

# Positive Solutions Bifurcating from Zero Solution in a Predator-Prey Reaction–Diffusion System\*

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**Abstract.** An elliptic system subject to the homogeneous Dirichlet boundary condition denoting the steady-state system of a two-species predator-prey reaction–diffusion system with the modified Leslie–Gower and Holling-type II schemes is considered. By using the Lyapunov–Schmidt reduction method, the bifurcation of the positive solution from the trivial solution is demonstrated and the approximated expressions of the positive solutions around the bifurcation point are also given according to the implicit function theorem. Finally, by applying the linearized method, the stability of the bifurcating positive solution is also investigated. The results obtained in the present paper improved the existing ones.

**Keywords:** predator-prey reaction–diffusion system, positive solution, existence, steady state bifurcation, stability.

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## 1 Introduction

This paper is concerned with the following elliptic system

$$\begin{cases} -\Delta u = u \left( a - u - \frac{mv}{k_1 + u} \right), & x \in \Omega, \\ -\Delta v = v \left( b - \frac{v}{k_2 + u} \right), & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Delta$  is the Laplacian operator and  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ . As a predator-prey model with the modified Leslie–Gower and Holling-type II schemes, the ODE model corresponding to system (1.1) was proposed and studied by Aziz-Alaoui and Okiye [1]. In model

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(1.1), the variables  $u$  and  $v$  define the population densities of prey and predator species in the habitat  $\Omega$ ;  $a, m, k_1, b$  and  $k_2$  are positive constants and the corresponding biological meaning can be referred to [1]. In addition, the homogeneous Dirichlet boundary condition in (1.1) implies that the exterior environment is hostile. Note that  $u$  and  $v$  are population densities, therefore the nonnegative solutions of system (1.1) are of importance and interest [7, 11, 12, 13].

Suppose that  $q(x) \in C(\bar{\Omega})$  and  $\lambda_1(q) < \lambda_2(q) \leq \lambda_3(q) \leq \dots$  are the eigenvalues of the eigenvalue problem

$$\begin{cases} -\Delta w + q(x)w = \lambda w, & x \in \Omega, \\ w = 0, & x \in \partial\Omega. \end{cases}$$

Then  $\lambda_1(q)$  is simple and  $\lambda_i(q_1) < \lambda_i(q_2)$  when  $q_1(x) \leq q_2(x), q_1(x) \not\equiv q_2(x)$ . In addition, we denote  $\lambda_i(0)$  by  $\lambda_i$ . Let  $f(x)$  be a positive continuous function defined on  $\bar{\Omega}$  and consider the following boundary value problem

$$\begin{cases} -\Delta w + q(x)w = aw - f(x)w^2, & x \in \Omega, \\ w = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Then  $w = 0$  is the unique nonnegative solution of (1.2) if  $a \leq \lambda_1(q)$  and (1.2) has a unique positive solution when  $a > \lambda_1(q)$ , see [2, 15]. In addition, we denote the positive solution of (1.2) when  $q = 0, f = 1$  and  $a > \lambda_1$  by  $\theta_a$ .

Recently, Peng and Wang [11] investigated the positive solutions of system (1.1) in the case that the parameter  $m$  is large, and obtained a complete understanding of the existence, multiplicity and stability of positive solutions. In addition, Wang and Wang [14] studied the existence and stability of positive solutions of system (1.1) and they found that (1.1) has at least a positive solution when  $b > \lambda_1$  and  $a > \lambda_1(m\theta_b/k_1)$  by the fixed point index theory in a positive cone. Meanwhile, by regarding  $a$  as the bifurcation parameter, they obtained the connected component connecting the semitrivial solution  $(0, \theta_b)$  with the unique positive solution of the limit equation of (1.1) as  $a \rightarrow \infty$  and the stability of positive solution of (1.1) close to  $(0, \theta_b)$  was also investigated by the linearized stability theory [4, 9]. In addition, when  $a, b > \lambda_1$  and  $0 < m \ll 1$  or  $k$  is big enough, they also gave the existence and stability of positive solutions bifurcating from the unique positive solution of (1.1) in the case when  $m = 0$ . However, the existence and stability of positive solutions of (1.1) bifurcating from the zero solution were not discussed in [14]. In this paper, we consider mainly the existence and stability of positive solutions of (1.1) bifurcating from the zero solution.

