_{Q_t} \alpha(x-y)_+^{\alpha-1}\big(|u_\varepsilon|^2+\varepsilon\big)^{\frac{n}{2}} p|u_\varepsilon|^{p-1} g_\varepsilon(u_\varepsilon)\partial_x u_\varepsilon dx\,ds.$$

Therefore

$$\int_0^1 (x-y)_+^\alpha G_\varepsilon(u_\varepsilon)dx - \int_0^1 (x-y)_+^\alpha G_\varepsilon(u_0)dx + \iint_{Q_t} (x-y)_+^\alpha(\partial_x^3 u_\varepsilon)^2 dx\,ds$$

$$+p\iint_{Q_t}|u_\varepsilon|^{p-1}(x-y)_+^\alpha(\partial_x u_\varepsilon)^2dx\,ds=-2\iint_{Q_t}\alpha(x-y)_+^{\alpha-1}\partial_x^3u_\varepsilon\partial_x^2u_\varepsilon dx\,ds$$
$$-\iint_{Q_t}\partial_x^3u_\varepsilon\alpha(\alpha-1)(x-y)_+^{\alpha-2}\partial_xu_\varepsilon dx\,ds$$
$$-\iint_{Q_t}(|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}}\partial_x^5u_\varepsilon g_\varepsilon(u_\varepsilon)\alpha(x-y)_+^{\alpha-1}dx\,ds$$
$$-\iint_{Q_t}\alpha(x-y)_+^{\alpha-1}(|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}}p|u_\varepsilon|^{p-1}g_\varepsilon(u_\varepsilon)\partial_xu_\varepsilon dxds\equiv I_a+I_b+I_c+I_d.$$

Hölder's inequality yields

$$|I_a|\le\frac{1}{8}\iint_{Q_t}(x-y)_+^\alpha(\partial_x^3u_\varepsilon)^2dx\,ds+C\iint_{Q_t}(x-y)_+^{\alpha-2}(\partial_x^2u_\varepsilon)^2dx\,ds.$$

Similarly, the $|I_b|$ can be handled,

$$|I_b|\le\frac{1}{8}\iint_{Q_t}(x-y)_+^\alpha(\partial_x^3u_\varepsilon)^2dx\,ds+C\iint_{Q_t}(x-y)_+^{\alpha-4}(\partial_xu_\varepsilon)^2dx\,ds.$$

Noticing that

$$\left(|u_\varepsilon|^2+\varepsilon\right)^{\frac{n}{2}}|g_\varepsilon(u_\varepsilon)|\le 2|u_\varepsilon|/(1-n),$$

using (2.8), we have

$$|I_c|\le\Big(\iint_{Q_t}(|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}}(\partial_x^5u_\varepsilon)^2dx\,ds\Big)^{\frac12}$$
$$\times\Big(\iint_{Q_t}(|u_\varepsilon|^2+\varepsilon)^{\frac{n}{2}}\alpha^2(x-y)_+^{2\alpha-2}(g_\varepsilon(u_\varepsilon))^2dx\,ds\Big)^{\frac12}$$
$$\le C\Big(\iint_{Q_t}(x-y)_+^{2\alpha-2}(|u_\varepsilon|^2+\varepsilon)^{-\frac{n}{2}}|u_\varepsilon|^2dx\,ds\Big)^{\frac12}\le C\Big(\iint_{Q_t}|u_\varepsilon|^{2-n}dx\,ds\Big)^{\frac12}.$$

By Hölder's inequality, Poincaré's inequality and Friedrichs's inequality, we obtain

$$|I_c|\le C\left(\iint_{Q_t}(u_\varepsilon)^2dx\,ds\right)^{\frac{2-n}{4}}\le C\left(\iint_{Q_t}(\partial_x^3u_\varepsilon)^2dx\,ds\right)^{\frac{2-n}{4}},$$
$$|I_d|\le C\iint_{Q_t}|\partial_xu_\varepsilon|^2(x-y)_+^{\alpha-4}dx\,ds.$$

From what are discussed above, we have

$$\int_0^1(x-y)_+^\alpha G_\varepsilon(u_\varepsilon)dx-\int_0^1(x-y)_+^\alpha G_\varepsilon(u_0)dx+\iint_{Q_t}(x-y)_+^\alpha(\partial_x^3u_\varepsilon)^2dx\,ds$$
$$+p\iint_{Q_t}|u_\varepsilon|^{p-1}(x-y)_+^\alpha(\partial_xu_\varepsilon)^2dx\,ds\le C\iint_{Q_t}(x-y)_+^{\alpha-4}(\partial_xu_\varepsilon)^2dx\,ds$$
$$+C\left(\iint_{Q_t}(\partial_x^3u_\varepsilon)^2dx\,ds\right)^{\frac{2-n}{4}}+C\iint_{Q_t}(x-y)_+^{\alpha-2}(\partial_x^2u_\varepsilon)^2dx\,ds.$$

Letting $\varepsilon \to 0$, and using Lemma 1, we immediately get the desired conclusion and complete the proof of the lemma. □

Proof. [Proof of Theorem 4] For any $y \geq x_2$, Lemma 2 and Hardy's inequality [14] imply that for any $t \in [0, T]$,

$$\begin{aligned}
&\int_0^1 (x-y)_+^\alpha u^{2-n} dx + \iint_{Q_t} (x-y)_+^\alpha |\partial_x^3 u|^2 dx\, ds \\
&\quad \leq C \iint_{Q_t} (x-y)_+^{\alpha-4} |\partial_x u|^2 dx\, ds + C \iint_{Q_t} (x-y)_+^{\alpha-2} |\partial_x^2 u|^2 dx\, ds \\
&\quad \leq C \iint_{Q_t} (x-y)_+^{\alpha-2} |\partial_x^2 u|^2 dx\, ds. \qquad (4.1)
\end{aligned}$$

