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A Strong Convergence of a Modified Krasnoselskii-Mann Method for Non-Expansive Mappings in Hilbert Spaces*

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Abstract. In this paper, we introduce a new method based on the well-known Krasnoselskii-Mann's method for non-expansive mappings in Hilbert spaces. We show that the proposed method has strong convergence for non-expansive mappings.
Keywords: non-expansive mapping, fixed point, modified Krasnoselskii-Mann's method, strong convergence, Hilbert space.

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1 Introduction

Let H be a real Hilbert space. Recall that a mapping $T : H \to H$ is said to be non-expansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in H.$$

A point $x \in H$ is a fixed point of T provided $Tx = x$. Let's denote the set of fixed points of T by $F(T)$, that is, $F(T) = \{x \in H : Tx = x\}$. Assume, throughout this paper, that $F(T) \neq \emptyset$.

Construction of fixed points of non-expansive mappings is an important subject in the theory of non-expansive mappings and its applications in a number of applied areas, in particular, image recovery and signal processing (see [1]–[28]). It is well-known that Picard's iteration

$$x_{n+1} = Tx_n = \cdots = T^{n+1}x$$

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of the mapping T at a point $x \in H$ may not behave well in general. This means that it may not converge even in the weak topology. One way to overcome this difficulty is to use Krasnoselskii-Mann's method that produces a sequence $\{x_n\}$ via the recursive manner:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \geq 0. \tag{1.1}$$

This Krasnoselskii-Mann's method is remarkably useful for finding fixed points of a non-expansive mapping and provides a unified framework for some kinds of algorithms from various different fields. In this respect, the following result is basic and important.

Theorem 1. ([19]) *Let T be a non-expansive mapping on H. Then the sequence $\{x_n\}$ defined by the iterative method* (1.1) *converges weakly to a fixed point of T provided $\alpha_n \in [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = +\infty$ whenever such fixed points exist.*

However, as in Theorem 1, Krasnoselskii-Mann's method for non-expansive mappings has only weak convergence. Thus a natural question rises: could we obtain a strong convergence result by using the well-known Krasnoselskii-Mann method for non-expansive mappings? In this connection, in 1975, Genel and Lindenstrass [7] gave a counterexample. Hence the modification is necessary in order to guarantee the strong convergence of Krasnoselskii-Mann's method.

Some attempts to construct iteration algorithm so that strong convergence is guaranteed have recently been made. For a sequence $\{\alpha_n\}$ of real numbers in $[0,1]$ and fixed $u \in C$, let the sequence $\{x_n\}$ in C be iteratively defined by $x_0 \in C$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad \forall n \geq 0. \tag{1.2}$$

The iterative method is now referred to as the Halpern iterative algorithm. The interest and importance of the Halpern iterative algorithm lie in the fact that strong convergence of the sequence $\{x_n\}$ is achieved under certain mild conditions on parameter $\{\alpha_n\}$ in a general Banach space. We recall some relevant important results as follows. In 1977, Lions [11] proved the strong convergence of $\{x_n\}$ generated by (1.2) to a fixed point of T, where the real sequence $\{\alpha_n\}$ satisfies the following conditions:

$$(C1) \ \lim_{n\to\infty} \alpha_n = 0, \quad (C2) \ \sum_{n=0}^{\infty} \alpha_n = \infty, \quad (C3) \ \lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0.$$

Based on this result, many authors considered the strong convergence of the Halpern algorithm under some restrictions on the parameters $\{\alpha_n\}$. For related works, see, e.g., [21, 25] and the references therein. Recently, Kim and Xu [9] proposed the following simpler modification of Mann iteration method:

Let C be a closed convex subset of a Banach space and $T : C \to C$ a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Define $\{x_n\}$ in the following way:

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ x_{n+1} = \beta_n u + (1-\beta_n)y_n, \qquad \forall n \geq 0, \end{cases} \tag{1.3}$$

where $u \in C$ is an arbitrary (but fixed) element in C, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$.

Remark 1. The modified Mann iteration scheme (1.3) is a convex combination of a fixed point in C and the Mann iteration method (1.1). There is no additional projection involved in iteration scheme (1.3).

A strong convergence of iteration scheme (1.3) is proved in [9].

Theorem 2. *Let C be a closed convex subset of a uniformly smooth Banach space X and let $T : C \to C$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Given a point $u \in C$ and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $(0,1)$, the following conditions are satisfied:*
(a) $\beta_n \to 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(b) $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.
Then $\{x_n\}_{n=0}^{\infty}$ *defined by* (1.2) *strongly converges to a fixed point of* T.

Motivated by the result of Kim and Xu [9], Yao, Chen and Yao [23] introduced a modified version of the algorithm (1.3) with the viscosity method and proved the strong convergence of the proposed algorithm under some mild assumptions on the parameters.

