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On the Solutions of Nonlinear Third Order Boundary Value Problems*

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Abstract. The author considers a three–point third order boundary value problem. Properties and the structure of solutions of the third order equation are discussed. Also, a connection between the number of solutions of the boundary value problem and the structure of solutions of the equation is established.

Keywords: nonlinear boundary value problems, structure of solutions, multiplicity of solutions, branches of initial values.

AMS Subject Classification: 34B15.

1 Introduction

Nonlinear boundary value problems arise in a variety of different areas of applied mathematics and physics. The theory of nonlinear boundary value problems is a very actual part of nonlinear analysis since it is aimed to applications. There are a lot of investigations on this subject. The classical results in the theory of nonlinear boundary value problems concern the existence and uniqueness of solutions. Let us mention books by P. Bailey, L. Shampine, P. Waltman [2], S. Bernfeld and V. Lakshmikantham [3], N.I. Vasilyev and Yu.A. Klokov [11], and modern treatises by C. Coster, P. Habets [4], W. Kelley, A. Peterson [7]. The more complicated, more actual questions are about the number of solutions of boundary value problems, of their properties etc. This type problems are insufficiently investigated in the literature even for the second order problems. There are few results on multiple solutions of the third order nonlinear problems. Results concerning two-point third order nonlinear boundary value problems were obtained by E. Rovderova [9], F. Sadyrbaev [10]. In [9] the author states some results on the number of solutions of two-point boundary value problems. In [10] the author established multiplicity results for certain classes

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of third order nonlinear boundary value problems. His approach was based on the Hanan's theory [5] of conjugate points for third order linear differential equations.

In the present paper we consider a three-point third order boundary value problem. The similar type problems were investigated by P. Hartman in [6] with respect to existence of a solution. The focal three–point boundary value problem was considered in [1] where the authors evaluated the number of solutions. The main object of investigation in our paper is three–point boundary value problem

$$x''' = f(x), \tag{1.1}$$

$$x(a) = x(b) = x(c) = 0, \quad a < b < c, \tag{1.2}$$

where $f(x)$ is strictly increasing, continuous function such that $f(0) = 0$. By a solution of (1.1) we mean $C^3(I)$ function $x(t)$, which satisfies the equation.

We are looking for multiple solutions of the problem (1.1), (1.2). The idea is based on the following argumentation. First, consider the auxiliary initial value problem

$$\begin{aligned} &x''' = f(x), \\ &x(b) = 0, \ \ x'(b) = \alpha, \ \ x''(b) = \beta. \end{aligned} \tag{1.3}$$

Let M^+ be a set of (α, β) such, that the solution of (1.3) vanishes at $t = c$. Let M^- be a set of (α, β) such, that the solution of (1.3) vanishes at $t = a$. If $M^+ \cap M^- \neq \emptyset$, then the problem (1.1), (1.2) has solutions. We can estimate the number of solutions if properties of the set $M^+ \cap M^-$ are known.

In our investigation we consider the structure and properties of solutions of equation (1.1). We are interested in solutions which vanish at $t = b$ and oscillate for $t > b$ and $t < b$. We show that the structure of solutions is similar to that of solutions of the linear equation

$$x''' = x. \tag{1.4}$$

Linear equations of type (1.4) were intensively studied by Hanan [5].

Section 2 contains some auxiliary results. In Section 3 we consider some important properties of solutions of the equation (1.1). In Section 4 we investigate the structure of solutions. Section 5 deals with the three-point boundary value problem, we state and prove some propositions and one theorem.

2 Preliminaries

Proposition 1. *Suppose $x(t) \in C^3(I)$. If the values $x(a)$, $x'(a)$, $x''(a)$ are non-negative (but not all zero) and $x'''(t)$ is positive if $x(t)$ is positive, then the functions $x(t)$, $x'(t)$, $x''(t)$ are positive for $t > a$.*

Remark 1. This result is analogous to Lemma 2.1 from [8], but it is true for general functions not only for solutions of linear differential equations.

