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Nonexistence of Solutions Results for Certain Fractional Differential Equations

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Abstract. This paper is meant to establish sufficient conditions for the nonexistence of weak solutions to nonlinear fractional diffusion equation in space and time with nonlinear convective term. The Fujita exponent is determined.
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1 Introduction

In his pioneering article [3], Fujita considered the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^{1+\tilde{p}} & \text{in } Q = \mathbb{R}^n \times \mathbb{R}^+, \\ u(0,x) = a(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\tilde{p} > 0$. If $p_c = 2/n$, he proved that:

1. If $0 < \tilde{p} < p_c$ and $a(x_0) > 0$ for some x_0, then any solution to (1.1) blows-up in a finite time.

2. If $\tilde{p} > p_c$ then there exist a solution in Q. The critical case $\tilde{p} = p_c$ was decided later by Hayakawa [6] for $n = 1, 2$ and by Kobayashi, Sirao and Tanaka [11] for $n \geq 3$.

Analogous blow-up results for (1.1) with fractional laplacian and the critical exponent for the non existence of solutions are contained, e.g., in [1, 2, 4]. Finally, Kirane and Qafsaoui [10] treated the more general equation

$$u_t + (-\Delta)^{\beta/2}(u^m) + a(x,t).\nabla u^q = h(x,t)u^{1+\tilde{p}}, \quad \text{in } Q.$$

Let us consider the following nonlinear fractional differential equation

$$\begin{cases} \mathbf{D}^{\alpha}_{0|t} u + (-\Delta)^{\frac{\beta}{2}} |u|^m + a(x).\nabla(|u|^{q-1}u) = h(x,t)|u|^p, & \text{in } Q, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

where $\mathbf{D}^{\alpha}_{0|t}$ denotes the fractional time-derivative of arbitrary order $\alpha \in (0,1)$ in the Caputo sense, $(-\Delta)^{\frac{\beta}{2}}$, $0 \le \beta \le 2$ is the $(\frac{\beta}{2})$-fractional power of the Laplacian $-\Delta_x$ in the x variable, defined by

$$(-\Delta)^{\frac{\beta}{2}} v(x,t) = \mathcal{F}^{-1}\left(|\xi|^{\beta}\mathcal{F}(v)(\xi)\right)(x,t),$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse, $a(x) := (a_1(x), \ldots, a_n(x))$, $h(x,t)$ is given function, $a(x).\nabla(|u|^{q-1}u)$ is the scalar product of $a(x)$ and $\nabla(|u|^{q-1}u)$ and the exponents p, q and m are positive constants $p > 1$, $q > 1$ and $m > 1$.

The main problem we encounter here arises from the nonlocal nature of the fractional operators. To overcome this difficulties, we use a versatile method which has been used by Mitidieri and Pohozaev [12], Pohozaev and Tesei [14], Hakem [5], Zhang [16, 17].

We recall here some definitions of fractional derivative. The left-handed derivative and the right-handed derivative in the Riemann–Liouville sense (see [8, 13]), for $\Psi \in L^1(0,T)$, $0 < \alpha < 1$ are defined as follows:

$$\left(D^{\alpha}_{0|t}\Psi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{\Psi(\sigma)}{(t-\sigma)^{\alpha}}\,d\sigma,$$

where the symbol Γ stands for the usual Euler gamma function, and

$$\left(D^{\alpha}_{t|T}\Psi\right)(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T \frac{\Psi(\sigma)}{(\sigma-t)^{\alpha}}\,d\sigma,$$

respectively. The Caputo fractional derivative

$$\left(\mathbf{D}^{\alpha}_{0|t}\Psi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\Psi'(\sigma)}{(t-\sigma)^{\alpha}}\,d\sigma,$$

requires $\Psi' \in L^1(0,T)$. Clearly we have

$$\left(D^{\alpha}_{0|t}g\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{g(0)}{t^{\alpha}} + \int_0^t \frac{g'(\sigma)}{(t-\sigma)^{\alpha}}\,d\sigma\right],$$

and

$$\left(D^{\alpha}_{t|T}f\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(T)}{(T-t)^{\alpha}} - \int_t^T \frac{f'(\sigma)}{(\sigma-t)^{\alpha}}\,d\sigma\right]. \tag{1.3}$$

The Caputo derivative is related to the Riemann–Liouville derivative by

$$\mathbf{D}^{\alpha}_{0|t}\Psi(t) = D^{\alpha}_{0|t}\left[\Psi(t) - \Psi(0)\right].$$

We have the formula of integration by parts (see [15, p. 46]).

$$\int_0^T f(t)\left(D^{\alpha}_{0|t}g\right)(t)\,dt = \int_0^T g(t)\left(D^{\alpha}_{t|T}f\right)(t)\,dt.$$

We need also the following result.

