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Oscillations for an Equation Arising in Groundwater Flow with the Relaxation Time*

Jiří Mls[a] and Leopold Herrmann[b]

[a] *Charles University in Prague, Faculty of Science*
Albertov 6, 128 43 Praha 2, Czech Republic
[b] *Institute of Technical Mathematics, Faculty of Mechanical Engineering, Czech Technical University in Prague*
Karlovo náměstí 13, 121 35 Praha 2, Czech Republic
E-mail(*corresp.*): mls@natur.cuni.cz

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Abstract. Groundwater flow problems are mostly formulated by means of mass-balance equation combined with Darcy's law. In this way, the flow is governed by a parabolic equation. To prevent inaccuracies which may result from this formulation, the Cattaneo approach can be utilized. The paper presents groundwater flow equation adopting the Cattaneo approach. In both 2D and 3D cases, the equation is of hyperbolic type and contains a constant known as relaxation time. The article focuses further on energy solutions defined on unbounded time interval. It is shown that under certain conditions, such solutions are oscillatory. The conditions sufficient to ensure the oscillatory solutions are derived. An upper bound for the oscillatory time is proved to be independent of the particular solution.

Keywords: modified Darcy's law, Cattaneo approach, relaxation time, oscillatory solutions of an autonomous problem, oscillatory time.

AMS Subject Classification: 35B05; 35L20; 76S05; 86A05.

1 Introduction

Solving groundwater flow problems, or more generally, problems of flow of fluids in porous media, we mostly solve an initial–boundary value problem governed by a second order parabolic equation. The governing equation is a balance equation of the fluid phase combined with a constitutive law – Darcy's law in our case.

Several other processes in nature are known to be also governed by a conservation law of a substance in a medium. Let the bulk density of the substance in the medium be a function $\sigma(t, \boldsymbol{x})$, $\sigma : (t_1, t_2) \times \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^3$ and

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$(t_1, t_2) \subset \mathbb{R}$ are the domain and the time interval in which the process is studied, respectively. That is, the amount of the substance in an arbitrary control volume $\omega \subset \Omega$ at time t is given by $M(\omega, t) = \int_\omega \sigma(t, \boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$.

The motion of the substance is represented by the vector field $\boldsymbol{w}$ (flux-density vector), $\boldsymbol{w} : (t_1, t_2) \times \Omega \to \mathbb{R}^3$. The time rate $M'(S, t)$ of amount of the substance crossing at time t a surface S oriented by the unit normal vector field $\boldsymbol{\nu}$ is given by $M'(S, t) = \int_S \boldsymbol{w} \cdot \boldsymbol{\nu}\,\mathrm{d}S$.

Denote by $q(t, \boldsymbol{x})$, $q : (t_1, t_2) \times \Omega \to \mathbb{R}$ the density of sources of the substance within the medium, that is, q is the amount of the substance generated by sources in the medium per unit volume and unit time. Then the time rate of change of the substance amount in any control volume ω is given by the flux of the substance through the boundary $\partial\omega$ of ω and the amount of the substance generated in ω per unit time, thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\omega \sigma \boldsymbol{x} = - \int_{\partial\omega} \boldsymbol{w} \cdot \boldsymbol{\nu} S + \int_\omega q \boldsymbol{x},$$

where $\boldsymbol{\nu}$, defined on $\partial\omega$, is outward unit normal of the volume ω. Since this integral identity is true for an arbitrary control volume ω, we obtain the fundamental balance law (conservation law of mass or energy)

$$\frac{\partial \sigma}{\partial t} + \operatorname{div} \boldsymbol{w} = q, \tag{1.1}$$

see, e.g., Feistauer [9], Fowler [10] or, for the case of two-phase systems, Mls [22]. This conservation principle is combined with a constitutive relation between σ and the flux-density vector $\boldsymbol{w}$. Classical constitutive relations (such as Fourier's law in the theory of heat conduction or Fick's law in the diffusion theory) assume the form

$$\boldsymbol{w} = -K \operatorname{grad} \sigma + \beta \boldsymbol{v}, \tag{1.2}$$

where K and β are functions of σ, t and $\boldsymbol{x}$, K is a second rank symmetric positive definite tensor or a positive scalar, and $\boldsymbol{v}$ denotes the velocity field of the medium in which the substance is moving.