Proof. Let $x(a) \geq 0$, $x'(a) \geq 0$, $x''(a) \geq 0$ and $(x(a))^2+(x'(a))^2+(x''(a))^2 > 0$. Then in all cases $x(t)$ will be positive in some open interval whose left boundary point is $t = a$. Suppose that $x(t)$, $x'(t)$, $x''(t)$ are not positive for all t such that $t > a$ and there exists a point $t = t_0$ such that $x(t_0) = 0$ and $x(t) > 0$ for $a < t < t_0$. Since $x(t_0) = 0$, there will exist a point $t = t_1$, $a \leq t_1 < t_0$ such that $x'(t_1) = 0$ and $x'(t) < 0$ for $t_1 < t < t_0$. Since $x'(t_1) = 0$, there will exist a point $t = t_2$, $a \leq t_2 < t_1$ such that $x''(t_2) = 0$ and $x''(t) < 0$ for $t_2 < t < t_1$. Since $x'''(t) > 0$ for $x(t) > 0$, it follows that $x'''(t) > 0$ for $a < t < t_0$.

Consider

$$x''(t) = x''(a) + \int_a^t x'''(s)\,ds, \quad a < t \leq t_0.$$

The right-hand side is positive, and increases in t, as long as $x'''(t)$ remains positive. We thus conclude that $x''(t)$ is positive for $a < t \leq t_0$. Next consider

$$x'(t) = x'(a) + \int_a^t x''(s)\,ds, \quad a < t \leq t_0.$$

The right-hand side is positive, and increases in t, as long as $x''(t)$ remains positive. We thus conclude that $x'(t)$ is positive for $a < t \leq t_0$. Consider

$$x(t) = x(a) + \int_a^t x'(s)\,ds, \quad a < t \leq t_0.$$

The right-hand side is positive, and increases in t, as long as $x'(t)$ remains positive. We thus conclude that $x(t)$ is positive for $a < t \leq t_0$. These contradictions prove the proposition. □

Proposition 2. *Suppose $x(t) \in C^3(I)$. If the values $x(a)$, $x'(a)$, $x''(a)$ are non-positive (but not all zero) and $x'''(t)$ is negative if $x(t)$ is negative, then the functions $x(t)$, $x'(t)$, $x''(t)$ are negative for $t > a$.*

The proof is analogous to the prove of Proposition 1.

Remark 2. Function $x(t)$ from Propositions 1 and 2 may be thought as a solution of differential equation (1.1).

3 Properties of Solutions

Corollary 1. If $x(t)$ is a nontrivial solution of (1.1) and $x(a) = x(b) = 0$, $a < b$, then $x'(a) \neq 0$.

Proof. Assume $x'(a) = 0$, and, without loss of generality, let $x''(a) > 0$. Then, by the Proposition 1 $x(t) > 0$ for $t > a$. But $x(b) = 0$, $a < b$. The contradiction proves the corollary. □

Corollary 2. If $x(t)$ is a nontrivial solution of (1.1) and $x(a) = x(b) = 0$ $(a < b)$, then $x'(a) \cdot x''(a) < 0$.

Proof. Assume $x'(a) \cdot x''(a) \geq 0$. If $x'(a) \geq 0$, $x''(a) \geq 0$, then, by the Proposition 1 $x(t) > 0$ for $t > a$. We have a contradiction, since $x(b) = 0$.

If $x'(a) \leq 0$, $x''(a) \leq 0$, then, by the Proposition 2 $x(t) < 0$ for $t > a$. We have a contradiction, since $x(b) = 0$. $\square$

Proposition 3. *If $x(t)$ is a nontrivial solution of* (1.1) *and $x(a) = 0$, then the function $x(t)$ changes sign for $t < a$.*