Lemma 1. *Let $0 \le \delta \le 2$ and let $l \ge 1$. Then, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi \ge 0$ and any $x \in \mathbb{R}^n$,*

$$(-\Delta)^{\delta/2} \psi^l(x) \le l\psi^{l-1}(x)(-\Delta)^{\delta/2}\psi(x).$$

Proof. (See Proposition 3.3 in [7].) The cases $\delta = 0$ and 1 are obvious. If $\delta \in (0,1)$, we have

$$(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta)\int_{\mathbb{R}^n} \frac{\psi(x+z)-\psi(x)}{|z|^{n+\delta}}\,dz, \quad \forall x \in \mathbb{R}^n,$$

where $c_n(\delta) = \delta\Gamma((n+\delta)/2)/2\pi^{n/2+\delta}\Gamma(1-\delta/2)$. Then

$$\psi^{l-1}(x)(-\Delta)^{\delta/2}\psi(x) = -c_n(\delta)\int_{\mathbb{R}^n}\frac{\psi^{l-1}(x)\psi(x+z)-\psi^l(x)}{|z|^{n+\delta}}\,dz.$$

The case $l = 1$ is clear. If $l > 1$, then by Young's inequality we have

$$\psi^{l-1}(x)\psi(x+z) \le \frac{l-1}{l}\psi^l(x) + \frac{1}{l}\psi^l(x+z).$$

Therefore

$$\psi^{l-1}(x)(-\Delta)^{\delta/2}\psi(x) \ge \frac{-c_n(\delta)}{l}\int\limits_{\mathbb{R}^n}\frac{\psi^l(x+z)-\psi^l(x)}{|z|^{n+\delta}}dz = \frac{1}{l}(-\Delta)^{\delta/2}\psi^l(x).$$

If $\delta \in (1,2)$, we have

$$(-\Delta)^{\delta/2}\psi(x) = -c_n(\delta)\int_{\mathbb{R}^n}\frac{\psi(x+z)-\psi(x)-\nabla\psi(x).z}{|z|^{n+\delta}}\,dz, \quad \forall x \in \mathbb{R}^n.$$

Then

$$\begin{aligned}
&\psi^{l-1}(x)(-\Delta)^{\delta/2}\psi(x)\\
&\quad \ge -c_n(\delta)\int_{\mathbb{R}^n}\frac{\left(\psi^l(x+z)-\psi^l(x)\right)/l-(\nabla\psi(x).z)\psi^{l-1}(x)}{|z|^{n+\delta}}\,dz\\
&\quad = \frac{-c_n(\delta)}{l}\int_{\mathbb{R}^n}\frac{\left(\psi^l(x+z)-\psi^l(x)\right)-\nabla\psi^l(x).z}{|z|^{n+\delta}}\,dz = \frac{1}{l}(-\Delta)^{\delta/2}\psi^l(x).
\end{aligned}$$

□

We adopt the following definition.

Definition 1. A function $u \in L^p_{loc}(Q)$ $(Q := \mathbb{R}^n \times (0,+\infty))$ is a weak solution to (2) defined on Q, if $uh^{1/p} \in L^1_{loc}(Q, dx\,dt)$ such that

$$\begin{aligned}
&\int_Q h(x,t)\varphi|u|^p\,dx\,dt + \int_Q u_0 D^\alpha_{t|T}\varphi\,dx\,dt = \int_Q uD^\alpha_{t|T}\varphi\,dx\,dt \qquad (1.4)\\
&\quad + \int\limits_Q |u|^m(-\Delta)^{\frac{\beta}{2}}\varphi\,dx\,dt - \sum_{i=1}^N\int\limits_Q|u|^{q-1}u\varphi\frac{\partial a_i}{\partial x_i}\,dx\,dt - \int\limits_Q|u|^{q-1}ua.\nabla\varphi\,dx\,dt,
\end{aligned}$$

for any test function $\varphi \in C^{2,1}_{x,t}(Q)$.