2 Equations of Groundwater Flow

In the theory of fluid flow in porous media, the fundamental balance law (1.1) is called the continuity equation. Unlike the above classical relations (1.2), the flux of the substance does not depend directly on the gradient of its amount. The applied constitutive law is Darcy's law, see, e.g., Bear and Cheng [1] or Fowler [10],

$$\boldsymbol{w} = -K\Big(\rho \operatorname{grad} x_3 + \frac{1}{g} \operatorname{grad} p\Big) + \sigma \boldsymbol{v},$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$ are space coordinates with x_3-axis oriented vertically upwards, g is gravity acceleration, p is fluid-phase pressure, K is hydraulic conductivity tensor, ρ is intrinsic density of the fluid phase, i.e. mass of unit volume of the fluid phase, σ is bulk density of the fluid phase and $\boldsymbol{v}$ is velocity field of the matrix (the solid phase); general relation between the flux densities of the liquid phase and the solid phase and the relative flux density of the liquid phase is presented in [22].

When the density ρ is constant, hydraulic head Φ is usually used to express Darcy's law,

$$\Phi = x_3 + h, \quad h = \frac{1}{\rho g} p, \tag{2.1}$$

where h is pressure head and x_3 is elevation head. Then Darcy's law reads

$$\boldsymbol{w} = -\rho K \operatorname{grad} \Phi + \sigma \boldsymbol{v}. \tag{2.2}$$

In the case of saturation, it is assumed that σ is a linear function of the pressure head

$$\sigma = \sigma_0 + \rho S_s h, \tag{2.3}$$

where S_s is the specific storativity defined as volume of water released from unit volume of porous medium in response to unit drop in pressure head, σ_0 is bulk density at zero pressure. Specific storativity is either a positive constant or a positive function. Inserting the constitutive relation (2.2) into the continuity equation (1.1), we obtain the dynamical equation

$$\rho S_s \frac{\partial \Phi}{\partial t} - \rho \operatorname{div}(K \operatorname{grad} \Phi) + \operatorname{div}(\sigma \boldsymbol{v}) = q, \tag{2.4}$$

see, e.g., Bear and Cheng [1]. This equation is commonly used in hydraulics of groundwater as the governing equation of groundwater flow problems. It contains one unknown function, hydraulic head Φ or pressure head h, other members are considered as given parameters.

Eq. (2.4) is of the parabolic type and, consequently, its solutions predict infinite velocity of the substance propagation. This means that the equation is not appropriate to describe high-speed processes and processes with large changes of momentum. Groundwater hydraulics studies mostly low-speed processes that are governed by Eq. (2.4) with a good accuracy. On the other hand, there are processes in groundwater that are modelled by means of jump conditions of the hydraulic head along certain specified surfaces, e.g. the domain boundaries. Moreover, groundwater hydraulics studies also processes that periodically change direction of groundwater flow. Such processes frequently concern large amounts of water. Small as the velocities are, the inertial effects may be significant. Consequently, applying the above introduced model, the involved inaccuracy may be considerable.

Bodvarsson [3] studied periodic changes in groundwater level observed in a borehole which were caused by tide forces or atmospheric pressure fluctuations. In such cases, the borehole serves as a piezometer for the hydraulic head in the aquifer in which the borehole is open. Bodvarsson found that, for relatively rapidly varying flow through wide openings of the fractured rock, the inertia effects cannot be disregarded. To prevent the inaccuracy due to commonly applied approach, Bodvarsson [3] suggested to modify Darcy's law by adding a linear inertia term of the form $\frac{\rho K}{\sigma g} \frac{\partial \boldsymbol{w}}{\partial t}$, where the notation introduced in this paper is used, to the left-hand side of Eq. (2.2). This suggestion is very close to that of Cattaneo [4] which we consider below. The Bodvarsson paper is often utilized in hydrogeological studies, e.g. [21, 26, 27].

