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Advanced Impulsive Differential Equations with Piecewise Constant Arguments

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Abstract. We prove the existence and uniqueness of the solutions of a class of first order nonhomogeneous advanced impulsive differential equations with piecewise constant arguments. We also study the conditions of periodicity, oscillation, nonoscillation and global asymptotic stability for some special cases.

Keywords: impulsive differential equation, differential equation with piecewise constant argument, oscillation, nonoscillation, periodicity, global asymptotic stability.

AMS Subject Classification: 34K11; 34K13; 34K25; 34K45.

1 Introduction

The theory of differential equations with piecewise constant arguments (DEPCA) of the type

$$x'(t) = f\ (t, x(t), x(h(t)))$$

was initiated in [8, 21] where $h(t) = [t], [t-n], [t+n]$, etc. These types of equations have been intensively investigated for twenty five years. Systems described by DEPCA exist in a widely expanded area such as biomedicine, chemistry, physics and mechanical engineering. Busenberg and Cooke [7] first established a mathematical model with a piecewise constant argument for analyzing vertically transmitted diseases. In physical and engineering systems, the phenomena related to stepwise or piecewise constant variables or motions under piecewise constant forces are common. These kinds of systems can usually be formulated as first or second order differential equations with piecewise constant arguments. Examples in practice include machinery driven by servo units, charged particles moving in a piecewise constantly varying electric field, and elastic systems impelled by a Geneva wheel.

DEPCA is closely related to difference and differential equations [8]. So, they describe hybrid dynamical systems and combine the properties of both differential and difference equations. The oscillation, periodicity and some asymptotic properties for various differential equations with piecewise constant arguments without impulses were methodically demonstrated in [1, 2, 3, 11, 12, 13, 17, 19, 20, 22, 24, 27]. Also, Wiener's book [25] is a distinguished source in this area.

The theory of impulsive differential equations developed rapidly, in recent years. This development, in particular, is due to the fact that many phenomena and process in natural sciences, such as physics, population dynamics, ecology, biology, *etc.*, can be simulated by these type of equations.

There are many papers that study the qualitative behaviours of the impulsive differential equations [5, 6, 9, 10, 14, 18, 28]. Among these investigations stability and instability problems are very interesting. Impulses can make unstable systems stable and stable systems can become unstable after impulse effects [4, 16, 23]. To the best of our knowledge, there are only a few papers involving impulsive differential equations with piecewise constant arguments [15, 26]. In [15], Li and Shen considered the problem

$$\begin{aligned} &y'(t) = f(t, y[t-k]), \quad t \neq n,\ t \in J, \\ &\Delta y(n^+) = I_n(y(n)), \quad n = 1, 2, \ldots, p, \quad y(0) = y(T). \end{aligned}$$

Using the method of upper and lower solutions, they proved that it has at least one solution. In [26], however, Wiener and Lakshmikantham proved the existence and uniqueness of solutions of the initial value problem

$$x'(t) = f\ (x(t), x(g(t))), \quad x(0) = x_0,$$

and they also studied the cases of oscillation and stability, where f is a continuous function and $g : [0, \infty) \to [0, \infty),\ g(t) \le t$ is a step function.

In this paper, we consider the first order linear nonhomogeneous advanced impulsive differential equation with piecewise constant arguments of the form

$$x'(t) + a(t)x(t) + b(t)x([t]) + c(t)x([t+1]) = f(t), \quad t \neq n, \tag{1.1}$$

$$\Delta x(n) = d_n x(n), \quad n \in \mathbb{N} = \{0, 1, 2, \ldots\}, \tag{1.2}$$

and the initial condition

$$x(0) = x_0, \tag{1.3}$$

where a, b, c, $f : [0, \infty) \to \mathbb{R}$ are continuous functions, $d_n : \mathbb{N} \to \mathbb{R}$, $\Delta x(n) = x(n^+) - x(n^-)$, $x(n^+) = \lim\limits_{t \to n^+} x(t)$, $x(n^-) = \lim\limits_{t \to n^-} x(t)$ and $[\cdot]$ denotes the greatest integer function.