Proof. First let $x'(a) = 0$. Without loss of generality, let $x''(a) > 0$. Assume that $x(t)$, $x'(t)$, $x''(t)$ do not change sign for $t < a$, that is $x(t) > 0$, $x'(t) < 0$, $x''(t) > 0$ for $t < a$. Then $x'''(t) > 0$ for $t < a$. Consider

$$x''(t) = x''(a) - \int_t^a x'''(s)\,ds.$$

Obviously, there exists $t_1 < a$ such, that

$$\int_{t_1}^a x'''(s)\,ds > x''(a).$$

Thus $x''(t) < 0$ for $t < t_1$. Consider

$$x'(t) = x'(t_1) - \int_t^{t_1} x''(s)\,ds.$$

Obviously, there exists $t_2 < t_1$ such, that

$$\int_{t_2}^{t_1} x''(s)\,ds < x'(t_1).$$

Thus $x'(t) > 0$ for $t < t_2$. Consider

$$x(t) = x(t_2) - \int_t^{t_2} x'(s)\,ds.$$

Obviously, there exists $t_3 < t_2$ such, that

$$\int_{t_3}^{t_2} x'(s)\,ds > x(t_2).$$

Thus $x(t) < 0$ for $t < t_3$. Let $x'(a) \neq 0$. Assume that $x(t)$ does not change sign for $t < a$. Four cases are possible.

1. $x'(a) > 0$, $x''(a) \geq 0$. Thus $x(t) < 0$ for $t < a$.
2. $x'(a) > 0$, $x''(a) \leq 0$. Thus $x(t) < 0$ for $t < a$.
3. $x'(a) < 0$, $x''(a) \geq 0$. Thus $x(t) > 0$ for $t < a$.
4. $x'(a) < 0$, $x''(a) \leq 0$. Thus $x(t) > 0$ for $t < a$.

We will give the prove for the first case. For the other cases the proof is similar. Since $x(t) < 0$ for $t < a$, then $x'''(t) < 0$ for $t < a$. Consider

$$x''(t) = x''(a) - \int_t^a x'''(s)\,ds.$$

The right-hand side is positive, hence $x''(t)$ is positive for $t < a$. Consider

$$x'(t) = x'(a) - \int_t^a x''(s)\,ds.$$

Obviously, there exists $t_1 < a$ such, that

$$\int_{t_1}^a x''(s)\,ds > x'(a).$$

Thus $x'(t) < 0$ for $t < t_1$. Consider

$$x(t) = x(t_1) - \int_t^{t_1} x'(s)\,ds.$$

Obviously, there exists $t_2 < t_1$ such, that

$$\int_{t_2}^{t_1} x'(s)\,ds < x(t_1).$$

Thus $x(t) > 0$ for $t < t_2$. These contradictions prove the proposition. □

Corollary 3. If $x(t)$ is a nontrivial solution of (1.1) and $t = t_0$ is a simple or a double zero of $x(t)$, then $x(t)$ has an infinity of simple zeros in $(-\infty, t_0)$. If $t = t_0$ is a double zero of $x(t)$, then $x(t)$ does not vanish in $(t_0, +\infty)$.

Proposition 4. *Assume that $x(t)$ is a solution of* (1.1) *and $x(a) = 0$, $x'(a) = \alpha > 0$, $x''(a) = \beta < 0$. If $\overline{x}(t)$ is a solution of* (1.1) *such that $\overline{x}(a) = 0$, $\overline{x}'(a) = \alpha > 0$, $\overline{x}''(a) = \overline{\beta} < 0$ and $\overline{\beta} > \beta$, then $\overline{x}(t) > x(t)$, $\overline{x}'(t) > x'(t)$, $\overline{x}''(t) > x''(t)$ for $t > a$.*

Proof. Consider the auxiliary function $z(t) = \overline{x}(t) - x(t)$. Obviously $z(a) = 0$, $z'(a) = 0$, $z''(a) = \overline{\beta} - \beta > 0$ and $z'''(t) > 0$, since $f(x)$ is an increasing function. Thus, by Proposition 1, we get that $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$ for $t > a$. Hence $\overline{x}(t) > x(t)$, $\overline{x}'(t) > x'(t)$, $\overline{x}''(t) > x''(t)$ for $t > a$. □

Proposition 5. *Assume that $x(t)$ is a solution of* (1.1) *and $x(a) = 0$, $x'(a) = \alpha > 0$, $x''(a) = \beta < 0$. If $\overline{x}(t)$ is a solution of* (1.1) *such that $\overline{x}(a) = 0$, $\overline{x}'(a) = \overline{\alpha} > 0$, $\overline{x}''(a) = \beta < 0$ and $\overline{\alpha} > \alpha$, then $\overline{x}(t) > x(t)$, $\overline{x}'(t) > x'(t)$, $\overline{x}''(t) > x''(t)$ for $t > a$.*

The proof is analogous to the proof of Proposition 4.