In order to get a theoretical derivation of Darcy's law, Whitaker [29] studied flow through porous media under general conditions and presented the nature of the Darcian simplification of the problem. Mls [22] introduced a general approach based on d'Alembert's principle. The approach, however, requires to study both phases and to define several constitutive relations. Mls [24] studied groundwater movement and hydraulic head oscillations in a tidally influenced confined aquifer without adding any inertial term.

In this paper, we consider a relatively simple approach proposed by Cattaneo [4] in order to remove the paradox of the infinite propagation velocity of disturbances. He proposed to modify the classical constitutive relation, Fourier law, by adding a time derivative term to the left-hand side of (1.2):

$$\tau \frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{w} = -K \operatorname{grad} \sigma + \beta \boldsymbol{v}, \tag{2.5}$$

where the parameter τ is a measure of inertia effect incorporated into the final model. It is called the relaxation time and could be described as the response time of the model, i.e. the time that the model would need, according to its immediate reaction, to reach the reaction of the original model without the time derivative term.

The modified constitutive relation (2.5) together with the conservation principle (1.1) yields then the equation of the hyperbolic type and the propagation velocity is finite. There is a number of papers dealing with hyperbolic Cattaneo-type heat models, non-Fickian diffusion, suspension thickening, etc., mainly from the numerical point of view (see, e. g., [6, 7, 12, 13, 23, 28]). In the next section we take up the fluid flow in porous medium (groundwater flow equation) using the just described modification (2.5) of Darcy's law.

3 Hyperbolic Groundwater Flow Equation

According to equations (2.1) and (2.3), Eq. (1.1) becomes

$$\rho S_s \frac{\partial \Phi}{\partial t} + \operatorname{div} \boldsymbol{w} = q, \tag{3.1}$$

and hence

$$\frac{\partial}{\partial t} \operatorname{div} \boldsymbol{w} = -\rho S_s \frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial q}{\partial \Phi} \frac{\partial \Phi}{\partial t} + \frac{\partial q}{\partial t}. \tag{3.2}$$

Now, we adopt the Cattaneo approach to generalize Darcy's law (2.2). In the same way as in Eq. (2.5) we include one linear inertia term proportional to the time derivative of the fluid-phase flux density to (2.2) and obtain

$$\tau \frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{w} = -\rho K \operatorname{grad} \Phi + \sigma \boldsymbol{v},$$

where τ is the relaxation time. Divergence of this vector equation reads

$$\tau \frac{\partial}{\partial t} \operatorname{div} \boldsymbol{w} + \operatorname{div} \boldsymbol{w} = -\rho \operatorname{div} (K \operatorname{grad} \Phi) + \operatorname{div}(\sigma \boldsymbol{v}). \tag{3.3}$$

As the balance equations (3.1) and (3.2) give

$$\tau\frac{\partial}{\partial t}\operatorname{div}\boldsymbol{w}+\operatorname{div}\boldsymbol{w}=-\tau\rho S_s\frac{\partial^2\Phi}{\partial t^2}+\tau\frac{\partial q}{\partial\Phi}\frac{\partial\Phi}{\partial t}+\tau\frac{\partial q}{\partial t}-\rho S_s\frac{\partial\Phi}{\partial t}+q, \tag{3.4}$$

the right-hand side of Eq. (3.4) can be substituted for the left-hand side of Eq. (3.3) giving general form of the final governing equation:

$$\tau\rho S_s\frac{\partial^2\Phi}{\partial t^2}+\left(\rho S_s-\tau\frac{\partial q}{\partial\Phi}\right)\frac{\partial\Phi}{\partial t}-\rho\operatorname{div}(K\operatorname{grad}\Phi)+\operatorname{div}(\sigma\boldsymbol{v})-\tau\frac{\partial q}{\partial t}-q=0. \tag{3.5}$$