This paper is organised as follows. In Section 2, we calculate the solutions of (1.1)–(1.2) and prove the existence and uniqueness of them. In Section 3, we present our main results. This section consist of three subsections. Firstly, the necessary and sufficient conditions are given for the existence of k – periodic solutions of (1.1)–(1.2) when $a(t)$, $b(t)$, $c(t)$ and d_n are constant functions. Secondly, we show the conditions for the oscillation and nonoscillation of solutions of (1.1)–(1.2) with $f(t) \equiv 0$. In this section, finally, we prove the global asymptotical stability of zero solution of (1.1)–(1.2) with constant coefficients and $f(t) = 0$. Section 4 covers some examples.

2 Preliminaries

DEFINITION 1. A function $x(t)$ defined on $[0,\infty)$ is said to be a solution of (1.1)–(1.2) if it satisfies the following conditions:

(i) $x:[0,\infty)\to\mathbb{R}$ is continuous for $t\in[0,\infty)$ with the possible exception of the points $[t]\in[0,\infty)$,

(ii) $x(t)$ is right continuous and has left-hand limits at the points $[t]\in[0,\infty)$,

(iii) $x'(t)$ exists for every $t\in[0,\infty)$ with the possible exception of the points $[t]\in[0,\infty)$ where one-sided derivatives exist,

(iv) $x(t)$ satisfies (1.1) with the possible exception of the points $[t]\in[0,\infty)$,

(v) $x(n)$ satisfies (1.2) for $n=0,1,2,\dots$.

Theorem 1. *Let*

$$1-d_{n+1}+\int_n^{n+1} c(u)\exp\Big(\int_{n+1}^u a(s)\,ds\Big)du\neq 0. \tag{2.1}$$

Then the initial value problem (1.1)–(1.3) *has a unique solution* $x(t)$ *on* $[0,\infty)$

$$\begin{aligned}x(t)=&\exp\Big(-\int_{[t]}^t a(s)\,ds\Big)\Big(1-\int_{[t]}^t b(u)\exp\Big(\int_{[t]}^u a(s)\,ds\Big)du\Big)y_{[t]} \\ &-\Big(\int_{[t]}^t c(u)\exp\Big(\int_t^u a(s)\,ds\Big)du\Big)y_{[t+1]}+\int_{[t]}^t f(u)\exp\Big(\int_t^u a(s)\,ds\Big)du,\end{aligned} \tag{2.2}$$

where $y_{[t]}=x([t])$ *for* $t\in[0,\infty)$ *and it is found by*

$$y_{[t]}=\Big(\prod_{i=0}^{[t]-1}\alpha(i)\Big)x_0+\sum_{j=0}^{[t]-1}\Big(\prod_{i=j+1}^{[t]-1}\alpha(i)\Big)\beta(j), \tag{2.3}$$

$$\alpha(i)=\frac{\exp\Big(-\int_i^{i+1}a(s)\,ds\Big)\Big(1-\int_i^{i+1}b(u)\exp\Big(\int_i^u a(s)ds\Big)\,du\Big)}{1-d_{i+1}+\int_i^{i+1}c(u)\exp\Big(\int_{i+1}^u a(s)\,ds\Big)\,du}, \tag{2.4}$$

$$\beta(i)=\frac{\int_i^{i+1}f(u)\exp\Big(\int_{i+1}^u a(s)\,ds\Big)\,du}{1-d_{i+1}+\int_i^{i+1}c(u)\exp\Big(\int_{i+1}^u a(s)\,ds\Big)\,du}. \tag{2.5}$$

Proof. Let $x_n(t)\equiv x(t)$ be a solution of (1.1)–(1.2) on $n\le t<n+1$.Then

$$x'(t)+a(t)x(t)=-b(t)x(n)-c(t)x(n+1)+f(t).$$

So,

$$\begin{aligned}x_n(t)=&\exp\Big(-\int_n^t a(s)ds\Big)\Big(1-\int_n^t b(u)\exp\Big(\int_n^u a(s)\,ds\Big)\,du\Big)x(n) \\ &-\Big(\int_n^t c(u)\exp\Big(\int_t^u a(s)\,ds\Big)du\Big)x(n+1)+\int_n^t f(u)\exp\Big(\int_t^u a(s)\,ds\Big)du.\end{aligned} \tag{2.6}$$