2 Preliminaries

In the sequel, we use the notation $\rightharpoonup$ for weak convergence and $\to$ for strong convergence. We will need some lemmas to prove our main results.

Lemma 1. ([8]) *Let H be a real Hilbert space. Let $T : H \to H$ be a nonexpansive mapping. Then $I - T$ is demi-closed at 0, i.e., if $x_n \rightharpoonup x \in H$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2. ([21]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n$ for all $n \geq 0$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that*
(a) $\sum_{n=0}^{\infty} \gamma_n = \infty$, *(b)* $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=0}^{\infty} |\delta_n\gamma_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

3 Main Results

Let H be a real Hilbert space and $T : H \to H$ be a non-expansive mapping. Let λ be a constant in $(0, 1)$. For each $t \in (0, 1)$, define a mapping $T_t : H \to H$ by

$$T_t x = t(\lambda x) + (1-t)Tx, \quad \forall x \in H.$$

For any $x, y \in H$, we have

$$\|T_t x - T_t y\| = \|[t(\lambda x) + (1-t)Tx] - [t(\lambda y) + (1-t)Ty]\| \leq [1-(1-\lambda)t]\|x-y\|,$$

which implies that T_t is a contraction. Using Banach's contraction principle, we get that there exists a unique fixed point x_t of T_t in H, i.e.,

$$x_t = t(\lambda x_t) + (1-t)Tx_t. \tag{3.1}$$

Next, we show the convergence of the net $\{x_t\}$.

Lemma 3. *Let H be a real Hilbert space and $T : H \to H$ be a non-expansive mapping with $F(T) \neq \emptyset$. Then, as $t \to 0$, the net $\{x_t\}$ defined by* (3.1) *converges strongly to a fixed point of T.*

Proof. First, we prove that $\{x_t\}$ is bounded. Take $u \in F(T)$. From (3.1), we have

$$\begin{aligned}\|x_t - u\| &= \|t(\lambda x_t) + (1-t)Tx_t - u\| \leq \lambda t\|x_t - u\| + (1-t)\|Tx_t - u\| + (1-\lambda)t\|u\| \\ &\leq [1-(1-\lambda)t]\|x_t - u\| + (1-\lambda)t\|u\|,\end{aligned}$$

which implies that $\|x_t - u\| \leq \|u\|$. Hence $\{x_t\}$ is bounded. Again, from (3.1), it follows that

$$\|x_t - Tx_t\| = t\|\lambda x_t - Tx_t\| \to 0. \tag{3.2}$$

Next, we show that $\{x_t\}$ is relatively norm compact as $t \to 0$. Let $\{t_n\} \subset (0, 1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$. From (3.2), we have

$$\|x_n - Tx_n\| \to 0. \tag{3.3}$$

From (3.1), we get

$$\begin{aligned}\|x_t - u\|^2 &= \langle t(\lambda x_t) + (1-t)Tx_t - u, x_t - u\rangle \\ &= \lambda t\langle x_t - u, x_t - u\rangle + (1-t)\langle Tx_t - u, x_t - u\rangle - (1-\lambda)t\langle u, x_t - u\rangle \\ &\leq [1-(1-\lambda)t]\|x_t - u\|^2 - (1-\lambda)t\langle u, x_t - u\rangle,\end{aligned}$$

which implies that $\|x_t - u\|^2 \leq \langle u, u - x_t\rangle$. In particular,

$$\|x_n - u\|^2 \leq \langle u, u - x_n\rangle, \quad \forall u \in F(T). \tag{3.4}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_n \rightharpoonup x^* \in H$. Noticing (3.3), we can use Lemma 1 to get $x^* \in F(T)$. Therefore, we can substitute x^* for u in (3.4) to get

$$\|x_n - x^*\|^2 \le \langle x^*, x^* - x_n \rangle.$$

Hence $x_n \rightharpoonup x^*$ implies that $x_n \to x^*$. This has proved the relative norm compactness of the net $\{x_t\}$ as $t \to 0$.

To show that the entire net $\{x_t\}$ converges to x^*, assume $x_{t_m} \to \tilde{x} \in F(T)$, where $t_m \to 0$. Put $x_m = x_{t_m}$. Similarly, we have

$$\|x_m - x^*\|^2 \le \langle x^*, x^* - x_m \rangle$$

and so

$$\|\tilde{x} - x^*\|^2 \le \langle x^*, x^* - \tilde{x} \rangle. \tag{3.5}$$

Interchanging x^* and $\tilde{x}$, we obtain

$$\|x^* - \tilde{x}\|^2 \le \langle \tilde{x}, \tilde{x} - x^* \rangle. \tag{3.6}$$

Adding up (3.5) and (3.6) yields

$$2\|x^* - \tilde{x}\|^2 \le \|x^* - \tilde{x}\|^2,$$

which implies that $\tilde{x} = x^*$. This completes the proof. □

Remark 2. It should be pointed out that Lemma 3 is a new result which is very important for proving the following theorem.