In most cases, there are no sources within the investigated aquifers. On the other hand, under conditions of nearly horizontal flow, hydraulics of aquifers makes use of Dupuit's assumption of constant value of hydraulic head along vertical lines and lowers the dimension of the solved problem. In such cases, any exchange of groundwater between the aquifer and its neighbourhood does not go through the boundary of the domain and makes a source term instead. Usually, see, e.g., [1], the source term is expressed in the form

$$q=D(H-\Phi), \tag{3.6}$$

where H is a constant and D is a positive function of space coordinates or a positive constant. We will further assume that the speed of the solid matrix is negligible and put $\boldsymbol{v}=0$. Denoting $u=\Phi-H$ and making use of Eq. (3.6) and of the assumption $\boldsymbol{v}=0$, we get Eq. (3.5) in the form

$$\tau\rho S_s\frac{\partial^2 u}{\partial t^2}+(\rho S_s+\tau D)\frac{\partial u}{\partial t}-\rho\operatorname{div}(K\operatorname{grad}u)+Du=0. \tag{3.7}$$

We will derive conditions under which any global solution, i.e. defined for $t\in\mathbb{R}^+=[0,+\infty)$, of the above problems has an oscillatory character for $t\to+\infty$. Moreover, we will find an estimate of the oscillatory time, a quantitative measure of oscillations, and show that the estimate is independent of those solutions.

4 Oscillation of Solutions

We deal with the autonomous equation (3.7) in a somewhat more general manner. Let $n\in\mathbb{N}$, $\Omega\subset\mathbb{R}^n$ be a bounded domain with sufficiently regular boundary $\partial\Omega$, τ and α_0 be given positive constants, $\boldsymbol{x}\mapsto\alpha(\boldsymbol{x})$, $\alpha:\bar{\Omega}\to\mathbb{R}$ and $(\boldsymbol{x},u)\mapsto f(\boldsymbol{x},u)$, $f:\bar{\Omega}\times\mathbb{R}\to\mathbb{R}$ be given functions.

Further, let $A(\boldsymbol{x})=\big(a_{jk}(x)\big)_{j,k=1}^n$ be a matrix of functions from $C^1(\bar{\Omega})$, which is symmetric ($a_{jk}(\boldsymbol{x})=a_{kj}(\boldsymbol{x})$, $\boldsymbol{x}\in\bar{\Omega}$) and positive definite uniformly with respect to $\boldsymbol{x}\in\bar{\Omega}$, i.e. there exists $\eta>0$ such that

$$\sum_{j,k=1}^n a_{jk}(\boldsymbol{x})\xi_j\xi_k\geqslant\eta\sum_{j=1}^n\xi_j^2,\quad(\xi_j)_{j=1}^n\in\mathbb{R}^n,\ \boldsymbol{x}\in\bar{\Omega}.$$

Let $\boldsymbol{B}(\boldsymbol{x})=\big(B_j(\boldsymbol{x})\big)_{j=1}^n$ be a vector of functions from $C^1(\bar{\Omega})$ and $c\in C(\bar{\Omega})$.

We are interested in oscillatory properties of a function $(t,\boldsymbol{x}) \mapsto u(t,\boldsymbol{x})$, $u : \mathbb{R}^+ \times \Omega \to \mathbb{R}$, that solves the equation

$$\tau\frac{\partial^2 u}{\partial t^2} + \big(\alpha_0 + \alpha(\boldsymbol{x})\tau\big)\frac{\partial u}{\partial t} + Lu + f(\boldsymbol{x},u) = 0 \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{4.1}$$

where

$$Lu = -\operatorname{div}\big(A(\boldsymbol{x})\operatorname{grad} u\big) + \boldsymbol{B}(\boldsymbol{x})\cdot\operatorname{grad} u + c(\boldsymbol{x})u,$$

supplemented, for definiteness and simplicity, with the homogeneous Dirichlet boundary condition $u = 0$ on $\mathbb{R}^+ \times \partial\Omega$. Under stronger assumptions more general boundary conditions in a special case $\alpha \equiv 0$ are dealt with in [18].