4 Structure of Solutions

Consider the auxiliary initial value problem (1.3).

Proposition 6. *For any $k = 1, 2, \dots$ there exists a continuous branch Γ_k of initial values (α, β) such that the respective initial value problem* (1.3) *has a solution $x_k(t, \alpha, \beta)$ with a double zero at some point $\tau_k(\alpha, \beta)$ and $k-1$ simple zeros in $(b; \tau_k)$.*

Proof. Consider the auxiliary Cauchy problem (1.1) with the following initial conditions

$$x(\eta) = x'(\eta) = 0, \quad x''(\eta) = \xi > 0.$$

The solutions continuously depend on ξ. These solutions have only simple zeros to the left of $t = \eta$ (see, Corollary 3). Let us denote zeros to the left of $t = \eta$ by $-\infty < \dots < t_2 < t_1 < \eta$. Let $x'(t_1) = \alpha$, $x''(t_1) = \beta$, η be fixed and ξ is varied. Initial values (α, β) at $t = t_1$ continuously vary with ξ. So Γ_1 (the set of all respective (α, β)) is a continuous one-parametric curve. As the equation (1.1) is autonomous, we can identify t_1 with $t = b$ and Γ_1 does not change. For the rest branches Γ_i the proof is similar. □

Remark 3. $x(t) \neq 0$ for $t > \tau_k$ (see, Corollary 3).

Proposition 7. *Branches Γ_n and Γ_m cannot intersect unless $n = m$.*

Proof. The proof follows from the existence and uniqueness theorem. □

Proposition 8. *For any $k = 0, 1, 2, \dots$ there exists a set F_k of initial values (α, β) such that the respective initial value problem* (1.3) *has a solution $x_k(t, \alpha, \beta)$ with exactly k simple zeros in $(b; \tau_k)$ and no zeros in $(\tau_k; +\infty)$.*

Consider the initial value problem (1.3), where $\alpha > 0$ is arbitrary and $\beta = \beta_0 < 0$ is fixed. Let α decrease from $+\infty$. If α is sufficiently large, then the solution $x(t, \alpha, \beta_0)$ does not vanish in $(b; +\infty)$. Then there exists such $\alpha = \alpha_1$ that the solution $x_1(t, \alpha_1, \beta_0)$ has a double zero at some point $\tau_1(\alpha_1; \beta_0)$ and no zero in $(b; \tau_1)$. Analogously, there exists such $\alpha = \alpha_3$ that the solution $x_3(t, \alpha_3, \beta_0)$ has a double zero at some point $\tau_3(\alpha_3; \beta_0) > \tau_1(\alpha_1; \beta_0)$ and two simple zeros in $(b; \tau_3)$. Then there exists such $\alpha = \alpha_5$ that the solution $x_5(t, \alpha_5, \beta_0)$ has a double zero at some point $\tau_5(\alpha_5; \beta_0) > \tau_3(\alpha_3; \beta_0)$ and four simple zeros in $(b; \tau_5)$. We can continue this process.

Let α increase from zero. If α is sufficiently small, then the solution $x(t, \alpha, \beta_0)$ has exactly one simple zero in $(b; \tau_1)$ and no zero in $(\tau_1; +\infty)$. Then there exists such $\alpha = \alpha_2$ that the solution $x_2(t, \alpha_2, \beta_0)$ has a double zero at some point $\tau_2(\alpha_2, \beta_0)$ $(\tau_1 < \tau_2 < \tau_3)$ and one simple zero in $(b; \tau_2)$. If t_1 is a simple zero of $x_2(t, \alpha_2, \beta_0)$, then $b < t_1 < \tau_1$. Then there exists such $\alpha = \alpha_4$ that the solution $x_4(t, \alpha_4, \beta_0)$ has a double zero at some point $\tau_4(\alpha_4, \beta_0)$ $(\tau_3 < \tau_4 < \tau_5)$ and three simple zeros in $(b; \tau_4)$. If $t_1,\ t_2,\ t_3$ are simple zeros of $x_4(t, \alpha_4, \beta_0)$, then $b < t_1 < \tau_1 < t_2 < \tau_2 < t_3 < \tau_4$. We can continue this process.