2 The Results

To begin, we set some hypotheses: the function h is assumed to satisfy

$$(\mathrm{H}_h) \quad h\big(yR, \tau T^{\beta/\alpha}\big) \geq C_h R^{\sigma} T^{\rho\beta/\alpha}, \quad C_h > 0,$$

for some σ, $\rho > 0$ to be determined later, R, T large and $\tau \geq 0$, y in a bounded domain. It can easily be seen that there is no conditions imposed on σ.

The vector functions $a(x) = (a_1(x), \dots, a_n(x))$ are required to satisfy

$$(\mathrm{H}_a) \quad |a_i(x)| \sim c\,|x|^{\delta_i} \quad \text{for } |x| \text{ large and } \delta_i > 2.$$

For later use, we define $\delta = \max(\delta_i)$. Our main result is the following.

Theorem 1. *Let $n \geq 1, p, q$ and m be such that $p > \max\{m, q\} > 1$. The exponent ρ satisfies*

$$(\rho + 1) > \max\{p/m, (1-\alpha)\,p, p/q\}.$$

Assume that $(\mathrm{H}_h,\ \mathrm{H}_a)$ are satisfied and $u_0(x)$ satisfies $u_0(x) \geq 0$. If

$$p \leq p_c = \min\left(1 + \frac{\alpha(\sigma+\beta) + \beta\rho}{\alpha n + \beta(1-\alpha)}, \frac{((\alpha n + \beta) + (\alpha\sigma + \beta\rho))q}{((\delta - 1)\alpha + (n\alpha + \beta))}\right),$$

then problem (1.2) *does not admit non trivial weak solutions defined on Q.*

Proof. Let u be a nontrivial solution and ξ be a smooth function such that $\xi(x, T) = 0$. From (1.4), if $Q_T := \mathbb{R}^n \times (0, T)$, we get

$$\int_{Q_T} h(x,t)\xi\,|u|^p\,dx\,dt + \int_{Q_T} u_0 D^{\alpha}_{t|T}\xi\,dx\,dt = \int_{Q_T} u D^{\alpha}_{t|T}\,\xi\,dx\,dt \tag{2.1}$$
$$+ \int_{Q_T} |u|^m (-\Delta)^{\frac{\beta}{2}}\,\xi\,dx\,dt - \sum_{i=1}^{n}\int_{Q_T} |u|^{q-1} u\xi \frac{\partial a_i}{\partial x_i}\,dx\,dt - \int_{Q_T} |u|^{q-1} ua.\nabla\xi\,dx\,dt.$$

For later use, let $\varPhi$ be a smooth nonincreasing function such that

$$\varPhi(z) = \begin{cases} 1 & \text{if } 0 \leq z \leq 1, \\ 0 & \text{if } z \geq 2, \end{cases}$$

$z|\varPhi'(z)| \leq C$ and $0 \leq \varPhi \leq 1$. Let R be fixed positive number, then we set

$$\xi(x,t) := \varPhi^l\left(\left(|x|^2 + t^{\theta}\right)/R^2\right),$$

where $l > \max\{\frac{p}{(p-m)}, \frac{p}{(p-q)}\}$ and θ is positive real number to be determined later. We see from the definition of ξ that for $T^{\theta} \geq 2R^2$ we have $\xi(x, T) = 0$.

To estimate the right-hand side of (2.1) on $Q_{TR^{2/\theta}}$, we have by using the Lemma 1,

$$\int_{Q_{TR^{2/\theta}}} |u|^m (-\Delta)^{\beta/2}\,\xi(x,t)\,dx\,dt = \int_{Q_{TR^{2/\theta}}} |u|^m (-\Delta)^{\beta/2}\,\varPhi^l\left(\frac{|x|^2 + t^{\theta}}{R^2}\right)dx\,dt$$
$$\leq l \int_{Q_{TR^{2/\theta}}} |u|^m \varPhi^{l-1} (-\Delta)^{\beta/2}\,\varPhi\left(\frac{|x|^2 + t^{\theta}}{R^2}\right)dx\,dt,$$