Because of the impulse conditions (1.2),

$$x_n\big((n+1)^-\big) = (1-d_{n+1})x_{n+1}(n+1).$$

This equality leads to the nonhomogenous difference equation

$$\begin{aligned} y_{n+1} =& \frac{\exp\big(-\int_n^{n+1} a(s)\,ds\big)\big(1-\int_n^{n+1} b(u)\exp\big(\int_n^u a(s)\,ds\big)du\big)}{1-d_{n+1}+\int_n^{n+1} c(u)\exp\big(\int_{n+1}^u a(s)\,ds\big)du}\,y_n \\ &+ \frac{\int_n^{n+1} f(u)\exp\big(\int_{n+1}^u a(s)\,ds\big)du}{1-d_{n+1}+\int_n^{n+1} c(u)\exp\big(\int_{n+1}^u a(s)\,ds\big)du}, \end{aligned} \tag{2.7}$$

where $y_n = x(n)$, $n = 0, 1, 2, \dots$. It is to be noted that the initial condition (1.3) takes the form

$$y_0 = x_0. \tag{2.8}$$

Therefore, the unique solution of the initial value problem (2.7)–(2.8) can be represented by

$$y_n = \Big(\prod_{i=0}^{n-1} \alpha(i)\Big)x_0 + \sum_{j=0}^{n-1}\Big(\prod_{i=j+1}^{n-1} \alpha(i)\Big)\beta(j),$$

where $\alpha(i)$ and $\beta(i)$ are as in (2.4) and (2.5), respectively. $\Box$

Theorem 2. *Let*

$$1-d_{-n+1}+\int_{-n}^{-n+1} c(u)\exp\Big(\int_{-n+1}^{u} a(s)\,ds\Big)du \neq 0.$$

Then the initial value problem (1.1)–(1.3) *has a unique backward continuation on* $(-\infty, 0]$ *given by* (2.2)–(2.3).

Proof. Let us denote the solution of (1.1)–(1.2) as $x_{-n}(t)$ on $[-n, -n+1)$, $n = 0, 1, 2, \dots$. Repeating the proof of Theorem 1, we have the unique solution $x_{-n}(t)$ which is the same as (2.6) with n replaced by $-n$. $\Box$

From Theorem 1 and 2 we have the following result.

Corollary 1. Assume that the condition (2.1) holds, then the initial value problem (1.1)–(1.3) has a unique solution on $(-\infty, \infty)$ given (2.2) and (2.3).

By the way, we need to restate the above results for (1.1)–(1.2) with constant coefficients:

$$x'(t) + ax(t) + bx([t]) + cx([t+1]) = f(t),\ t \neq n, \tag{2.9}$$

$$\varDelta x(n) = dx(n), \quad n \in \mathbb{N}, \tag{2.10}$$

where a, b, c, d are real constants.

Theorem 3. *Let $a \neq 0$ and $d \neq 1 - b_1$. Then the unique solution of the problem* (2.9)–(2.10), (1.3) *is formulated on* $[0, \infty)$ *by*

$$x(t) = m_0(\{t\})y_{[t]} + m_1\big(\{t\}\big)y_{[t+1]} + \int_{[t]}^{t} e^{-a(t-u)} f(u)\, du, \tag{2.11}$$

where $\{t\} = t - [t]$,

$$y_{[t]} = \Big(\frac{b_0}{1 - d - b_1}\Big)^{[t]} x_0 + \sum_{j=0}^{[t]-1} \Big(\frac{b_0}{1 - d - b_1}\Big)^{[t]-j-1} \gamma(j), \tag{2.12}$$

$$m_0(t) = e^{-at} + (e^{-at} - 1)ba^{-1},\ m_1(t) = (e^{-at} - 1)ca^{-1}, \tag{2.13}$$

$$m_0(1) = b_0, \quad m_1(1) = b_1, \tag{2.14}$$

$$\gamma(j) = \frac{1}{1 - d - b_1} \int_j^{j+1} e^{-a(j+1-u)} f(u)\, du. \tag{2.15}$$

Proof. In this case the difference equation (2.7) reduces to

$$y_{n+1} = b_0 y_n / (1 - d - b_1) + \gamma(n). \tag{2.16}$$

From (2.2) and (2.3) , we obtain (2.11) and (2.12), respectively. □

Theorem 4. *Let $a \neq 0$, $d \neq 1 - b_1$ and $b_0 \neq 0$. Then the problem* (2.9)–(2.10), (1.3) *has a unique backward continuation on* $(-\infty, 0]$ *given by* (2.11), (2.12).