Thus, let t_n is the n-th simple zero of $x_k(t, \alpha_k, \beta_0)$ in $(b; \tau_k)$, then $\tau_{n-1} < t_n < \tau_n$, $n = 1, 2, \dots, k$, $k \in \mathbb{N}$.

Let $\alpha \in (\alpha_3; \alpha_1)$, then the solution $x(t, \alpha, \beta_0)$ has exactly two simple zeros in $(b; \tau_2)$ and no zero in $(\tau_2; +\infty)$. If $t_1,\ t_2$ are simple zeros of $x(t, \alpha, \beta_0)$, then $b < t_1 < \tau_1 < t_2 < \tau_2$. Let $\alpha \in (\alpha_2; \alpha_4)$, then the solution $x(t, \alpha, \beta_0)$ has exactly three simple zeros in $(b; \tau_3)$ and no zero in $(\tau_3; +\infty)$. If $t_1,\ t_2,\ t_3$ are simple zeros of $x(t, \alpha, \beta_0)$, then $b < t_1 < \tau_1 < t_2 < \tau_2 < t_3 < \tau_3$. We can continue this process. A structure of the solution is presented in Fig. 1.

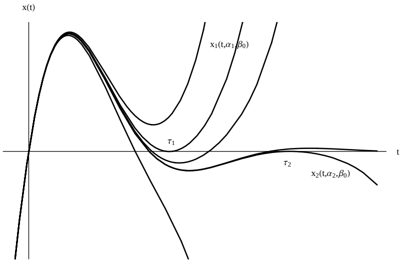

Figure 1. Structure of the solutions of (1.3), $\alpha > 0$ is arbitrary, $\beta = \beta_0 < 0$ is fixed.

Remark 4. Analogous argumentation is valid, if we chose $\alpha = \alpha_0 > 0$ fixed and let $\beta < 0$ to be arbitrary.

Remark 5. It follows that Γ_i are monotone curves.

Remark 6. Suppose, that $f(x) = kx$. If we change $\alpha^2 + \beta^2$ (staying on a fixed branch Γ_i), then the points $\tau_1,\ \tau_2, \ldots$ do not change the location (because of the linear dependence of solutions). These points are called conjugate points [5]. If $f(x)$ is not a linear function, the points $\tau_1,\ \tau_2, \ldots$ can change their location.

Let $l_k(\alpha, \beta)$ be simple zeros of the initial value problem (1.3) to the left of $t = b$.

Definition 1. We shall say that the function $f(x)$ in (1.1) has the property P^{sup}, if the following statement holds: let $(\alpha, \beta) \in \Gamma_k$ and $\alpha^2 + \beta^2$ tends to infinity, then $\tau_k(\alpha, \beta)$ and $l_k(\alpha, \beta)$ continuously and monotonically tend to $t = b$ and if $\alpha^2 + \beta^2$ tends to zero, then $\tau_k(\alpha, \beta)$ and $l_k(\alpha, \beta)$ continuously and monotonically tend to infinity.

Definition 2. We shall say that the function $f(x)$ from (1.1) has the property P_{sub}, if the following statement holds: let $(\alpha, \beta) \in \Gamma_k$ and $\alpha^2 + \beta^2$ tends to infinity, then $\tau_k(\alpha, \beta)$ and $l_k(\alpha, \beta)$ continuously and monotonically tend to infinity and if $\alpha^2 + \beta^2$ tends to zero, then $\tau_k(\alpha, \beta)$ and $l_k(\alpha, \beta)$ continuously and monotonically tend to $t = b$.