we remark that

$$\int\limits_{Q_{TR^{2/\theta}}} |u|^m \varPhi^{l-1} (-\varDelta)^{\beta/2} \varPhi\left(\frac{|x|^2+t^\theta}{R^2}\right) dx\,dt$$
$$= \int\limits_{Q_{TR^{2/\theta}}} h^{\frac{m}{p}} |u|^m \varPhi^{l\frac{m}{p}} \varPhi^{-l\frac{m}{p}} h^{-\frac{m}{p}} \varPhi^{l-1} (-\varDelta)^{\beta/2} \varPhi\left(\frac{|x|^2+t^\theta}{R^2}\right) dx\,dt,$$

and by ε-Young's inequality,

$$XY \le \varepsilon X^p + C(\varepsilon)\, Y^{p'}, \quad p+p' = pp',\ X \ge 0,\ Y \ge 0,$$

we find

$$\int\limits_{Q_{TR^{2/\theta}}} |u|^m \varPhi^{l-1} (-\varDelta)^{\beta/2} \varPhi\left(\frac{|x|^2+t^\theta}{R^2}\right) dx\,dt \le \varepsilon \int\limits_{Q_{TR^{2/\theta}}} h\,|u|^p\, \xi\, dx\,dt$$
$$+ C(\varepsilon) \int\limits_{Q_{TR^{2/\theta}}} h^{-\frac{m}{p-m}} \varPhi^{-l\frac{m}{p-m}} \left|\varPhi^{l-1} (-\varDelta)^{\beta/2} \varPhi\right|^{\frac{p}{p-m}} dx\,dt$$
$$= \varepsilon \int\limits_{Q_{TR^{2/\theta}}} h\,|u|^p\, \xi\, dx\,dt + C(\varepsilon) \int\limits_{Q_{TR^{2/\theta}}} h^{-\frac{m}{p-m}} \varPhi^{l-\frac{p}{p-m}} \left|(-\varDelta)^{\beta/2} \varPhi\right|^{\frac{p}{p-m}} dx\,dt$$
$$= \varepsilon \int\limits_{Q_{TR^{2/\theta}}} h\,|u|^p\, \xi\, dx\,dt + C(\varepsilon) \int\limits_{\operatorname{supp}\xi} h^{-\frac{m}{p-m}} \xi^{1-\frac{p}{(p-m)l}} \left|(-\varDelta)^{\beta/2} \varPhi\right|^{\frac{p}{p-m}} dx\,dt,$$

the requirement $l > \frac{p}{(p-m)} > 1$ (i.e. $1 - \frac{p}{(p-m)l} > 0$) ensure that

$$\int\limits_{\operatorname{supp}\xi} h^{-\frac{m}{p-m}} \xi^{1-\frac{p}{(p-m)l}} \left|(-\varDelta)^{\beta/2} \varPhi\right|^{\frac{p}{p-m}} dx\,dt < \infty.$$

Similarly,

$$\int\limits_{Q_{TR^{2/\theta}}} u D^\alpha_{t|TR^{2/\theta}} \xi\, dx\,dt = \int\limits_{Q_{TR^{2/\theta}}} u\,(h\xi)^{\frac{1}{p}}\,(h\xi)^{\frac{-1}{p}}\, D^\alpha_{t|TR^{2/\theta}} \xi\, dx\,dt$$
$$\le \varepsilon \int\limits_{Q_{TR^{2/\theta}}} h\xi\,|u|^p\, dx\,dt + C(\varepsilon) \int\limits_{Q_{TR^{2/\theta}}} (h\xi)^{\frac{-1}{p-1}} \left|D^\alpha_{t|TR^{2/\theta}} \xi\right|^{\frac{p}{p-1}} dx\,dt.$$

Integrating by parts, one obtains

$$\int\limits_{Q_{TR^{2/\theta}}} a.\nabla\left(|u|^{q-1} u\right) \xi\, dx\,dt = -\int\limits_{Q_{TR^{2/\theta}}} |u|^{q-1} u a.\nabla\xi\, dx\,dt$$
$$- \int\limits_{Q_{TR^{2/\theta}}} \sum_{i=1}^n |u|^{q-1} u\xi \frac{\partial a}{\partial x_i}\, dx\,dt.$$