Corollary 2. Under the conditions of Theorem 4, the problem (2.9)–(2.10), (1.3) has a unique solution on $(-\infty, \infty)$ given by (2.11), (2.12).

3 Main Results

This section contains the statements and proofs of our main results about periodicity, oscillation, nonoscillation and global asymptotic stability.

3.1 Periodicity

Theorem 5. *Assume that $a \neq 0$, $1 - d - b_1 \neq 0$ and $f(t)$ is a periodic function with period* 1. *Then the solution $x(t)$ of the problem* (2.9)–(2.10), (1.3) *is k-periodic if and only if*

$$y_k = y_0, \tag{3.1}$$

where $k = 1, 2, 3, \dots$ and y_n is a solution of the difference equation (2.16).

Proof. Let the solution $x(t)$ of (2.9)–(2.10), (1.3) be periodic with period k. This implies that

$$x_n(t - k) = x_{n+k}(t),\ \ n + k \leq t < n + k + 1,\ \ n = 0, 1, 2, \dots. \tag{3.2}$$

For $n = 0$, Eqn. (3.2) becomes

$$x_0(t - k) = x_k(t), \quad k \leq t < k + 1. \tag{3.3}$$

By using (2.11), we obtain

$$x_0(t-k) = \big(e^{-a(t-k)} + \big(e^{-a(t-k)}-1\big)ba^{-1}\big)y_0 + \big(e^{-a(t-k)}-1\big)ca^{-1}y_1 \\ + \int_0^{t-k} e^{-a(t-u-k)} f(u)\,du, \quad k \le t < k+1, \tag{3.4}$$

$$x_k(t) = (e^{-a(t-k)} + (e^{-a(t-k)}-1)ba^{-1})y_k + (e^{-a(t-k)}-1)ca^{-1}y_{k+1} \\ + \int_k^{t} e^{-a(t-u)} f(u)\,du, \quad k \le t < k+1. \tag{3.5}$$

If we take $u = z + k$ in the integral on the right-hand side of (3.5), we deduce that

$$\int\limits_k^t e^{-a(t-u)} f(u)du = \int\limits_0^{t-k} e^{-a(t-u-k)} f(u)du.$$

Combining (3.4) and (3.5), we have

$$y_0 = y_k, \quad y_1 = y_{k+1}.$$

Because of (2.16), the equality $y_1 = y_{k+1}$ implies (3.1). For $n = 1$, Eqn. (3.2) reduces to

$$x_1(t-k) = x_{k+1}(t), \ \ k+1 \le t < k+2,$$

and this equality implies (3.1), again. If we continue the same procedure, we always obtain (3.1) from (3.2) for each $n = 2, 3, 4 \ldots$. Now we assume that $y_0 = y_k$. From the difference equation (2.16) for $n = 0$ and $n = k$, we get

$$y_1 = \frac{b_0}{1-d-b_1} y_0 + \gamma(0), \quad y_{k+1} = \frac{b_0}{1-d-b_1} y_k + \gamma(k),$$

respectively. Let $u = z+k$ in $\gamma(k)$ given by (2.15). Then we obtain $\gamma(0) = \gamma(k)$. Combining (3.4) and (3.5), we get

$$x_0(t-k) = x_k(t), \quad k \le t < k+1.$$

Moreover, by taking $n = 1$ and $n = k+1$ in (2.16), respectively, and using the same procedure above, we have

$$x_1(t-k) = x_{k+1}(t), \quad k+1 \le t < k+2.$$

By the induction principle, we conclude

$$x_n(t-k) = x_{n+k}(t), \quad k+n \le t < k+n+1.$$

Finally, it can be shown that for $n = 0, -1, -2, \ldots$

$$x_n(t-k) = x_{n+k}(t), \quad k+n \le t < k+n+1.$$

Hence, the proof is complete. $\square$

3.2 Oscillation and Nonoscillation

In this section, we are interested in the following homogeneous case:

