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Construction of Chaotic Dynamical System

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Abstract. The first-order difference equation $x_{n+1} = f(x_n)$, $n = 0, 1, \ldots$, where $f\colon \mathbf{R} \to \mathbf{R}$, is referred as an one-dimensional discrete dynamical system. If function f is a chaotic mapping, then we talk about chaotic dynamical system. Models with chaotic mappings are not predictable in long-term. In this paper we consider family of chaotic mappings in symbol space Σ_2. We use the idea of topological semi-conjugacy and so we can construct a family of mappings in the unit segment such that it is chaotic.

Keywords: chaotic mapping, infinite symbol space, increasing mapping, topological semi-conjugacy, binary expansion.

AMS Subject Classification: 37B10; 37B05; 37C15.

1 Introduction

The theory of discrete dynamical systems and difference equations developed greatly during the last twenty five years of the twentieth century, following the T.Y. Li and J.A. Yorke paper [9] in 1975. In 1986 R. Devaney published [4], the first book on the subject. Discrete dynamical systems are applied in many scientific disciplines (for example, in biology, see article [11]).

Starting from a point x_0, one may generate the sequence

$$x_0,\ f(x_0),\ f(f(x_0)) = f^2(x_0),\ \ldots, f(f^n(x_0)) = f^n(x_0),\ \ldots$$

This iterative procedure is an example of a discrete dynamical system. If we let $x(n) = f^n(x_0)$, then we obtain the first-order difference equation

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots.$$

We may conclude that difference equations and discrete dynamical systems represent two sides of the same coin. Difference equations represent analytic theory of the subject but discrete dynamical systems represent its geometrical and topological aspects. If we consider the logistic difference equation

$$x_{n+1} = h_4(x_n) = 4x_n(1 - x_n)$$

or the doubling equation

$$x_{n+1} = D(x_n) = 2x_n \pmod 1,$$

then we know that mappings h_4 and D are chaotic in $[0;1]$. In 1994 H.O. Peitgen, H. Juergen, D. Saupe [12] have shown that doubling mapping D is topological semi-conjugate with the shift map in one-sided infinite sequences space Σ_2. The shift map is chaotic too. We use this idea of topological semi-conjugacy and show a possibility of construction of chaotic discrete dynamical systems. Models with chaotic mappings are not predictable in long-term.

2 Basic Definitions

Definition 1. ([7, 13]). The set of all infinite sequences of symbols 0 and 1 is called *the symbol space of* 0 *and* 1 and is denoted by Σ_2, i.e.,

$$\Sigma_2 = \left\{ s_0 s_1 s_2 \ldots \,\middle|\, s_i = 0 \text{ or } s_i = 1,\ i = 0, 1, 2, \ldots \right\}.$$

We introduce a metric structure on Σ_2 by

$$\forall s = s_0 s_1 s_2 \ldots,\ t = t_0 t_1 t_2 \ldots \in \Sigma_2\colon \quad d(s,t) = \sum_{i=0}^{+\infty} \frac{|s_i - t_i|}{2^i}.$$

The space (Σ_2, d) has many specific and interesting properties (see, [7, 10, 14]).

Let (X, ρ) be metric space.

Definition 2. (Devaney, [4]) The function $f\colon X \to X$ is *chaotic* if a) the periodic points of f are dense in X, b) f is topologically transitive, c) f exhibits sensitive dependence on initial conditions.

Devaney's definition is not the unique classification of a chaotic map. For example, another definition can be found in Robinson [13]. Also mappings with only one property, i.e. sensitive dependence on initial conditions, frequently are considered as chaotic (see, Gulick [5]).

Definition 3. ([4, 6, 7, 13]). The function $f\colon X \to X$ is *topologically transitive* on X if

$$\forall x, y \in X \ \ \forall \varepsilon > 0 \ \ \exists z \in X \ \ \exists n \in \mathbf{N}\colon \ \ \rho(x,z) < \varepsilon \ \&\ \rho(f^n(z), y) < \varepsilon.$$

Definition 4. ([4, 6, 7, 13]). The function $f\colon X \to X$ *exhibits sensitive dependence on initial conditions* if

$$\exists \delta > 0 \ \ \forall x \in X \ \ \forall \varepsilon > 0 \ \ \exists y \in X \ \ \exists n \in \mathbf{N} \qquad \rho(x,y) < \varepsilon \ \&\ \rho(f^n(x), f^n(y)) > \delta.$$

Definition 5. ([4, 6, 7, 13]). Let $A, B \subseteq X$ and $A \subseteq B$. Then A is *dense* in B if for each point $x \in B$ and each $\varepsilon > 0$, there exists $y \in A$ such that $d(x, y) < \varepsilon$.