5 Three–Point Boundary Value Problem

Let Z_k^r $(k = 1, 2, \ldots)$ be a branch of initial values (α, β) such that the respective initial value problem (1.3) has a solution $x(t, \alpha, \beta)$ which vanishes at $t = c$ and has $(k - 1)$ simple zeros in (b, c). $Z_k^r \subset M^+$ and $\cup Z_k^r = M^+$.

Let Z_k^l $(k = 1, 2, \ldots)$ be a branch of initial values (α, β) such that the respective initial value problem (1.3) has a solution $x(t, \alpha, \beta)$ which vanishes at $t = a$ and has $k-1$ simple zeros in (a, b). $Z_k^l \subset M^-$ and $\cup Z_k^l = M^-$. So Z_k^l and Z_k^r are subsets of the fourth quadrant of the plane (α, β).

If $Z_k^r \cap Z_m^l \neq \emptyset$ for some integers k and m, then the problem (1.1), (1.2) has a solution with (α, β) belonging to the above intersection. The number of these intersections is the number of solutions of the main problem.

So knowledge of properties of the sets Z_k^r and Z_m^l is essential for solvability of the problem (1.1), (1.2).

Proposition 9. *If the right side function in* (1.1) *has the property* P^{sup} *(or* P_{sub}*), then the set* Z_k^l *is a continuous line connecting the rays* $\varphi = -\pi/2$ $(\beta \in (-\infty, 0),\ \alpha = 0)$ *and* $\varphi = 0$ $(\alpha \in (0, +\infty),\ \beta = 0)$ *for any* $k = 1, 2, \ldots$. *For any* $M > 0$ *there exists integer* k *such that* Z_k^l *are in the outer region with respect to the sector with radius* M*.*

Proof. We will prove the proposition for the P^{sup} case, for P_{sub} case the proof is analogous. Let choose angle φ close to $\varphi = -\pi/2$. Denote by r_φ the ray emanating from the origin in direction of angle φ (in the fourth quadrant). By property P^{sup} there are countably many points P_k on the ray with the properties: for $P_k(\alpha_k, \beta_k)$ $k = 1, 2, \ldots$ the respective solution of (1.3) $x(t, \alpha_k, \beta_k)$ has exactly $k-1$ zeros in the interval (a, b) and $x(a, \alpha_k, \beta_k) = 0$. Consider now the family of rays r_φ parameterized by the angle $\varphi \in (-\pi/2, 0)$. We got a sequence $P_k(\varphi)$ for any angle φ. It is evident that the $Z_k^l = \cup_\varphi P_k(\varphi)$ is a continuous line connecting the ray $\varphi = -\pi/2$ and $\varphi = 0$ for any $k = 1, 2, \ldots$ and for any $M > 0$ there exists integer k such that Z_k^l are in the outer region with respect to the sector with radius M. $\square$

Proposition 10. *If the right side function in* (1.1) *has the property* P^{sup} *(or* P_{sub}*), then the set* $\cup_k Z_k^r$ *is a continuous line which emanates from the origin and stretches to infinity (in the fourth quadrant).*

Proof. We will prove the proposition for the P^{sup} case, for P_{sub} case the proof is analogous. Let choose radius ρ close to zero (ρ is so small that initial conditions (α, β) corresponding to any double zero of solutions of (1.3) are in the outer region with respect to the sector with radius ρ). We parameterize the respective arc by the angle $\varphi \in (-\pi/2, 0)$. We know that for small enough $\varphi < 0$ the solution $x(t, \alpha, \beta)$ has no zeros at all for $t > 0$. A point of intersection with Γ_1 gives a solution with the double zero $\tau_1 > c$. We continue this process and for some φ_2 the arc intersects Γ_2 $(\tau_2 > c)$. Now consider φ between $-\pi/2$ and φ_2. Any solution with the initial values α, β on the arc has exactly one zero $t_1(\varphi)$. It is evident that $t_1(\varphi) \to 0$ as $\varphi \to -\pi/2$. It follows also that $t_1(\varphi) = c$ for some $\varphi \in (-\pi/2, \varphi_2)$ where φ_2 is angle corresponding to point of intersection with Γ_2. Let increase radius ρ. There exists ρ such that the respective solution has a double zero at $t = c$ and no zeros in (b, c). Thus, we got the set Z_1^r with the property: any solution with $(\alpha, \beta) \in Z_1^r$ has the first zero at $t = c$. Repeating this process with respect to the second zeros we got the set Z_2^r with the property: any solution with $(\alpha, \beta) \in Z_2^r$ has the second zero at $t = c$.