$$x'(t) + a(t)x(t) + b(t)x([t]) + c(t)x([t+1]) = 0, \quad t \neq n, \tag{3.6}$$

$$\Delta x(n) = d_n x(n), \quad n \in \mathbb{N} = \{0, 1, 2 \ldots\}, \tag{3.7}$$

$$x(0) = x_0. \tag{3.8}$$

By taking $f(t) = 0$ in (2.2) and (2.3) , we obtain the unique solution of the initial value problem (3.6)–(3.8) as

$$x(t) = \exp\Big(- \int_{[t]}^{t} a(s)\, ds\Big)\Big(1 - \int_{[t]}^{t} b(u) \exp\big(\int_{[t]}^{u} a(s)\, ds\big) du\Big) y_{[t]} - \Big(\int_{[t]}^{t} c(u) \exp\big(\int_{t}^{u} a(s)\, ds\big) du\Big) y_{[t+1]}, \tag{3.9}$$

where

$$y_{[t]} = \Big(\prod_{i=0}^{[t]-1} \alpha(i)\Big) x_0 \tag{3.10}$$

and $\alpha(i)$ is defined in (2.4). In this case, the difference equation (2.7) reduces to

$$y_{n+1} = \frac{\exp\Big(- \int_n^{n+1} a(s)\, ds\Big)\Big(1 - \int_n^{n+1} b(u) \exp\Big(\int_n^u a(s)\, ds\Big) du\Big)}{1 - d_{n+1} + \int_n^{n+1} c(u) \exp\Big(\int_{n+1}^{u} a(s)\, ds\Big) du} \, y_n. \tag{3.11}$$

Definition 2. A function $x(t)$ defined on $[0, \infty)$ is said to be oscillatory if there exist two real valued sequences $(t_n)_{n\geq 0}$, $(t'_n)_{n\geq 0} \subset [0, \infty)$ such that $t_n \to +\infty$, $t'_n \to +\infty$ as $n \to +\infty$ and $x(t_n) \leq 0 \leq x(t'_n)$ for $n \geq N$, where N is sufficiently large. Otherwise, the solution is called nonoscillatory.

Remark 1. According the to Definition 2 given above, a piecewise continuous function $x : [0, \infty) \to \mathbb{R}$ can be oscillatory even if $x(t) \neq 0$ for all $t \in [0, \infty)$. Also, zero function on $[0, \infty)$ is oscillatory with subject to the same definition.

Definition 3. A solution y_n of the difference equation (3.11) is called oscillatory if $y_n y_{n+1} \leq 0$. Otherwise, y_n is called nonoscillatory.

Theorem 6. *Let $x(t)$ be the unique solution of the problem* (3.6)–(3.8) *on* $[0, \infty)$. *Assume that $c(t) > 0$ and $1 - d_{n+1} > 0$.*

(i) If the solution y_n of the problem (3.11) , (2.8) *is oscillatory, then the solution $x(t)$ of* (3.6)–(3.8) *is also oscillatory;*

(ii) If y_n is nonoscillatory, then $x(t)$ is nonoscillatory if and only if

$$\frac{y_{n+1}}{y_n} < \frac{1 - \int_n^t b(u) \exp\Big(\int_n^u a(s)ds\Big) du}{\int_n^t c(u) \exp\Big(\int_n^u a(s)\, ds\Big) du}, \quad n \leq t < n+1, \quad n \geq N', \tag{3.12}$$

where N' is sufficiently large.

Proof. (i) From (3.9), $x(t)$ can be written on the interval $n \leq t < n+1$, $n = 0, 1, 2, \ldots$, as

$$x(t) = \exp\Big(-\int_n^t a(s)ds\Big)\Big(1 - \int_n^t b(u)\exp\Big(\int_n^u a(s)ds\Big)du\Big)y_n \\ - \Big(\int_n^t c(u)\exp\Big(\int_t^u a(s)\,ds\Big)du\Big)y_{n+1}. \tag{3.13}$$