Theorem 3. *Let H be a real Hilbert space and $T : H \to H$ be a non-expansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\lambda_n\}$ be two real sequences in $(0,1)$. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)Tx_n, \quad \forall n \ge 0. \tag{3.7}$$

Suppose the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0$; *(b)* $\lim_{n\to\infty} \lambda_n = 1$;

(c) $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty} (1-\lambda_n)\alpha_n = \infty$.

Then the sequence $\{x_n\}$ generated by (3.7) strongly converges to a fixed point of T.

Proof. First, we prove that the sequence $\{x_n\}$ is bounded. Take $u \in F(T)$. From (3.7), we have

$$\begin{aligned}\|x_{n+1} - u\| &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)Tx_n - u\| \\ &\le \alpha_n\lambda_n\|x_n - u\| + (1-\lambda_n)\alpha_n\|u\| + (1-\alpha_n)\|x_n - u\| \\ &= [1-(1-\lambda_n)\alpha_n]\|x_n - u\| + (1-\lambda_n)\alpha_n\|u\| \le \max\{\|x_n - u\|, \|u\|\}.\end{aligned}$$

Hence $\{x_n\}$ is bounded and so is $\{Tx_n\}$. From (3.7), it follows that

$$\begin{aligned}\|x_{n+1}-x_n\|&=\|\alpha_n(\lambda_n x_n)+(1-\alpha_n)Tx_n-\alpha_{n-1}(\lambda_{n-1}x_{n-1})-(1-\alpha_{n-1})Tx_{n-1}\|\\&=\|\alpha_n\lambda_n(x_n-x_{n-1})+\alpha_n(\lambda_n-\lambda_{n-1})x_{n-1}+(\alpha_n-\alpha_{n-1})(\lambda_{n-1}x_{n-1})\\&\quad+(1-\alpha_n)(Tx_n-Tx_{n-1})+(\alpha_{n-1}-\alpha_n)Tx_{n-1}\|\\&\le\alpha_n\lambda_n\|x_n-x_{n-1}\|+(1-\alpha_n)\|Tx_n-Tx_{n-1}\|\\&\quad+|\alpha_n-\alpha_{n-1}|(\lambda_{n-1}\|x_{n-1}\|+\|Tx_{n-1}\|)+\alpha_n|\lambda_n-\lambda_{n-1}|\|x_{n-1}\|\\&\le[1-(1-\lambda_n)\alpha_n]\|x_n-x_{n-1}\|+(|\alpha_n-\alpha_{n-1}|+\alpha_n|\lambda_n-\lambda_{n-1}|)M_1,\end{aligned}\tag{3.8}$$

where M_1 is a constant such that $\sup_n\{\|x_{n-1}\|+\|Tx_{n-1}\|\}\le M_1$. Hence, from (3.8) and Lemma 2, we deduce

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

At the same time, we note that

$$\|x_{n+1}-Tx_n\|=\alpha_n\|(\lambda_n x_n)-Tx_n\|\to 0.$$

Therefore, we have

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.$$

Next, we prove $\limsup_{n\to\infty}\langle x^*,x^*-x_n\rangle\le 0$, where $x^*=\lim_{t\to 0}x_t$ and $\{x_t\}$ is the net defined by (3.1). As a matter of fact, we have

$$\begin{aligned}\|x_t-x_n\|^2&=\langle x_t-x_n,x_t-x_n\rangle=t\langle x_t-x_n,x_t-x_n\rangle-(1-\lambda)t\langle x_t,x_t-x_n\rangle\\&\quad+(1-t)\langle Tx_t-Tx_n,x_t-x_n\rangle+(1-t)\langle Tx_n-x_n,x_t-x_n\rangle\\&\le\|x_t-x_n\|^2-(1-\lambda)t\langle x_t,x_t-x_n\rangle+(1-t)\langle Tx_n-x_n,x_t-x_n\rangle\\&\le\|x_t-x_n\|^2-(1-\lambda)t\langle x_t,x_t-x_n\rangle+M_2\|Tx_n-x_n\|,\end{aligned}$$

where $M_2>0$ such that $\sup\{\|x_t-x_n\|,t\in(0,1),n\ge 0\}\le M_2$, which implies that

$$\langle x_t,x_t-x_n\rangle\le\frac{M_2}{(1-\lambda)t}\|Tx_n-x_n\|.$$

Therefore, we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t,x_t-x_n\rangle\le 0.\tag{3.9}$$