This paper is organized as follows. In Section 2, by applying Lyapunov-Schmidt reduction process, we demonstrate the existence of positive solutions of (1.1) bifurcating from the zero solution. In Section 3, according to the implicit function theorem, the asymptotic expression of positive solutions of (1.1) bifurcating from the zero solution is given. By analyzing the spectrum of the linearized operator of (1.1) at the positive solutions obtained in Section 3, we analyze the stability of the bifurcating positive solutions of (1.1) in Section 4.

## 2 Existence of Positive Solutions Bifurcating from the Zero Solution

In this section, we discuss the existence of positive solutions of system (1.1) bifurcating from zero solution according to the Lyapunov–Schmidt reduction method [6, 8]. For system (1.1), we have the following result.

**Lemma 1.** *If (1.1) has positive solutions, then  $a, b > \lambda_1$ .*

*Proof.* Suppose that  $\phi_1$  is the eigenfunction corresponding to the eigenvalue  $\lambda_1$ , that is,  $\phi_1$  satisfies the following boundary value problem

$$-\Delta\phi_1 = \lambda_1\phi_1, \quad x \in \Omega, \quad \phi_1 = 0, \quad x \in \partial\Omega. \tag{2.1}$$

Then from [5] we know that  $\phi_1 > 0, x \in \Omega$ . Let  $(u, v)$  be a positive solution of (1.1). It follows from the first equation of (1.1) that

$$\begin{cases} -\phi_1\Delta u = \phi_1u \left( a - u - \frac{mv}{k_1 + u} \right) < a\phi_1u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

Thus the Green’s identity, (2.1) and (2.2) imply that

$$0 = \int_{\Omega} (\phi_1\Delta u - u\Delta\phi_1) dx > (\lambda_1 - a) \int_{\Omega} \phi_1u dx.$$

The positivity of  $u$  and  $\phi_1$  in  $\Omega$  gives  $a > \lambda_1$ .

Similarly, one can obtain that  $b > \lambda_1$  and thus the proof is complete.  $\square$

Assume that  $p > n$  and the Banach spaces  $X$  and  $Y$  are defined by

$$X = [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)] \times [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)], \quad Y = L^p(\Omega) \times L^p(\Omega).$$

By virtue of the Sobolev embedding theorem, we know that  $X \subset C^1(\bar{\Omega}) \times C^1(\bar{\Omega})$ . If we define an operator  $H : \mathbb{R} \times \mathbb{R} \times X \rightarrow Y$  by

$$H(a, b, u, v) = \begin{pmatrix} -\Delta u - u(a - u - mv/(k_1 + u)) \\ -\Delta v - v(b - v/(k_2 + u)) \end{pmatrix},$$

then system (1.1) is equivalent to the following nonlinear equation

$$H(a, b, u, v) = 0. \tag{2.3}$$

Let

$$\begin{aligned} L_0(u, v) &= \begin{pmatrix} -\Delta u - \lambda_1 u \\ -\Delta v - \lambda_1 v \end{pmatrix}, \quad F(a, b, u, v) = \begin{pmatrix} (\lambda_1 - a)u \\ (\lambda_1 - b)v \end{pmatrix}, \\ G(u, v) &= \begin{pmatrix} u(u + mv/(k_1 + u)) \\ v^2/(k_2 + u) \end{pmatrix}. \end{aligned}$$

Then equation (2.3) is transformed into

$$L_0(u, v) + F(a, b, u, v) + G(u, v) = 0.$$

Denote the adjoint operator of  $L_0$  by  $L_0^*$ . It is easy to see that

$$\mathcal{N}(L_0) = \mathcal{N}(L_0^*) = \text{span} \left\{ \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} \right\},$$

where  $\mathcal{N}(A)$  is the null space of the operator  $A$ . Therefore,  $L_0$  is a Fredholm operator and thus  $X$  and  $Y$  can be decomposed as