On the branches Γ_k ($k = 0, 1 \ldots$) mark the points $T_k(\alpha_k; \beta_k)$ such, that the solution of (1.3) $x_k(t, \alpha_k, \beta_k)$ has a double zero at $t = c$ and $k-1$ simple zeros in $(b; c)$. Obviously, if $f(x)$ has the property P^{sup}, then $\alpha_k^2 + \beta_k^2 < \alpha_{k+1}^2 + \beta_{k+1}^2$ for any $k = 1, 2, \ldots$. Z_1^r emanates from the origin (origin does not belong to any branch), goes through the region F_1, then intersects all branches Γ_k (and regions between them) and ends in the point T_1. Z_2^r emanates from the point T_1, intersects all branches Γ_k between Γ_1 and Γ_2 and ends in the point T_2. Generally, Z_k^r emanates from the point T_{k-1}, intersects all branches Γ_k between Γ_{k-1} and Γ_k and ends in the point T_k (see, Fig.2). So, the branch $\cup Z_k^r = M^+$ is continuous line which emanates from the origin and tends to the minus infinity. □

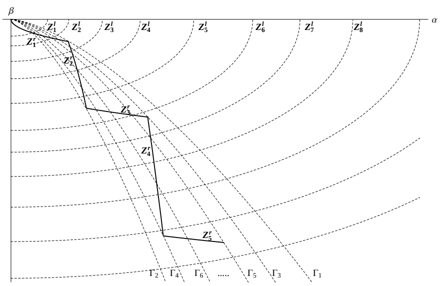

Figure 2. Sets Z_k^l and Z_k^r for P^{sup} function (schematically).

Theorem 1. *Suppose, that the right side function $f(x)$ in* (1.1) *has the property P^{sup} (or P_{sub}). Then the problem* (1.1), (1.2) *has a countable set of solutions.*

Proof. It follows from Propositions 9 and 10 that any set Z_n^l intersects $\cup_i Z_i^r$. Sets Z_n^l are countably many. Any point of intersection by definitions of the sets Z_i^r and Z_n^l corresponds to the solution of boundary value problem (1.1), (1.2). Hence we get the proof. □

Remark 7. Any point of intersection of Z_n^l with $\cup_i Z_i^r$ yields a solution to the problem (1.1), (1.2). Suppose this point belongs to $Z_n^l \cap Z_m^r$ for certain n and m, then the corresponding solution $x(t)$ has exactly $(m-1)$ simple zeros in (b, c) and $(n-1)$ simple zeros in (a, b).

Example 1. Consider the problem

$$x''' = x^3, \quad x(-1) = x(0) = x(1) = 0.$$

Numerical experiments show that the branches Z_1^l and Z_1^r intersect at the point $(9.85, -21.2) \in F_1$, branches Z_2^l and Z_1^r intersect at the point $(26.8, -73) \in F_1$, branches Z_3^l and Z_1^r intersect at the point $(38.18, -117.83) \in F_1$ (coordinates of points are given approximately). The respective solutions are depicted in. Fig. 3. So it is shown that the sets Z_i^r may intersect with multiple sets Z_j^l. The converse is not true.

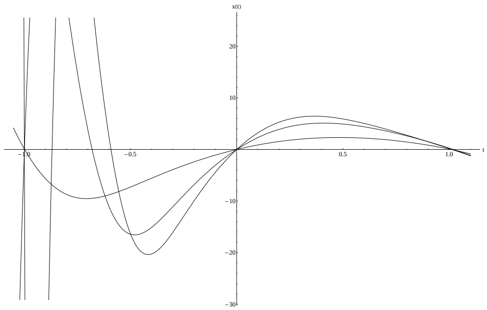

Figure 3. Some solutions of the problem $x''' = x^3$, $x(-1) = x(0) = x(1) = 0$.

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