The shift map

$$\sigma\colon \Sigma \to \Sigma \ \forall \ s = s_0 s_1 s_2 \ldots \in \Sigma_2\colon \ \ \sigma(s) = s_1 s_2 \ldots$$

is a well known example of a chaotic map (see, [7, 10, 13]). But it is not unique chaotic map in space (Σ_2, d).

Definition 6. ([3]) The α_m*-mapping* $(m = 2, 3, \ldots)$ $\alpha_m\colon \Sigma_2 \to \Sigma_2$ is defined by

$$\alpha_m(s_0 s_1 s_2 \ldots) = s_1 s_2 \ldots s_{m-1} s_{m+1} s_{m+2} \ldots$$

This mapping is not the kth iteration of the shift map, the α-mapping "forgets" two symbols of the sequence in every iteration.

It is possible to prove that the every α_m-mapping $(m \geq 2)$ is continuous, the set of periodic points of the α_m-mapping is dense in Σ_2 and the α_m-mapping is topologically transitive on Σ_2 too. It follows from [1] that the α_m-mapping is chaotic mapping. If we observe that every α_m-mapping $(m \geq 2)$ is increasing mapping, then this gives a much shorter proof of the fact that α_m-mapping is chaotic.

3 Increasing Mappings

Let A be a finite *alphabet*, i.e., a finite nonempty set $\{a_0, a_1, a_2, \ldots, a_n\}$ and the elements are called *symbols*. We assume that A contains at least two symbols. We consider infinite sequences of symbols over a finite set A. *One-sided* infinite sequence over A is any total map $\omega\colon \mathbf{N} \to A$.

The set A^ω contains all infinite sequences. Let $f_\omega\colon A^\omega \to A^\omega$ and $\exists f\colon \mathbf{N} \to \mathbf{N}$ such that

$$f_\omega(x) = x_{f(0)} x_{f(1)} x_{f(2)} \ldots x_{f(i)} \ldots, \quad i \in \mathbf{N},\ x \in A^\omega.$$

In this case the function f is called *the generator function* of mapping f_ω.

Definition 7. ([3]) A function $f\colon \mathbf{N} \to \mathbf{N}$ is called *positively increasing function* if

$$0 < f(0) \text{ and } \forall i, j\colon \ \ i < j \Rightarrow f(i) < f(j).$$

Definition 8. ([3]) The mapping $f_\omega\colon A^\omega \to A^\omega$ is called *increasing mapping* if its generator function $f\colon \mathbf{N} \to \mathbf{N}$ is positively increasing.

The shift map

$$\sigma\colon \Sigma \to \Sigma \quad \forall \ s = s_0 s_1 s_2 \ldots \in \Sigma_2\colon \ \sigma(s) = s_1 s_2 \ldots$$

is positively increasing mapping because its generator function $f\colon \mathbf{N} \to \mathbf{N}$ is positively increasing:

$$f(x) = x + 1, \quad x = 0, 1, 2, \ldots.$$

Theorem 1. ([3]) *The increasing mapping* $f_\omega \colon A^\omega \to A^\omega$ *is chaotic in the set* A^ω.

In our case $A^\omega = \Sigma_2$ and α_m-mapping is increasing mapping because its generator function $f \colon \mathbf{N} \to \mathbf{N}$ is positively increasing:

$$f(x) = \begin{cases} x+1, & x = 0, 1, 2, \ldots, m-2, \\ x+2, & x = m-1, m, m+1, \ldots \end{cases}$$

Corollary 1. The increasing mapping $f_\Sigma \colon \Sigma_2 \to \Sigma_2$ is chaotic in the symbol space Σ_2.

Corollary 2. The α_m-mapping is chaotic in the symbol space Σ_2, $m = 2, 3, ...$

But on the other hand we have the following theorem.

Theorem 2. ([3]) *If generator function* f *of mapping* $f_\omega \colon A^\omega \to A^\omega$ *is such that* $f(0) = 0$, *then the generated mapping* f_ω *is not topologically transitive in the set* A^ω, *i.e., it is not chaotic in the set* A^ω.

In other words chaotic generated mapping always "forgets" the first symbol of the sequence.

4 Topological Semi–Conjugacy

Definition 9. ([13]) Let $f \colon A \to A$ and $g \colon B \to B$ be functions. A map $h \colon A \to B$ is called a *topological semi-conjugacy from* f *to* g provided 1) h is continuous, 2) h is onto, and 3) $h \circ f = g \circ h$. The map h is called a *topological conjugacy* if it is homeomorphism and $h \circ f = g \circ h$.

One essential result for our purpose is the following theorem.

Theorem 3. ([12]) *Let* A *and* B *be subsets of the metric spaces,* $f \colon A \to A$, $g \colon B \to B$, *and* $\tau \colon A \to B$ *be a topological semi-conjugacy of* f *to* g. *If* f *is chaotic on* A, *then* g *is topologically transitive on* B *and has dense set of periodic points in* B. *If* $\tau \colon A \to B$ *be a topological conjugacy of* f *and* g, *then* f *is chaotic on* A *if and only if* g *is chaotic on* B.