$$X = X_1 \oplus X_2, \quad Y = Y_1 \oplus Y_2, \tag{2.4}$$

where  $X_1 = Y_1 = \mathcal{N}(L_0)$ ,  $Y_2 = \mathcal{R}(L_0)$ , here  $\mathcal{R}(L_0)$  denotes the range of  $L_0$ . According to the decomposition (2.4), we define the projection operators  $P : Y \rightarrow Y_1$ ,  $Q = I - P : Y \rightarrow Y_2$ . It follows from the Lyapunov–Schmidt reduction method that equation (2.1) is equivalent to the following equations

$$PH(a, b, u, v) = 0, \tag{2.5}$$

$$QH(a, b, u, v) = 0. \tag{2.6}$$

Suppose that  $(u, v) \in X$ . Then from the decomposition (2.4) we know that  $u$  and  $v$  can be expressed as  $u = \alpha(\phi_1 + u_2)$ ,  $v = \beta(\phi_1 + v_2)$ , where  $\alpha, \beta \in \mathbb{R}$ ,  $(u_2, v_2) \in X_2$ . Thus

$$H(a, b, u, v) = L_0(\alpha u_2, \beta v_2) + F(a, b, \alpha(\phi_1 + u_2), \beta(\phi_1 + v_2)) + G(\alpha(\phi_1 + u_2), \beta(\phi_1 + v_2)),$$

and hence equation (2.6) is equivalent to

$$K(u_2, v_2; a, b, \alpha, \beta) = 0,$$

where

$$K(u_2, v_2; a, b, \alpha, \beta) = L_0(\alpha u_2, \beta v_2) + F(a, b, \alpha(\phi_1 + u_2), \beta(\phi_1 + v_2)) + G(\alpha(\phi_1 + u_2), \beta(\phi_1 + v_2)).$$

It is easy to see  $K(0, 0; \lambda_1, \lambda_1, 0, 0) = 0$ . In addition, notice that the product of the matrix  $\begin{pmatrix} 1/\alpha & 0 \\ 0 & 1/\beta \end{pmatrix}$  and the Frechét derivative [16] of  $K(u_2, v_2; a, b, \alpha, \beta)$  at  $(0, 0; \lambda_1, \lambda_1, 0, 0)$  is

$$\begin{pmatrix} 1/\alpha & 0 \\ 0 & 1/\beta \end{pmatrix} K_{(u_2, v_2)}(u_2, v_2; a, b, \alpha, \beta) \Big|_{(0, 0; \lambda_1, \lambda_1, 0, 0)} = L_0.$$

Clearly,  $L_0 : X_2 \rightarrow Y_2$  has a bounded inverse. Thus, the implicit function theorem implies that there exist sufficiently small positive number  $s_0$  and two continuous differential functions  $\bar{u}_2(a, b, \alpha, \beta)$ ,  $\bar{v}_2(a, b, \alpha, \beta)$  such that when  $|a - \lambda_1|$ ,  $|b - \lambda_1|$ ,  $|\alpha|$ ,  $|\beta| < s_0$ ,  $\bar{u}_2(a, b, \alpha, \beta)$  and  $\bar{v}_2(a, b, \alpha, \beta)$  satisfy equation (2.6) and

$$\bar{u}_2(\lambda_1, \lambda_1, 0, 0) = 0, \quad \bar{v}_2(\lambda_1, \lambda_1, 0, 0) = 0.$$

Now suppose that  $|a - \lambda_1|, |b - \lambda_1|, |\alpha|, |\beta| < s_0$  and let

$$\bar{u} = \alpha(\phi_1 + \bar{u}_2(a, b, \alpha, \beta)), \quad \bar{v} = \beta(\phi_1 + \bar{v}_2(a, b, \alpha, \beta)).$$

Then

$$T(a, b; \alpha, \beta) := PH(a, b, \bar{u}, \bar{v}) = PF(a, b, \bar{u}, \bar{v}) + PG(\bar{u}, \bar{v}).$$

Notice that  $T(\lambda_1, \lambda_1; 0, 0) = 0$  and

$$\left( \begin{matrix} 1/\alpha & 0 \\ 0 & 1/\beta \end{matrix} \right) T_{ab}(a, b; \alpha, \beta) \Big|_{(\lambda_1, \lambda_1, 0, 0)} = \left( \begin{matrix} -\phi_1 & 0 \\ 0 & -\phi_1 \end{matrix} \right).$$