This implies $x(t) = y(n)$ for $t = n$. From the theory of the difference equations it is well known that y_n is oscillatory if and only if $y_n \cdot y_{n+1} \leq 0$ for $n \geq N'$, where N' is a sufficiently large integer. Thus $x(t)$ is an oscillatory solution.
(ii) Now, let y_n be a nonoscillatory solution. According to this, we can assume that $y_n > 0$ for $n \geq N'$ where N' is large enough. If $x(t)$ is a nonoscillatory solution, then we can take $x(t) > 0$ for $t \geq T$ where T is sufficiently large. Hence, from (3.13)

$$x(t) = \exp\Big(-\int_n^t a(s)\,ds\Big)\Big[\Big(1 - \int_n^t b(u)\exp\Big(\int_n^u a(s)\,ds\Big)du\Big)y_n \\ - \Big(\int_n^t c(u)\exp\Big(\int_n^u a(s)\,ds\Big)du\Big)y_{n+1}\Big] \tag{3.14}$$

for $n \geq n'$ where $n' = \max\{N', T\}$. Since $x(t) > 0$ and $\exp(-\int_n^t a(s)ds) > 0$, we have

$$\Big(1 - \int_n^t b(u)\exp\Big(\int_n^u a(s)ds\Big)du\Big)y_n - \Big(\int_n^t c(u)\exp\Big(\int_n^u a(s)ds\Big)du\Big)y_{n+1} > 0,$$

which implies (3.12). Now, let us assume that (3.12) is true. We should show that $x(t)$ is nonoscillatory. For a contradiction assume that $x(t)$ is an oscillatory solution. Therefore, there must exist two sequences $(t_k), (t'_k)$ such that $t_k \to +\infty$, $t'_k \to +\infty$ as $k \to +\infty$ and $x(t_k) \leq 0 \leq x(t'_k)$. From (3.14),

$$x(t_k) = \exp\Big(-\int_{n_k}^{t_k} a(s)\,ds\Big)\Big[\Big(1 - \int_{n_k}^{t_k} b(u)\exp\Big(\int_{n_k}^u a(s)\,ds\Big)du\Big)y_{n_k} \\ - \Big(\int_{n_k}^{t_k} c(u)\exp\Big(\int_{n_k}^{u} a(s)\,ds\Big)du\Big)y_{n_k+1}\Big],$$

where $n_k = [t_k]$ and it is obvious that $n_k \to +\infty$ as $k \to +\infty$. Hence,

$$\Big(1 - \int_{n_k}^{t_k} b(u)\exp\Big(\int_{n_k}^u a(s)\,ds\Big)du\Big)y_{n_k} - \Big(\int_{n_k}^{t_k} c(u)\exp\Big(\int_{n_k}^u a(s)\,ds\Big)du\Big)y_{n_k+1} \leq 0,$$

$$\frac{1 - \int_{n_k}^{t_k} b(u)\exp\Big(\int_{n_k}^u a(s)\,ds\Big)du}{\int_{n_k}^{t_k} c(u)\exp\Big(\int_{n_k}^u a(s)\,ds\Big)du} \leq \frac{y_{n_k+1}}{y_{n_k}},$$

which contradicts to (3.12). The proof is similar when $y_n < 0$ for $n > N'$. $\square$

Theorem 7. *Suppose that* $1-d_{n+1}>0$ *for* $n\in\mathbb{N}$ *and* $c(t)>0$ *for* $t\geq 0$. *Then all solutions of Eqn.* (3.11) *are oscillatory if and only if*

$$\lim_{n\to\infty}\inf\int_n^{n+1} b(u)\exp\Big(\int_n^u a(s)\,ds\Big)du\geq 1. \tag{3.15}$$

Proof. First, assume that all solutions of Eqn. (3.11) are oscillatory. So, every solution y_n of (3.11) satisfies the inequality $y_n y_{n+1}\leq 0$. By using (3.10), we obtain $\alpha(n)\leq 0$, where $\alpha(n)$ is given by (2.4) for $i=n$. Substituting the hypotheses $1-d_{n+1}>0$ and $c(t)>0$ into (2.4), we get

$$1-\int_n^{n+1} b(u)\exp\Big(\int_n^u a(s)\,ds\Big)du\leq 0$$

that leads to (3.15). Now assume that (3.15) is true. Then, we have

$$1-\int_n^{n+1} b(u)\exp\Big(\int_n^u a(s)ds\Big)du\leq 0. \tag{3.16}$$

Taking into account (3.16) together with the hypotheses $1-d_{n+1}>0$ and $c(t)>0$ into (2.4), we obtain $\alpha(n)\leq 0$. So, $y_n y_{n+1}\leq 0$ and this completes the proof. □

Corollary 3. Under the hypotheses of Theorem 7, all solutions of (3.6)–(3.7) are oscillatory.