Eq. (3.7) is subsumed into Eq. (4.1) and it is clear how the parameters (coefficients, source terms) of these equations are related. Hence, in the sequel we specify all further assumptions in terms of parameters of Eq. (4.1) (the sufficient smoothness of which is always assumed). It can be seen below that these assumptions do not contradict the properties of parameters of Eq. (3.7).

By local existence theory under smoothness properties of α and f, the initial–boundary value problem given by equation (4.1) and initial conditions $u(0,\cdot) = u_0$, $\frac{\partial u}{\partial t}(0,\cdot) = u_1$ has for any $(u_0,u_1) \in \overset{\circ}{W}{}_2^1(\Omega) \times L_2(\Omega)$ a unique solution $(t,x) \mapsto u(t,x)$ with finite energy defined on a maximal interval, $u \in C([0,t_{\max}), \overset{\circ}{W}{}_2^1(\Omega)) \cap C^1([0,t_{\max}), L_2(\Omega))$. These local solutions can be extended to exist for all time $t \in \mathbb{R}^+$ (under some growth condition on f in u, e.g., if the growth of f is at most linear for $|u| \to +\infty$) and we denote by $\mathcal{U}$ the non-empty set of all global solutions (see, e.g., [14]).

It is known (e.g. [2, 8, 11]) that the operator

$$L^+v = -\operatorname{div}\big(A(\boldsymbol{x})\operatorname{grad} v\big) - \operatorname{div}(\boldsymbol{B}(\boldsymbol{x})v) + c(\boldsymbol{x})v,$$

under homogeneous Dirichlet boundary condition has principal eigenvalue λ_1 and an associated (principal) eigenfunction v_1 which are both *real* (even if L^+ is not symmetric). The function v_1 is bounded, in fact $v_1 \in C(\bar{\Omega})$ (if $\partial\Omega$ has some degree of regularity), v_1 is positive in Ω, $L^+v_1 = \lambda_1 v_1$. Moreover, for any eigenvalue λ, $\operatorname{Re}\lambda \geqslant \lambda_1$ and λ_1 is a simple eigenvalue, i.e. v_1 spans the null space $\ker(L^+ - \lambda_1)$.

Let us recall that (in accordance with [5, 16, 17]), in general, a measurable function $u : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is said to be globally oscillatory (about zero at $+\infty$) if there exists a constant $\Theta > 0$, the oscillatory time, such that for any interval $J \subset \mathbb{R}^+$, the length $|J|$ of which is greater than Θ, u changes the sign on $J \times \Omega$, i.e. we have simultaneously $\operatorname{meas}\{(t,\boldsymbol{x}) \in J \times \Omega \mid u(t,\boldsymbol{x}) > 0\} > 0$ and $\operatorname{meas}\{(t,\boldsymbol{x}) \in J \times \Omega \mid u(t,\boldsymbol{x}) < 0\} > 0$. Roughly speaking, this means, for a continuous function $u (\not\equiv 0)$, that u has a zero in any domain $J \times \Omega$, where $J \subset \mathbb{R}^+$ is an interval the length of which is sufficiently large and this length can be chosen independently of J. The oscillatory time Θ defines the maximal length of time intervals on which non-zero solutions u can remain non-negative (non-positive) throughout the domain Ω.