Now writing

$$\int_{Q_{TR^{2/\theta}}} |u|^{q-1} u\xi \sum_{i=1}^{n} \frac{\partial a}{\partial x_i}\, dx\, dt = \int_{Q_{TR^{2/\theta}}} |u|^{q-1} u\, (h\xi)^{\frac{q}{p}}\, h^{\frac{-q}{p}} \xi^{\frac{(p-q)}{p}} \sum_{i=1}^{n} \frac{\partial a}{\partial x_i}\, dx\, dt,$$

and using the ε-Young inequality again, we get

$$\int_{Q_{TR^{2/\theta}}} |u|^{q-1} u\xi \sum_{i=1}^{n} \left(\frac{\partial a}{\partial x_i}\right) dx\, dt$$
$$\leq \varepsilon \int_{Q_{TR^{2/\theta}}} h\xi\, |u|^{p}\, dx\, dt + C(\varepsilon) \int_{Q_{TR^{2/\theta}}} h^{\frac{-q}{p-q}} \xi \left|\sum_{i=1}^{n} \frac{\partial a}{\partial x_i}\right|^{\frac{p}{p-q}} dx\, dt.$$

Similarly, we obtain

$$\int_{Q_{TR^{2/\theta}}} |u|^{q-1} u\, (a.\nabla\xi) \leq \varepsilon \int_{Q_{TR^{2/\theta}}} h\xi\, |u|^{p}\, dx\, dt$$
$$+ C(\varepsilon) \int_{Q_{TR^{2/\theta}}} (h\xi)^{\frac{-q}{p-q}}\, |a.\nabla\xi|^{\frac{p}{p-q}}\, dx\, dt.$$

Gathering the above estimates and taking ε small, we obtain via (1.3)

$$\int_{Q_{TR^{2/\theta}}} |u|^{p}\xi h\, dx\, dt \leq \int_{Q_{TR^{2/\theta}}} u_0 D^{\alpha}_{t|TR^{2/\theta}}\xi\, dx\, dt + \int_{Q_{TR^{2/\theta}}} |u|^{p}\xi h\, dx\, dt$$
$$\leq C(\varepsilon)\Bigg(\int_{Q_{TR^{2/\theta}}} (h\xi)^{\frac{-1}{p-1}} \left|D^{\alpha}_{t|TR^{2/\theta}}\xi\right|^{\frac{p}{p-1}} dx\, dt$$
$$+ \int_{Q_{TR^{2/\theta}}} h^{\frac{-q}{p-q}} \xi \left(\left|\sum_{i=1}^{n} \frac{\partial a}{\partial x_i}\right|^{\frac{p}{p-q}} + |a.\nabla\xi|^{\frac{p}{p-q}} \right) dx\, dt$$
$$+ \int_{Q_{TR^{2/\theta}}} h^{-\frac{m}{p-m}} \xi^{1-\frac{p}{(p-m)l}} \left|(-\Delta)^{\beta/2}\Phi\right|^{\frac{p}{p-m}} dx\, dt\Bigg). \quad (2.2)$$

Now, let us perform the change of variables $\tau = t/R^{\frac{2}{\theta}}$, $y = x/R$, and set

$$\Omega := \left\{(y,\tau) \in \mathbb{R}^n \times \mathbb{R}^+,\ |y|^2 + \tau^{\theta} < 2\right\}, \quad \mu\,(y,\tau) := |y|^2 + \tau^{\theta}.$$

We have

$$\int_{Q_{TR^{2/\theta}}} \left|D^{\alpha}_{t|TR^{2/\theta}}\,\xi\right|^{p/(p-1)} (h\xi)^{-1/(p-1)}\, dx\, dt$$
$$\leq CR^{-\frac{2}{\theta}\alpha\frac{p}{p-1}+n+\frac{2}{\theta}-\frac{1}{p-1}\left(\sigma+\frac{2\rho}{\theta}\right)} \times \int_{\Omega} \left|D^{\alpha}_{\tau|T}\Phi^{l} o\mu\right|^{\frac{p}{p-1}} \left(\Phi^{l} o\mu\right)^{-\frac{1}{p-1}}\, dy\, d\tau,$$

$$\int\limits_{Q_{TR^{2/\theta}}} h^{-\frac{m}{p-m}}\xi^{1-\frac{p}{(p-m)l}} \left|(-\Delta)^{\beta/2}\Phi\right|^{\frac{p}{p-m}}\,dx\,dt$$
$$\leq CR^{\frac{-\beta p}{p-m}+n+\frac{2}{\theta}-\frac{m}{p-m}\left(\sigma+\frac{2\rho}{\theta}\right)}\int_{\Omega}\left|(-\Delta)^{\frac{\beta}{2}}\Phi o\mu\right|^{\frac{p}{p-m}}(\Phi o\mu)^{l-\frac{p}{(p-m)}}\,dy\,d\tau,$$