In 1994 Peitgen, Juergen, Saupe [12] had shown that for chaotic shift map the corresponding chaotic mapping in a unit segment is

$$S(x) = \begin{cases} 2x \bmod 1, & x \in [0, 1[, \\ 1, & x = 1. \end{cases}$$

This result suggests to find for chaotic increasing mapping (in Σ_2) corresponding chaotic mapping in unit segment.

5 Construction of Chaotic Mappings in Unit Segment

We consider binary expansion of numbers from segment $[0,1]$. It is possible to write every number x from $[0,1]$ in form $x = a_0a_1a_2\ldots$, where $a_k \in \{0,1\}$ and $x = a_0 2^{-1} + a_1 2^{-2} + a_2 2^{-3} + \ldots$. For example, $\frac{1}{2} = 1000\ldots$ or $\frac{1}{7} = \overline{001}\ldots$ (infinite sequence which periodically repeats after some fixed length will be denoted by the finite length sequence with an over line).

The number $\frac{1}{2}$ has two binary expansions $1\overline{0}$ and $0\overline{1}$. We assume that we consider only first variant of binary expansion. Therefore we consider the set $I = \Sigma_2 \setminus J$, where

$$J = \{s_0s_1s_2\ldots \in \Sigma_2 | \exists N \geq 0\ \forall i \geq N\ s_i = 1\}.$$

Then we has a second problem with number 1, its binary expansion $\overline{1} \notin I$. But $\alpha_m(\overline{1}) = \overline{1}$ and this point is fixed point for every mapping α_m, $m = 2,3,\ldots$ and all iterations are same. The sequence $\overline{1}$ is fixed point for every increasing mapping too: $f_\omega(\overline{1}) = \overline{1}$.

The mapping $\tau\colon \Sigma_2 \to [0,1[$ defined by equality

$$\forall s = s_0s_1s_2\ldots \in \Sigma_2 \quad \tau(s) = s_0 2^{-1} + s_1 2^{-2} + s_2 2^{-3} + \ldots$$

is onto, continuous (see, for example, [8, 12]) but it is not one-to-one. The mapping $\tau\colon I \to [0,1[$ is onto, continuous and one-to-one.

Here are more possibilities how the number from segment $[0,1[$ transforms to binary expansion. We use the method from [12]:

$$x \in [0,1[, \quad \tau^{-1}(x) = s_0s_1s_2\ldots, \text{ where } s_i = \begin{cases} 0, & z(x)_i < \frac{1}{2}, \\ 1, & z(x)_i \geq \frac{1}{2}, \end{cases}$$
$$z(x)_0 = x,\ z(x)_i = 2z(x)_{i-1} \bmod 1,\ i = 1,2,\ldots.$$

If we consider $\tau\colon I \to [0,1[$, then the inverse map τ^{-1} is not continuous. For example, the sequence $x_n = \dfrac{1}{2} - \dfrac{1}{2^n}$, $n = 1,2,\ldots$, converges to $\dfrac{1}{2}$ but the sequence $\tau^{-1}(x_n)$, $n = 1,2,\ldots$, converges to $0\overline{1} \notin I$. Therefore $\tau\colon \Sigma_2 \to [0,1[$ and $\tau\colon I \to [0,1[$ are not homeomorphisms and not topological conjugacy.

We have shown that α_m-mapping is chaotic in symbol space Σ_2. Notice that $\alpha_m\colon J \to J$ and $\alpha_m\colon I \to I$. The α_m-mapping is increasing mapping in I too. It follows that α_m-mapping is chaotic in subset $I \subset \Sigma_2$.

We assume that for the α_m-mapping exists corresponding to chaotic mapping in segment $[0,1]$. We find for α_m-mapping corresponding mapping E_m in segment $[0,1]$. For this, we make the following numerical experiment: first, we write number x from segment $[0,1[$ in its binary expansion $s \in I$, second, we consider $\alpha_m(s)$, third, we write $\alpha_m(s)$ in its decimal expansion $E_m(x)$ and make a graph. Finally for $m = 2$ we find (see Figure 1)

$$E_2(x) = \begin{cases} 4x, & 0 \leq x < \frac{1}{8}, \quad 4x - \frac{1}{2}, \quad \frac{1}{8} \leq x < \frac{3}{8}, \\ 4x - 1, & \frac{3}{8} \leq x < \frac{4}{8}, \quad 4x - 2, \quad \frac{4}{8} \leq x < \frac{5}{8}, \\ 4x - \frac{5}{2}, & \frac{5}{8} \leq x < \frac{7}{8}, \quad 4x - 3, \quad \frac{7}{8} \leq x < 1. \end{cases}$$