Again, from the implicit function theorem, we know that for  $|\alpha|, |\beta|$  small enough, there exist two  $C^1$  functions  $\bar{a}(\alpha, \beta), \bar{b}(\alpha, \beta)$  such that

$$PH(\bar{a}(\alpha, \beta), \bar{b}(\alpha, \beta), \alpha(\phi_1 + \bar{u}_2(\bar{a}(\alpha, \beta), \bar{b}(\alpha, \beta), \alpha, \beta)), \beta(\phi_1 + \bar{v}_2(\bar{a}(\alpha, \beta), \bar{b}(\alpha, \beta), \alpha, \beta))) = 0,$$

and  $\bar{a}(0, 0) = \bar{b}(0, 0) = \lambda_1$ . From Lemma 1, we can see that when  $a, b > \lambda_1$  and  $\alpha, \beta > 0$ ,

$$((\bar{a}(\alpha, \beta), \bar{b}(\alpha, \beta), \alpha(\phi_1 + \bar{u}_2(\bar{a}(\alpha, \beta), \bar{b}(\alpha, \beta), \alpha, \beta)), \beta(\phi_1 + \bar{v}_2(\bar{a}(\alpha, \beta), \bar{b}(\alpha, \beta), \alpha, \beta))))$$

is the positive solution of (1.1) bifurcating from the zero solution.

Next, we give the linear approximation of  $\bar{a}(\alpha, \beta), \bar{b}(\alpha, \beta)$  when  $\alpha, \beta > 0$  are small enough. Let

$$\tilde{u}_2(\alpha, \beta) = \bar{u}_2(\bar{a}(\alpha, \beta), \bar{b}(\alpha, \beta), \alpha, \beta) \quad \text{and} \quad \tilde{v}_2(\alpha, \beta) = \bar{v}_2(\bar{a}(\alpha, \beta), \bar{b}(\alpha, \beta), \alpha, \beta).$$

Then the positive solutions of (1.1) bifurcating from the zero solution can be expressed as

$$\bar{u}(\alpha, \beta) = \alpha(\phi_1 + \tilde{u}_2(\alpha, \beta)) \quad \text{and} \quad \bar{v}(\alpha, \beta) = \beta(\phi_1 + \tilde{v}_2(\alpha, \beta)).$$

Now, substituting  $\bar{a}(\alpha, \beta), \bar{b}(\alpha, \beta), \bar{u}(\alpha, \beta)$  and  $\bar{v}(\alpha, \beta)$  into the first equation of (1.1), one can get

$$-\Delta(\phi_1 + \tilde{u}_2(\alpha, \beta)) = (\phi_1 + \tilde{u}_2(\alpha, \beta)) \times \left( \bar{a}(\alpha, \beta) - \alpha(\phi_1 + \tilde{u}_2(\alpha, \beta)) - \frac{m\beta(\phi_1 + \tilde{v}_2(\alpha, \beta))}{k_1 + \alpha(\phi_1 + \tilde{u}_2(\alpha, \beta))} \right). \tag{2.7}$$

Differentiating two sides of (2.7) with respect to  $\alpha$  and  $\beta$  at  $(\alpha, \beta) = (0, 0)$ , respectively, we have

$$(\Delta + \lambda_1) \frac{\partial \tilde{u}_2(0, 0)}{\partial \alpha} + \phi_1 \left( \frac{\partial \bar{a}(0, 0)}{\partial \alpha} - \phi_1 \right) = 0, \tag{2.8}$$

$$(\Delta + \lambda_1) \frac{\partial \tilde{u}_2(0, 0)}{\partial \beta} + \phi_1 \left( \frac{\partial \bar{a}(0, 0)}{\partial \beta} - \frac{m\phi_1}{k_1} \right) = 0. \tag{2.9}$$