For any positive number m, define

$$f_m(y) = \int_0^t \int_0^1 (x-y)_+^m |\partial_x^3 u(x,s)|^2 dx\, ds, \quad f_0(y) = \int_0^t \int_y^1 |\partial_x^3 u|^2 dx\, ds.$$

Then, the weighted Nirenberg's inequality [3] and the estimate (4.1) imply that

$$\begin{aligned}
f_{2p+1}(y) &\leq C \iint_{Q_t} (x-y)_+^{2p-1} |\partial_x^2 u|^2 dx\, ds \\
&\leq C \int_0^t \Big(\int_0^1 (x-y)_+^{2p-1} |\partial_x^3 u|^2 dx \Big)^a \Big(\int_0^1 (x-y)_+^{2p-1} |u|^q dx \Big)^{2(1-a)/q} ds \\
&\leq C \sup_{0<s<t} \Big(\int_0^1 (x-y)_+^{2p-1} |u|^q dx \Big)^{\frac{2(1-a)}{q}} t^{1-a} \Big(\iint_{Q_t} (x-y)_+^{2p-1} |\partial_x^3 u|^2 dx\, ds \Big)^a.
\end{aligned}$$

Using (4.1) and Hardy's inequality, we have

$$\sup_{0<s<t} \int_0^1 (x-y)_+^{2p-1} |u|^q dx \leq C \iint_{Q_t} (x-y)_+^{2p-1} |\partial_x^3 u|^2 dx\, ds$$

and hence

$$f_{2p+1}(y) \leq C t^{1-a} \Big(\iint_{Q_t} (x-y)_+^{2p-1} |\partial_x^3 u|^2 dx\, ds \Big)^{a+2(1-a)/q},$$

where $q = 2-n$ and $a = (\frac{1}{2} - \frac{1}{p} - \frac{1}{q})/(\frac{1}{2} - \frac{3}{2p} - \frac{1}{q})$. Denote $\lambda = 1-a, \mu = a + 2(1-a)/q$, then $\lambda > 0$, $1 < \mu$. Applying Hölder's inequality, we have

$$\begin{aligned}
f_{2p+1}(y) &\leq C t^\lambda \Big[\iint_{Q_t} (x-y)_+^{2p-1} |\partial_x^3 u|^2 dx\, ds \Big]^\mu \\
&\leq C t^\lambda \Big[\iint_{Q_t} (x-y)_+^{2p+1} |\partial_x^3 u|^2 dx\, ds \Big]^{\frac{(2p-1)\mu}{(2p+1)}} \Big[\int_0^t \int_y^1 |\partial_x^3 u|^2 dx\, ds \Big]^{\frac{2\mu}{2p+1}} \\
&\leq C t^\lambda \Big[f_{2p+1}(y) \Big]^{(2p-1)\mu/(2p+1)} \Big[f_0(y) \Big]^{2\mu/(2p+1)}.
\end{aligned}$$

Therefore

$$f_{2p+1}(y) \leq Ct^{\lambda/\sigma}\Big[f_0(y)\Big]^{2\mu/(2p+1)\sigma}, \qquad \sigma = 1 - \frac{2p-1}{2p+1}\mu > 0.$$

Using Hölder's inequality again, we get

$$f_1(y) \leq \big[f_0(y)\big]^{2p/2p+1}\big[f_{2p+1}(y)\big]^{1/2p+1} \leq Ct^{\gamma}\big[f_0(y)\big]^{1+\theta},$$

where

$$\gamma = \frac{\lambda}{(2p+1)\sigma}, \quad \theta = \frac{2\mu}{(2p+1)^2\sigma} - \frac{1}{2p+1} > 0.$$

Noticing that $f_1'(y) = -f_0(y)$, we obtain

$$f_1'(y) \leq -Ct^{-\gamma/(\theta+1)}\big[f_1(y)\big]^{1/(\theta+1)}.$$

If $f_1(x_2) = 0$, then $\operatorname{supp} u \subset [0, x_2]$. If $f_1(x_2) > 0$, then there exists a maximal interval (x_2, x_2^*) in which $f_1(y) > 0$ and

$$\Big[f_1(y)^{\theta/(\theta+1)}\Big]' = \frac{\theta}{\theta+1}\frac{f_1'(y)}{[f_1(y)]^{1/(\theta+1)}} \leq -Ct^{-\gamma/(\theta+1)}.$$

Integrating the above inequality over (x_2, x_2^*), we have

$$f_1(x_2^*)^{\theta/(\theta+1)} - f_1(x_2)^{\theta/(\theta+1)} \leq -Ct^{-\gamma/(\theta+1)}(x_2^* - x_2),$$

which implies that

$$x_2^* \leq x_2 + Ct^{\gamma}\big(f_0(x_2)\big)^{\theta}.$$

Lemma 1 implies that $f_0(y)$ can be controlled by a constant C independent of y. Therefore

$$\sup \operatorname{supp} u(\cdot, t) \leq x_2 + Ct^{\gamma} \equiv x_2(t).$$

We have thus completed the proof of Theorem 4. □

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