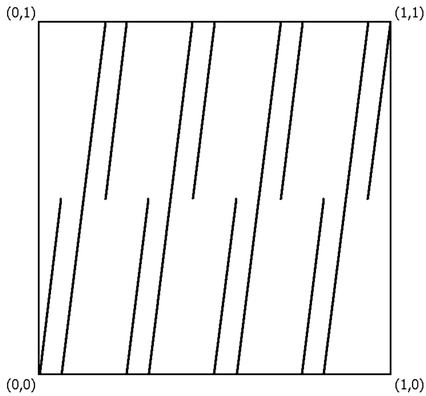

Figure 1. The graph of E_2.

Similarly we can find E_3, E_4, E_5,... and finally we give formula for E_m, $m \geq 2$, in general case (see [2]):

$$E_m(x) = \begin{cases} 4x, & 0 \leq x < 1/2^{m+1}, \\ 4x - 1/2^{m-1}, & 1/2^{m+1} \leq x < 3/2^{m+1} \\ 4x - 2/2^{m-1}, & 3/2^{m+1} \leq x < 5/2^{m+1}, \\ \dots\dots\dots & \dots\dots\dots, \\ 4x - i/2^{m-1}, & (2i-1)/2^{m+1} \leq x < (2i+1)2^{m+1}, \\ \dots\dots\dots & \dots\dots\dots, \\ 4x - 1, & (2^m - 1)/2^{m+1} \leq x < 1/2 = 2^m/2^{m+1}, \\ 4x - 2, & 1/2 \leq x < 2^m + 1/2^{m+1}, \\ 4x - 1/2^{m-1} - 2, & (2^m + 1)/2^{m+1} \leq x < (2^m + 3)/2^{m+1}, \\ \dots\dots\dots & \dots\dots\dots, \\ 4x - 3, & (2^{m+1} - 1)/2^{m+1} \leq x < 1. \end{cases}$$

We can prove equality $\tau \circ \alpha_m = E_m \circ \tau$, i.e., $\tau \colon I \to [0, 1[$ is topological semi-conjugacy from α_m to E_m. Similarly as for tent map it is possible to prove that $E_m \colon [0; 1] \to [0; 1]$ ($E_m(1) = 1$) exhibits sensitive dependence on initial conditions.

Theorem 4. ([2]) *Let* $E_m(1) = 1$. *Then every mapping* $E_m \colon [0, 1] \to [0, 1]$, $m = 2, 3, \dots$, *is chaotic in* $[0, 1]$.

Construction of a corresponding chaotic mapping in segment $[0, 1]$ for every increasing mapping $f_\Sigma \colon \Sigma_2 \to \Sigma_2$ can be done in a similar way. We have developed the computer tool that draws graphs of such mappings in the unit segment. The software is presented in http://www.webtech.lv/projects/math/. In this program we indicate only indices of symbols which are "forgotten" in the sequence. In finish we have the graph of corresponding mapping in

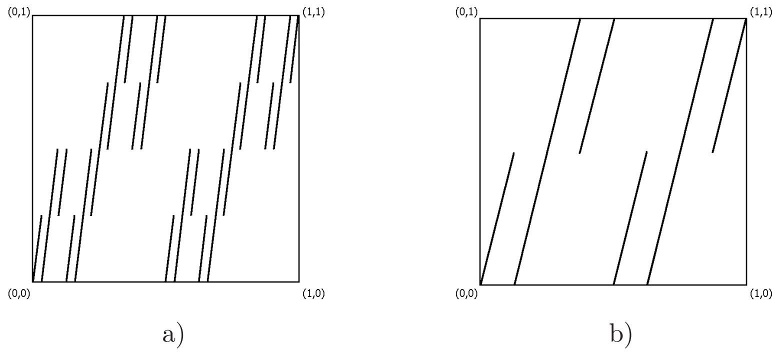

Figure 2. The graph of function in $[0;1]$ that corresponds to: a) increasing mapping that "forgets" first, third and fifth symbols of the sequence, b) increasing mapping that "forgets" first, second and fourth symbols of the sequence.

the unit segment. Similarly, as above we conclude that acquired mapping is chaotic. Two examples are considered. In Figure 2(a) we present a graph that corresponds to increasing mapping that "forgets" first, third and fifth symbols of the sequence and in Figure 2(b) we show a graph that corresponds to increasing mapping that "forgets" first, second and fourth symbols of the sequence.

6 Conclusions

We have considered the class of chaotic mappings in the space of infinite sequences. In particular, we have considered increasing mappings in the space of infinite sequences of two symbols (the space Σ_2). In this case we have developed a method of construction of corresponding mappings in unit segment. These mappings are chaotic, therefore models with such mappings are not predictable in long-term.

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