Multiplying two sides of (2.8) and (2.9) by  $\phi_1$  and then integrating them on  $\Omega$ , according to the Green’s identity, one can get

$$\frac{\partial \bar{a}(0, 0)}{\partial \alpha} = \int_{\Omega} \phi_1^3(x) \, dx, \quad \frac{\partial \bar{a}(0, 0)}{\partial \beta} = \frac{m}{k_1} \int_{\Omega} \phi_1^3(x) \, dx.$$

Similarly, we have

$$\frac{\partial \bar{b}(0, 0)}{\partial \alpha} = 0, \quad \frac{\partial \bar{b}(0, 0)}{\partial \beta} = \frac{1}{k_2} \int_{\Omega} \phi_1^3(x) \, dx.$$

Therefore, when  $\alpha, \beta > 0$  are sufficiently small,

$$\bar{a}(\alpha, \beta) = \lambda_1 + \int_{\Omega} \phi_1^3(x) \, dx \left( \alpha + \frac{m}{k_1} \beta \right) + o(|\alpha|, |\beta|), \tag{2.10}$$

$$\bar{b}(\alpha, \beta) = \lambda_1 + \frac{1}{k_2} \int_{\Omega} \phi_1^3(x) \, dx \beta + o(|\alpha|, |\beta|). \tag{2.11}$$

Thus, we have the following theorem.

**Theorem 1.** *If  $0 < a - \lambda_1 \ll 1, 0 < b - \lambda_1 \ll 1$ , then system (1.1) can bifurcate a small positive solution from the zero solution parameterized by  $\alpha, \beta > 0$  small enough as*

$$\begin{aligned} u(\alpha, \beta) &= \alpha(\phi_1 + \bar{u}_2(\bar{a}(\alpha, \beta), \bar{b}(\alpha, \beta), \alpha, \beta)), \\ v(\alpha, \beta) &= \beta(\phi_1 + \bar{v}_2(\bar{a}(\alpha, \beta), \bar{b}(\alpha, \beta), \alpha, \beta)), \end{aligned}$$

where  $\bar{a}(\alpha, \beta), \bar{b}(\alpha, \beta)$  are given respectively by (2.10), (2.11), and

$$\bar{u}_2(\lambda_1, \lambda_1, 0, 0) = 0, \quad \bar{v}_2(\lambda_1, \lambda_1, 0, 0) = 0.$$

*Remark 1.* By using the fixed point index theory in a positive cone, the reference [14] showed that system (1.1) has no positive solution when  $a \leq \lambda_1(m\theta_b/k_1)$ . In fact, system (1.1) can bifurcate a positive solution from the zero solution when  $a - \lambda_1 > 0$  and  $b - \lambda_1 > 0$  are sufficiently small.

### 3 Asymptotic Expression of Small Bifurcating Positive Solutions

In the previous section, we have obtained the existence of positive solutions of system (1.1) bifurcating from the zero solution by using the Lyapunov–Schmidt reduction method. In this section, by virtue of the implicit function theorem, we give a more accurate asymptotic expression for the small positive solutions of system (1.1) bifurcating from the zero solution.

Suppose that  $0 < a - \lambda_1, b - \lambda_1 \ll 1$  and let  $a - \lambda_1 = b - \lambda_1 =: r$ . If  $(u_r, v_r)$  is a positive solution of (1.1) when  $0 < r \ll 1$ , then  $(u_r, v_r)$  should be

a solution of the following elliptic boundary value problem

$$\begin{cases} \Delta u + u \left( a - u - \frac{mv}{k_1 + u} \right) = 0, & x \in \Omega, \\ \Delta v + v \left( b - \frac{v}{k_2 + u} \right) = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \\ u, v > 0, & x \in \Omega. \end{cases} \tag{3.1}$$