$$\int\limits_{Q_{TR^{2/\theta}}} h^{\frac{-q}{p-q}}\left|\sum_{i=1}^{n}\frac{\partial a}{\partial x_i}\right|^{\frac{p}{p-q}}\xi\,dx\,dt$$
$$\leq CR^{\frac{-q}{p-q}\left(\sigma+\frac{2}{\theta}\rho\right)+\frac{p}{p-q}(\delta-2)+n+\frac{2}{\theta}}\int_{\Omega}\left|\sum_{i=1}^{n}\frac{\partial a_i}{\partial y_i}\right|^{\frac{p}{p-q}}\Phi^l o\mu\,dy\,d\tau.$$

Similarly

$$\int\limits_{Q_{TR^{2/\theta}}}(h\xi)^{-\frac{q}{p-q}}\left|a.\nabla\xi\right|^{\frac{p}{p-q}}\,dx\,dt$$
$$\leq CR^{\frac{-q}{p-q}\left(\sigma+\frac{2}{\theta}\rho\right)+(\delta-1)\frac{p}{p-q}+n+\frac{2}{\theta}}\int_{\Omega}\left(\Phi^l o\mu\right)^{\frac{-q}{p-q}}\left|a.\nabla\xi\right|^{\frac{p}{p-q}}\,d\tau\,dy.$$

Since

$$-\beta p\theta+(p-m)\,\theta n+2\,(p-m)-m\,(\theta\sigma+2\rho)$$
$$\leq -\beta p\theta+(p-1)\,\theta n+2\,(p-1)-(\theta\sigma+2\rho)\,,$$

and choosing θ such that

$$-\beta p\theta+(p-1)\,\theta n+2\,(p-1)-(\theta\sigma+2\rho)$$
$$=-2\alpha p+(p-1)\,\theta n+2\,(p-1)-(\theta\sigma+2\rho)\,,$$

we find $\theta=2\alpha/\beta$. Thus we have the estimate

$$\int\limits_{Q_{TR^{2/\theta}}} h\,|u|^p\,\xi\,dx\,dt\leq C\,(\varepsilon)\,(R^{s_1}+R^{s_2}+R^{s_3})\,,\tag{2.3}$$

where $C\,(\varepsilon)$ is a constant which depends on the ε.

$$(p-1)\,\frac{2\alpha}{\beta}s_1=-2\alpha p+(p-1)\,\frac{2\alpha}{\beta}n+2\,(p-1)-\left(\frac{2\alpha}{\beta}\sigma+2\rho\right),$$
$$(p-q)\,\frac{2\alpha}{\beta}s_2=(\delta-2)\,p+\left(n\frac{2\alpha}{\beta}+2\right)(p-q)-q\left(\sigma+\frac{2}{\theta}\rho\right),$$
$$(p-q)\,\frac{2\alpha}{\beta}s_3=(\delta-1)\,p\frac{2\alpha}{\beta}+\left(n\frac{2\alpha}{\beta}+2\right)(p-q)-q\left(\frac{2\alpha}{\beta}\sigma+2\rho\right).$$

Consequently, if we choose $\max\left(s_{1,}s_2,s_3\right)<0$, that is

$$p\leq p_c=\min\left(1+\frac{\alpha\,(\sigma+\beta)+\beta\rho}{\alpha n+\beta\,(1-\alpha)},\frac{((\alpha n+\beta)+(\alpha\sigma+\beta\rho))}{((\delta-1)\,\alpha+(n\alpha+\beta))}q\right),$$

let $R \to \infty$ in (2.3), we obtain $\int_{\mathbb{R}^n\times\mathbb{R}^+} h\,|u|^p\;dx\,dt \le 0$. This implies that $u = 0$, which is a contradiction.