Now, consider the homogeneous impulsive equation with constant coefficients

$$x'(t)+ax(t)+bx([t])+cx([t+1])=0,\quad t\neq n, \tag{3.17}$$
$$\Delta x(n)=dx(n),\quad n\in\mathbb{N} \tag{3.18}$$

where $a,\ b,\ c,\ d$ are real constants.

Theorem 8. *Let* $a\neq 0,\ c>0$ *and* $1-d>0$. *Then, all solutions of* (3.17)–(3.18) *are oscillatory if and only if* $b\geq a/(e^a-1)$.

Proof. The proof comes out from Theorem 7 and Corollary 3. □

3.3 Global Asymptotic Stability

Theorem 9. *Let* $a\neq 0,\ 1-d-b_1\neq 0$. *Then the zero solution of* (3.17)–(3.18) *is globally asymptotically stable if and only if*

$$|b_0/(1-d-b_1)|<1, \tag{3.19}$$

where b_0 *and* b_1 *are as in* (2.14).

Proof. A solution $x(t)$ of (3.17)–(3.18) is

$$x(t) = m_0(\{t\})y_{[t]} + m_1(\{t\})y_{[t+1]},$$

where $y_{[t]} = (b_0/(1-d-b_1))^{[t]}\, x_0$,

$$m_0(\{t\}) = e^{-a\{t\}} + (e^{-a\{t\}} - 1)ba^{-1}, \quad m_1(\{t\}) = (e^{-a\{t\}} - 1)ca^{-1}. \tag{3.20}$$

Since $\{t\} \in [0,1)$, $e^{-a\{t\}} \le e^{|a|}$. It means that $m_0(\{t\})$ and $m_1(\{t\})$ are bounded on the interval $0 \le t < \infty$. Therefore, $y_{[t]} \to 0$ as $t \to +\infty$ when (3.19) holds. So $\lim\limits_{t\to+\infty} x(t) = 0$. Conversely, if $\lim\limits_{t\to+\infty} x(t) = 0$, then $y_{[t]} \to 0$ as $t \to +\infty$ which implies (3.19). □

The next theorem gives only a necessary condition for the global asymptotic stability of zero solution of (3.17)–(3.18).

Theorem 10. *Let $a \ne 0,\ c > 0$ and $1 - d > 0$. If the zero solution of* (3.17)–(3.18) *is globally asymptotically stable, then*

$$\frac{a}{e^a - 1} + \frac{ae^a}{e^a - 1}(d-1) - c < b < \frac{a}{e^a - 1} - \frac{ae^a}{e^a - 1}(d-1) + c. \tag{3.21}$$

Proof. Let the solution $x = 0$ of (3.17)–(3.18) be globally asymptotically stable. Therefore, (3.19) is satisfied. Substituting $b_0 = e^{-a} + (e^{-a} - 1)ba^{-1}$ and $b_1 = (e^{-a} - 1)ca^{-1}$ into (3.19), we obtain (3.21). □

4 Examples

In this section, we give some examples to illustrate our results.

Example 1. Let us consider the initial value problem

$$x'(t) + x(t) + 2x([t]) + 3x([t+1]) = \sin 2\pi t,\ t \ne n, \tag{4.1}$$

$$\Delta x(n) = 2x(n), \quad n \in \mathbb{N}, \quad x(0) = x_0, \tag{4.2}$$

that is a special case of (2.9)–(2.10) with $a = 1,\ b = 2,\ c = 3,\ d = 2$ and $f(t) = \sin 2\pi t$, Here, $a \ne 0$, $1 - d - b_1 = 2e - 3/e \ne 0$ and $f(t+1) = f(t)$. Moreover, by (2.12), $y_2 = y_0$. Thus all hypotheses of Theorem 5 are verified. So, by Theorem 5, the problem (4.1)–(4.2) has a unique 2−periodic solution. Indeed, this solution is