Define the operator  $\mathcal{D}$  by

$$\mathcal{D} = \begin{pmatrix} \Delta + \lambda_1 & 0 \\ 0 & \Delta + \lambda_1 \end{pmatrix},$$

and let  $\mathcal{N}(\mathcal{D})$  and  $\mathcal{R}(\mathcal{D})$  denote the null space and the range of  $\mathcal{D}$ , respectively. Then it is easy to see

$$\begin{aligned} \mathcal{N}(\mathcal{D}) &= \text{span}\{\eta_1, \eta_2\}, \\ \mathcal{R}(\mathcal{D}) &= \left\{ y = (y_1, y_2)^T \in Y: \langle \eta_i, y \rangle \stackrel{\text{def}}{=} \int_{\Omega} y_i(x)\phi_1(x) dx = 0, i = 1, 2 \right\} \end{aligned}$$

and  $X$  can be decomposed as  $X = \mathcal{N}(\mathcal{D}) \oplus \mathcal{R}(\mathcal{D})$ , where  $\eta_1 = (\phi_1, 0)^T$  and  $\eta_2 = (0, \phi_1)^T$ . Let  $c_* = \int_{\Omega} \phi_1^2(x) dx / \int_{\Omega} \phi_1^3(x) dx$  and define  $\alpha_0, \beta_0$  by  $\alpha_0 = (1 - mk_2/k_1)c_*$ ,  $\beta_0 = k_2c_*$ . If the condition

$$m < \frac{k_1}{k_2} \tag{3.2}$$

holds, then  $\alpha_0 > 0$  and  $\beta_0 > 0$ . Now, we consider the following boundary value problem in  $X \cap \mathcal{R}(\mathcal{D})$ :

$$\begin{cases} (\Delta + \lambda_1)\xi + \phi_1 - \left( \alpha_0 + \frac{m}{k_1}\beta_0 \right) \phi_1^2 = 0, & x \in \Omega, \\ (\Delta + \lambda_1)\eta + \phi_1 - \frac{1}{k_2}\beta_0\phi_1^2 = 0, & x \in \Omega, \\ \xi = \eta = 0, & x \in \partial\Omega. \end{cases} \tag{3.3}$$

In virtue of the definitions of  $\alpha_0$  and  $\beta_0$ , and notice that  $\mathcal{D}$  is a bijective mapping from  $X \cap \mathcal{R}(\mathcal{D})$  to  $\mathcal{R}(\mathcal{D})$ , we can obtain easily the following result:

**Lemma 2.** *Boundary value problem (3.3) has a unique solution  $(\xi_0(x), \eta_0(x))$  in  $Z \cap \mathcal{R}(\mathcal{D})$ , where  $Z = H_0^2(\overline{\Omega}) \times H_0^2(\overline{\Omega})$  and*

$$H_0^2(\overline{\Omega}) = \{y \in L^2(\overline{\Omega}): y', y'' \in L^2(\overline{\Omega}), y = 0 \text{ on } \partial\Omega\}.$$

**Theorem 2.** *If the condition (3.2) holds, then there exists a constant  $r^* > 0$  and a unique continuously differential mapping  $r \rightarrow (\xi_r, \eta_r, \alpha_r, \beta_r)$  from  $[0, r^*]$  to  $(X \cap \mathcal{R}(\mathcal{D}))^2 \times (\mathbb{R}^+)^2$  such that system (3.1) has a unique positive solution parameterized by  $r$  as*

$$u_r = \alpha_r r(\phi_1 + r\xi_r), \quad v_r = \beta_r r(\phi_1 + r\eta_r), \quad r \in (0, r^*], \tag{3.4}$$

and

$$\langle \phi_1, \xi_r \rangle = \langle \phi_1, \eta_r \rangle = 0.$$

*Proof.* Define  $F = (F_1, F_2, F_3, F_4) : Y \times \mathbb{R}^3 \rightarrow X \times \mathbb{R}^2$  by

$$\begin{aligned}
 F_1(\xi, \eta, \alpha, \beta, r) &= (\Delta + \lambda_1)\xi + \phi_1 + r\xi \\
 &\quad - (\phi_1 + r\xi) \left[ \alpha(\phi_1 + r\xi) + \frac{m\beta(\phi_1 + r\eta)}{k_1 + r\alpha(\phi_1 + r\xi)} \right], \\
 F_2(\xi, \eta, \alpha, \beta, r) &= (\Delta + \lambda_1)\eta + \phi_1 + r\eta - (\phi_1 + r\eta) \frac{\beta(\phi_1 + r\eta)}{k_2 + r\alpha(\phi_1 + r\xi)}, \\
 F_3(\xi, \eta, \alpha, \beta, r) &= \langle \phi_1, \xi \rangle, \quad F_4(\xi, \eta, \alpha, \beta, r) = \langle \phi_1, \eta \rangle.
 \end{aligned}$$