We prove that the following assumptions, together with $\tau \in \mathcal{T}$ introduced in (4.10) (or (4.11)), ensure the oscillatory properties of solutions. Let there exist a non-negative constant α_1 such that

$$0 \leqslant \alpha(\boldsymbol{x}) \leqslant \alpha_1, \quad \boldsymbol{x} \in \bar{\Omega}. \tag{4.2}$$

Further, let there exist a constant f_0 such that (α_0 is a positive constant)

$$\lambda_1 + f_0 > \alpha_0\alpha_1, \quad uf(\boldsymbol{x}, u) \geqslant f_0 u^2, \quad (\boldsymbol{x}, u) \in \bar{\Omega} \times \mathbb{R}. \tag{4.3}$$

In the linear case and with constant coefficients, the oscillations can be proved similarly as in [19] and [20] with the aid of results in [25], but the following approach solves more general problems which are nonlinear and with coefficients depending on spatial variables.

We prove the following theorem.

Theorem 1. *Let the assumptions* (4.2) *and* (4.3) *be satisfied. Then there exists an open interval $\mathcal{T} \subset \mathbb{R}^+$ such that for any $\tau \in \mathcal{T}$ Eq.* (4.1) (*under homogeneous Dirichlet condition*) *is uniformly globally oscillatory, i.e. there exists $\Theta > 0$ such that any solution $u \in \mathcal{U}$ is globally oscillatory with the oscillatory time Θ.*

In order to prove that a solution $u \in \mathcal{U}$ is globally oscillatory we shall assume that u does not change the sign on $J \times \Omega$, where $J = (\tau_1, \tau_2) \subset \mathbb{R}^+$ is any interval with the length $|J| = \tau_2 - \tau_1$ sufficiently large (this quantity, oscillatory time, can be estimated and is independent of u), and we prove that then necessarily $u \equiv 0$ on $J \times \Omega$ and hence on $\mathbb{R}^+ \times \Omega$ (in virtue of the unique solvability of the initial–boundary value problem).

Of considerable importance in the proof is the function $t \mapsto \gamma(t)$ with the following properties. Let τ, q, m and M be constants satisfying

$$\tau > 0, \quad q > 0, \quad -\sqrt{q} < m \leqslant M < \sqrt{q}. \tag{4.4}$$

Then there exist constants $\Theta > 0$ and $\varepsilon > 0$ such that for any interval (τ_1, τ_2) with $\tau_2 - \tau_1 > \Theta$ there exists a function $t \mapsto \gamma(t)$ with the properties

$$\gamma \in C^2\left([\tau_1, \tau_2]\right), \tag{4.5a}$$

$$\gamma > 0 \quad \text{in } (\tau_1, \tau_2), \quad \gamma(\tau_1) = \gamma(\tau_2) = 0, \tag{4.5b}$$

$$\dot{\gamma}(\tau_1) > 0, \quad \dot{\gamma}(\tau_2) < 0, \tag{4.5c}$$

and satisfying the equation

$$\ddot{\gamma} + 2\left(-M\dot{\gamma}^+ + m\dot{\gamma}^-\right) + (q - \varepsilon/\tau)\gamma = 0 \quad \text{in } (\tau_1, \tau_2). \tag{4.6}$$

Let us recall that $\dot{\gamma}^{\pm}(t) = \max\{\pm\dot{\gamma}(t), 0\}$, hence $\dot{\gamma}(t) = \dot{\gamma}^+(t) - \dot{\gamma}^-(t)$ and $|\dot{\gamma}(t)| = \dot{\gamma}^+(t) + \dot{\gamma}^-(t)$. Explicit formulae for the function γ and the constant Θ can be found in [15] and [16]. Now, using $\gamma(t)v_1(\boldsymbol{x})$ as a test function in (4.1) we obtain in virtue of (4.5a) and (4.5b)

$$\begin{aligned} 0 = \tau \int_\Omega &\big[\dot{\gamma}(\tau_1)u(\tau_1, \boldsymbol{x}) - \dot{\gamma}(\tau_2)u(\tau_2, \boldsymbol{x})\big] v_1(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\ &+ \int_{\tau_1}^{\tau_2} \int_\Omega \Big[\big(\tau\ddot{\gamma} - \big(\alpha_0 + \alpha(\boldsymbol{x})\tau\big)(\dot{\gamma}^+ - \dot{\gamma}^-) + \lambda_1\gamma\big)uv_1 + f(\boldsymbol{x}, u)\gamma v_1\Big]\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t. \end{aligned}$$