If $p = p_c$ (i.e. $\max(s_1, s_2, s_3) = 0$) the critical case, we have from (2.3)

$$\int\limits_{\mathbb{R}^n\times\mathbb{R}^+} h\,|u|^p\;dx\,dt \le C. \tag{2.4}$$

We modify the test function ξ by introducing a new fixed constant S ($0 < S < R$), such that

$$\xi(x,t) := \Phi^l\left(|x|^2/R^2 + t^\theta/(SR)^2\right).$$

We set

$$C_{R,S} = \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \colon R^2 \le |x|^2 + t^\theta/S^2 \le 2R^2\right\}.$$

Observe that because of the convergence of the integral in (2.4), then

$$\lim_{R\to\infty} \int_{C_{R,S}} h\,|u|^p\,\xi\,dx\,dt = 0. \tag{2.5}$$

By using the Hölder inequality, we get

$$\begin{aligned}
&\int\limits_{Q_{T(SR)^{2/\theta}}} |u|^{q-1}\,ua.\nabla\xi\,dx\,dt = \int\limits_{C_{R,S}} |u|^{q-1}\,ua.\nabla\xi\,dx\,dt\\
&\le \left(\int_{C_{R,S}} |u|^p\,h\xi\,dx\,dt\right)^{\frac{q}{p}} \left(\int_{C_{R,S}} (h\xi)^{-\frac{q}{p-q}}\,|a.\nabla\xi|^{\frac{p}{p-q}}\;dx\,dt\right)^{\frac{p-q}{p}},
\end{aligned}$$

where we have used that the support of $a.\nabla\xi$ is $C_{R,S}$. Taking into account of the scaled variables: $t = (RS)^{\frac{2}{\theta}}\,\tau, x = Ry\;\xi(x,t) = \xi(Ry, (RS)^{\frac{2}{\theta}}\,\tau) = \chi(y,\tau)$ and the fact that $p = p_c$ then instead of estimate (2.2), we get

$$\begin{aligned}
&(1-3\varepsilon) \int\limits_{Q_{T(SR)^{2/\theta}}} h\,|u|^p\,\xi\,dx\,dt\\
&\le \left(\int_{C_{R,S}} |u|^p\,h\xi\,dx\,dt\right)^{\frac{q}{p}} \left(\int_{C_{R,S}} (h\xi)^{-\frac{q}{p-q}}\,|a.\nabla\xi|^{\frac{p}{p-q}}\;dx\,dt\right)^{\frac{p-q}{p}} \qquad (2.6)\\
&+ C(\varepsilon)\left(L_1 S^{-\frac{1}{p-1}\left(\frac{2\rho}{\theta}\right)-\frac{2}{\theta}\alpha\frac{p}{p-1}+\frac{2}{\theta}} + L_2 S^{-\frac{m}{p-m}\left(\frac{2\rho}{\theta}\right)+\frac{2}{\theta}} + L_3 S^{\frac{-q}{p-q}\left(\frac{2}{\theta}\rho\right)+\frac{2}{\theta}}\right),
\end{aligned}$$

where

$$L_1 := \int_\Omega \chi^{\frac{-1}{p-1}}\left|D^\alpha_{t|T}\,\chi\right|^{\frac{p}{p-1}}\;dy\,d\tau, \quad L_2 := \int_\Omega \chi^{-\frac{m}{p-m}}\left|(-\Delta)^{\frac{\beta}{2}}\,\chi\right|^{\frac{m}{p-m}}\;dy\,d\tau,$$

$$L_3 := \int_\Omega \chi\left|\sum_{i=1}^n \frac{\partial a}{\partial y_i}\right|^{\frac{p}{p-q}}\;dy\,d\tau.$$

Using (2.6), we obtain via (2.5), after passing to the limit as $R \to \infty$

$$\int\limits_{\mathbb{R}^n \times \mathbb{R}^+} h\,|u|^p \; dx\,dt \leq C\left(S^{-\frac{1}{p-1}\left(\frac{2\rho}{\theta}\right) - \frac{2}{\theta}\alpha\frac{p}{p-1} + \frac{2}{\theta}} + S^{-\frac{m}{p-m}\left(\frac{2\rho}{\theta}\right) + \frac{2}{\theta}} + S^{\frac{-q}{p-q}\left(\frac{2}{\theta}\rho\right) + \frac{2}{\theta}} \right). \tag{2.7}$$

Finally, we remark that the left-hand side of (2.7) is independent of S, then by passing to the limit when S goes to infinity, we obtain $u = 0$, which is contradiction and this completes the proof. □

Remark 1. When the vector $a = 0$ and $q = m = 1$, we recover the case studied by Kirane-Tatar [9]. When $a = 0$, $q = m = 1$, $\sigma = \rho = 0, \alpha = 1$ and $\beta = 2$, the critical exponent coincides with the well known Fujita exponent [3].

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