Then we have

$$F(\xi_0, \eta_0, \alpha_0, \beta_0, 0) = \begin{pmatrix} (\Delta + \lambda_1)\xi_0 + \phi_1 - \left(\alpha_0 + \frac{m}{k_1}\beta_0\right)\phi_1^2 \\ (\Delta + \lambda_1)\eta_0 + \phi_1 - \frac{1}{k_2}\beta_0\phi_1^2 \\ \langle \phi_1, \xi_0 \rangle \\ \langle \phi_1, \eta_0 \rangle \end{pmatrix} = 0,$$

and from [16] we know that the Frechét derivative of  $F$  at  $(\xi_0, \eta_0, \alpha_0, \beta_0, 0)$  is

$$D_{(\xi, \eta, \alpha, \beta)}F(\xi_0, \eta_0, \alpha_0, \beta_0, 0) = \begin{pmatrix} \Delta + \lambda_1 & 0 & -\phi_1^2 & -\frac{m}{k_1}\phi_1^2 \\ 0 & \Delta + \lambda_1 & 0 & -\frac{1}{k_2}\phi_1^2 \\ \langle \phi_1, \cdot \rangle & 0 & 0 & 0 \\ 0 & \langle \phi_1, \cdot \rangle & 0 & 0 \end{pmatrix}.$$

Notice that  $\phi_1^2(x) \notin \mathcal{R}(\Delta + \lambda_1)$ . Therefore,  $D_{(\xi, \eta, \alpha, \beta)}F(\xi_0, \eta_0, \alpha_0, \beta_0, 0)$  is a bijective mapping from  $X \times \mathbb{R}^2$  to  $Y \times \mathbb{R}^2$ . Thus, it follows from the implicit function theorem [3, 6, 16] that there exist a  $r^* > 0$  and a unique continuously differential mapping  $r \rightarrow (\xi_r, \eta_r, \alpha_r, \beta_r)$  from  $[0, r^*]$  to  $(Y \cap \mathcal{R}(\mathcal{D})) \times \mathbb{R}^2$  such that

$$F(\xi_r, \eta_r, \alpha_r, \beta_r, r) \equiv 0, \quad r \in [0, r^*].$$

An easy calculation shows that  $(u_r, v_r)$  given by (3.4) solves the boundary value problem (3.1) and this completes the proof.  $\square$

### 4 Stability of Small Bifurcating Positive Solutions

Suppose that  $0 < r \ll 1$  and  $(u_r, v_r)$  is the positive solution of system (1.1) given by (3.4). Then the linearized system of system (1.1) at  $(u_r, v_r)$  is

$$\begin{cases} \Delta u + \left( a - 2u_r - \frac{mv_r}{k_1 + u_r} + \frac{mu_r v_r}{(k_1 + u_r)^2} \right) u - \frac{mu_r}{k_1 + u_r} v = 0, \\ \Delta v + \frac{v_r^2}{(k_2 + u_r)^2} u + \left( b - \frac{2v_r}{k_2 + u_r} \right) v = 0. \end{cases} \tag{4.1}$$

Define the operator  $A(r) : \mathcal{D}(A(r)) \rightarrow Y$  with domain  $\mathcal{D}(A(r)) \subseteq X$  by

$$A(r) = \begin{pmatrix} \Delta + \left( a - 2u_r - \frac{mv_r}{k_1 + u_r} + \frac{mu_r v_r}{(k_1 + u_r)^2} \right) & -\frac{mu_r}{k_1 + u_r} \\ \frac{v_r^2}{(k_2 + u_r)^2} & \Delta + \left( b - \frac{2v_r}{k_2 + u_r} \right) \end{pmatrix}.$$



From [10] we know that  $A(r)$  is an infinitesimal generator of a strong continuous semigroup and  $A(r)$  is also a self-adjoint operator. In addition, the eigenvalue problem corresponding to system (4.1) is given by

$$(A(r) - \lambda I)(y, z)^T = 0, \quad 0 \neq (y, z) \in \mathcal{D}(A(r)), \tag{4.2}$$

where  $I$  is the identity operator. It is well known that  $(u_r, v_r)$  is asymptotically stable if all the eigenvalues of (4.2) locate in the left-half complex plane and  $(u_r, v_r)$  is unstable if (4.2) has at least one root locating the right-half complex plane. In the following, we give a stability result on the positive solution of (1.1) given by (3.4).