Estimating further, for u of one sign in $J \times \Omega$, we get in view of (4.2), (4.3) and (4.5c)

$$0 \geqslant (\operatorname{sgn} u) \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\tau\ddot{\gamma} + (-\alpha_0 - \alpha_1\tau)\dot{\gamma}^+ + \alpha_0\dot{\gamma}^- + (\lambda_1 + f_0)\gamma\right] u v_1 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$= \varepsilon(\operatorname{sgn} u) \int_{\tau_1}^{\tau_2} \int_{\Omega} \gamma u v_1 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \tag{4.7}$$

The last equality is true, with some $\varepsilon > 0$, after the use of (4.6) with

$$q = \frac{\lambda_1 + f_0}{\tau}, \quad M = \frac{\alpha_0 + \alpha_1\tau}{2\tau}, \quad m = \frac{\alpha_0}{2\tau}.$$

In view of (4.2) and (4.3) the assumption $q > 0$ is fulfilled since

$$\lambda_1 + f_0 > \alpha_0\alpha_1 \geqslant 0. \tag{4.8}$$

The remaining assumptions in (4.4) are fulfilled if for the parameters α_0, α_1, f_0, λ_1 and τ the following quadratic inequality in τ is valid (assume first $\alpha_1 > 0$):

$$\alpha_1^2\tau^2 + 2\left[\alpha_0\alpha_1 - 2(\lambda_1 + f_0)\right]\tau + \alpha_0^2 < 0. \tag{4.9}$$

The assumption (4.8) ensures also the positivity of the discriminant

$$\delta = 4(\lambda_1 + f_0)\,(\lambda_1 + f_0 - \alpha_0\alpha_1) > 0$$

and the inequality (4.9) holds if

$$\tau \in \mathcal{T} = (\tau_-, \tau_+), \quad \tau_\pm = \frac{-\alpha_0\alpha_1 + 2(\lambda_1 + f_0) \pm \sqrt{\delta}}{\alpha_1^2}, \tag{4.10}$$

where the roots $\tau_\pm$ of the corresponding quadratic equation fulfil

$$\tau_- + \tau_+ = 2\left[-\alpha_0\alpha_1 + 2(\lambda_1 + f_0)\right]/\alpha_1^2, \quad \tau_-\tau_+ = \alpha_0^2/\alpha_1^2,$$

hence they are positive (if $\alpha_1 > 0$). If $\alpha_1 = 0$ then the assumption (4.4) is ensured by

$$\tau \in \mathcal{T} = \left(\frac{\alpha_0^2}{4(\lambda_1 + f_0)}, +\infty\right). \tag{4.11}$$

Returning to the inequality (4.7) for $\tau \in \mathcal{T}$ it is easily seen that according to the positivity of the constant ε, the positivity of the function v_1 on Ω, the positivity of the function γ on J we get $u \equiv 0$ on $J \times \Omega$, hence on $\mathbb{R}^+ \times \Omega$, and this completes the proof.

Conclusions

The direct application of Darcy's law to a mass-balance equation, which is the mostly used approach, leads to a parabolic equation. In order to prevent the possible inaccuracy of the model, the Cattaneo correction can be adopted.

In this article, we focused on the character of the Cattaneo model of flow in porous media in which the relaxation time τ is introduced. For any autonomous problem sufficient conditions are formulated which guarantee the existence of an interval $\mathcal{T} \subset \mathbb{R}^+$ such that solutions are oscillatory provided that $\tau \in \mathcal{T}$.

Finally, we showed that the oscillatory time can be estimated and does not depend on the particular solution.

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