$$x(t) = \big(3e^{-\{t\}} - 2\big)y_{[t]} + 3\big(e^{-\{t\}} - 1\big)y_{[t+1]} + \int_{[t]}^{t} e^{-(t-u)} \sin 2\pi u\, du \tag{4.3}$$

and it is easy to check that $x_n(t-2) = x_{n+2}(t)$, $n+2 \le t < n+3, n = 0,1,2,\dots,$ where

$$y_{[t]} = (-1)^{[t]} y_0 + \frac{e}{2e-3} \sum_{j=0}^{[t]-1} (-1)^{[t]-j-1} \int_j^{j+1} e^{-(j+1-u)} \sin 2\pi u\, du.$$

Example 2. Consider the equation

$$x'(t) + x(t) + \frac{2}{e-1} x([t]) + x([t+1]) = 0, \ t \neq n, \tag{4.4}$$

$$\Delta x(n) = \frac{1}{2} x(n), \quad n \in \mathbb{N}, \tag{4.5}$$

that is a special case of (3.17)–(3.18) with $a = 1$, $b = \dfrac{2}{e-1}$, $c = 1$ and $d = \dfrac{1}{2}$. Since all hypotheses of Theorem 8 are satisfied for (4.4)–(4.5), every solution of this equation is oscillatory. Indeed, a solution $x(t)$ of (4.4)–(4.5) that satisfies the initial condition $x(0) = x_0 > 0$ has the form

$$x(t) = \Big(\Big(\frac{e+1}{e-1} e^{-\{t\}} - \frac{2}{e-1} \Big) \Big(\frac{2}{2-3e} \Big)^{[t]} + \Big(e^{-\{t\}} - 1 \Big) \Big(\frac{2}{2-3e} \Big)^{[t+1]} \Big) x_0.$$

Here, it is possible to choose two sequences as $(t_n) = (2n)$ and $(t'_n) = (2n+1)$ that lead to the inequality $x(2n+1) < 0 < x(2n)$. If $x_0 < 0$, then $x(2n) < 0 < x(2n+1)$. So, by Definition 2, we deduce that $x(t)$ is oscillatory. On the other hand, all assumptions of Theorem 9 are satisfied for (4.4)–(4.5). Because, $a = 1 \neq 0$, $1 - d - b_1 = \dfrac{3e-2}{2e} \neq 0$ and $\dfrac{b_0}{1-d-b_1} = \dfrac{2}{2-3e} \in (-1, 0)$. Thus the zero solution of (4.4)–(4.5) is globally asymptotically stable. As a conclusion, it can be said that any solution of (4.4)–(4.5) goes to zero as $t \to +\infty$ by oscillating.

Example 3. Now, consider the equation

$$x'(t) + x(t) - \frac{3}{2} x([t]) + x([t+1]) = 0, \quad t \neq n, \tag{4.6}$$

$$\Delta x(n) = \frac{1}{2} x(n), \quad n \in \mathbb{N}, \tag{4.7}$$

that is also a special case of (3.17)–(3.18) with $a = 1$, $b = -\dfrac{3}{2}$, $c = 1$, $d = \dfrac{1}{2}$. In this case, $\dfrac{b_0}{1-d-b_1} = \dfrac{3e-1}{3e-2} > 1$. Therefore, the condition (3.19) is not fulfilled. So, due to Theorem 9, the zero solution of (4.6)–(4.7) is not globally asymptotically stable. Indeed, a solution $x(t)$ of (4.6)–(4.7) has the form

$$x(t) = \Big(\Big(\frac{3}{2} - \frac{1}{2} e^{-\{t\}} \Big) \Big(\frac{3e-1}{3e-2} \Big)^{[t]} + \Big(e^{-\{t\}} - 1 \Big) \Big(\frac{3e-1}{3e-2} \Big)^{[t+1]} \Big) x_0$$

and for a sequence such as $(t_n) = (n)$ we have

$$\lim_{t \to +\infty} x(t) = \lim_{n \to +\infty} x(n) = +\infty.$$

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