**Theorem 3.** *If  $m < k_1/k_2$  and  $0 < r^* \ll 1$ , then the bifurcating positive solution  $(u_r, v_r)$  of system (1.1) is asymptotically stable for  $r \in (0, r^*]$ .*

*Proof.* If we ignore a scalar factor, then for  $r \in (0, r^*]$  the solution  $(y, z)$  of the eigenvalue problem (4.2) can be represented as

$$y = \phi_1 + r\gamma, \langle \phi_1, \gamma \rangle = 0, \quad z = c\phi_1 + r\delta, \langle \phi_1, \delta \rangle = 0, \tag{4.3}$$

where  $c$  is a complex number. Rewrite the eigenvalue problem (4.2) into

$$\begin{cases} \left( \Delta + a - 2u_r - \frac{mv_r}{k_1 + u_r} + \frac{mu_r v_r}{(k_1 + u_r)^2} \right) y - \frac{mu_r}{k_1 + u_r} z = \lambda y, \\ \frac{v_r^2}{(k_2 + u_r)^2} y + \left( \Delta + b - \frac{2v_r}{k_2 + u_r} \right) z = \lambda z, \end{cases} \tag{4.4}$$

with  $0 \neq (y, z) \in X$ . From Theorem 2 and (4.3), we have

$$u_r = \alpha_r r \phi_1 + O(r^2), \quad v_r = \beta_r r \phi_1 + O(r^2), \quad y = \phi_1 + O(r), \quad z = c_r \phi_1 + O(r),$$

where  $c_r$  is a complex number satisfying  $c_r \rightarrow c_0$  as  $r \rightarrow 0$ . Therefore, after multiplying both sides of the first equality of (4.4) by  $\phi_1(x)$  and integrating on  $\Omega$ , we can get that

$$(\lambda - r) \int_{\Omega} \phi_1^2(x) dx = -r \left( 2\alpha_r + \frac{m}{k_1} \beta_r + \frac{m}{k_1} \alpha_r c_r \right) \int_{\Omega} \phi_1^3(x) dx + O(r^2).$$

Let  $\bar{\lambda} = \frac{\lambda}{r} c_*$ . Then the above equality can be rewritten as

$$\bar{\lambda} - c_* = - \left( 2\alpha_r + \frac{m}{k_1} \beta_r + \frac{m}{k_1} \alpha_r c_r \right) + O(r). \tag{4.5}$$

Noting that

$$\alpha_0 + \frac{m}{k_1} \beta_0 = c_*,$$

and  $\alpha_r \rightarrow \alpha_0, \beta_r \rightarrow \beta_0, c_r \rightarrow c_0$  as  $r \rightarrow 0$ , hence, (4.5) can be further rewritten as

$$\bar{\lambda} = - \left( \alpha_0 + \frac{m}{k_1} \alpha_0 c_0 \right) + O(r). \tag{4.6}$$

Similarly, from the fact that  $\beta_0/k_2 = c_*$ , and the second equality of (4.4), we can obtain

$$\bar{\lambda}c_0 = -\frac{\beta_0}{k_2}c_0 + O(r). \quad (4.7)$$

If  $c_0 = 0$ , then from (4.6) we can observe

$$\bar{\lambda} = -\alpha_0 + O(r).$$

Therefore,  $\bar{\lambda} < 0$  when  $0 < r \ll 1$ . If  $c_0 \neq 0$ , then from (4.7) one also can get easily that  $\bar{\lambda} < 0$  when  $0 < r \ll 1$ . Thus we know that when  $m < \frac{k_1}{k_2}$  and  $0 < r^* \ll 1$ , the bifurcating positive solution  $(u_r, v_r)$  of system (1.1) is asymptotically stable for  $r \in (0, r^*]$  and the proof is complete.  $\square$

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