We note that

$$\begin{aligned}\langle x^*,x^*-x_n\rangle&=\langle x^*,x^*-x_t\rangle+\langle x^*-x_t,x_t-x_n\rangle+\langle x_t,x_t-x_n\rangle\\&\le\langle x^*,x^*-x_t\rangle+\|x^*-x_t\|\|x_t-x_n\|+\langle x_t,x_t-x_n\rangle\\&\le\langle x^*,x^*-x_t\rangle+\|x^*-x_t\|M_2+\langle x_t,x_t-x_n\rangle.\end{aligned}$$

This together with $x_t \to x^*$ and (3.9) implies that

$$\limsup_{n\to\infty}\langle x^*, x^* - x_n\rangle \le 0.$$

Finally, we show that $x_n \to x^*$. From (3.7), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \langle x_{n+1}-x^*, x_{n+1}-x^*\rangle = \alpha_n\lambda_n\langle x_n-x^*, x_{n+1}-x^*\rangle \\ &+ (1-\lambda_n)\alpha_n\langle x^*, x^*-x_{n+1}\rangle + (1-\alpha_n)\langle Tx_n - x^*, x_{n+1}-x^*\rangle \\ &\le [1-(1-\lambda_n)\alpha_n]\|x_n-x^*\|\|x_{n+1}-x^*\| + (1-\lambda_n)\alpha_n\langle x^*, x^*-x_{n+1}\rangle \\ &\le \frac{1-(1-\lambda_n)\alpha_n}{2}(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)+(1-\lambda_n)\alpha_n\langle x^*, x^*-x_{n+1}\rangle,\end{aligned}$$

which implies that

$$\|x_{n+1}-x^*\|^2 \le [1-(1-\lambda_n)\alpha_n]\|x_n-x^*\|^2 + 2(1-\lambda_n)\alpha_n\langle x^*, x^*-x_{n+1}\rangle.$$

We can check that all the assumptions of Lemma 2 are satisfied. Therefore, we deduce $x_n \to x^*$. This completes the proof. □

Remark 3. The prototype for the iteration parameters are as follows: $\alpha_n = \frac{1}{\sqrt{n}}$ and $\lambda_n = 1 - \frac{1}{\sqrt{n}}$ for all $n \ge 0$. It is easy to check that these choices satisfy all the conditions of Theorem 3.

Remark 4. We conclude the paper with the following observations:

(1) Our algorithm (3.7) is similar to Mann's algorithm (1.1). It is well-known that Mann's method for non-expansive mappings has only weak convergence. However, our iterative method (3.7) has strong convergence. Our iterative scheme (3.7) may be interesting which is very different from the algorithm of Kim and Xu (1.3). As a matter of fact, the algorithm (1.3) is a convex combination of u and the Mann iteration method (1.1).

(2) In order to prove the strong convergence of the algorithm (1.3), Kim and Xu [9] used the convergence of the net $x_t = tu + (1-t)Tx_t$. It is worthy to note that we prove the strong convergence of the algorithm (3.7) using the new Lemma 3. In this respect, our methods are independent from those given in the literature.

4 Conclusions

The problem of finding fixed points of nonexpansive mappings has attracted much attention because of its extraordinary utility and broad applicability in many branches of mathematical science and engineering.

Mann's algorithm is a widely used method for solving a fixed point equation of the form $Tx = x$, where C is a nonempty closed convex subset of a Hilbert space H, and $T : C \to C$ is a nonexpansive mapping. This algorithm generates from an arbitrary initial guess $x_0 \in C$, a sequence $\{x_n\}$ by the recursive manner:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad \forall n \ge 1,$$

where $\{\alpha_n\}$ is a sequence in the interval $[0, 1]$.

In an infinite-dimensional space X, Mann's algorithm has only weak convergence, in general. In fact, it is known that if the sequence $\{\alpha_n\}$ is such that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then Mann's algorithm converges weakly to a fixed point of T provided the underlying space X is a Hilbert space or more general, a uniformly convex Banach space which has a Frechet differentiable norm or satisfies Opial's property. It is a very interesting topic of constructing some algorithms such that the strong convergence of proposed algorithms are guaranteed. For this purpose, in this article we present a modified Krasnoselskii-Mann method (3.7) for nonexpansive mappings in Hilbert spaces and show that the proposed method (3.7) has strong convergence. However, we note that in order to obtain the main result of Theorem 3, we have imposed some additional conditions $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ and $\sum_{n=0}^{\infty}(1 - \lambda_n)\alpha_n = \infty$. Hence this brings us a nature problem: could we weaken or drop these additional assumptions?

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