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Homogenization of Monotone Problems with Uncertain Coefficients*

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Abstract. The homogenization problem for a nonlinear elliptic equation modelling some physical phenomena set in a periodically heterogeneous medium is studied. Contrary to the usual approach, the coefficients in the equation are supposed to be uncertain functions from a given set of admissible data satisfying suitable monotonicity and continuity conditions. The problem with uncertainties is treated by means of the worst scenario method.

Keywords: homogenization, nonlinear elliptic equation, uncertain input data, worst scenario method.

AMS Subject Classification: 35B27; 35B30; 35B40; 35J25; 35R05; 49J20.

1 Introduction

Composite materials play an important role in industry due to their specific mechanical and physical properties. On the other hand, mathematical models involving composites are difficult to solve (especially from the numerical point of view) due to the strong heterogeneity. One of useful mathematical methods designed for modelling problems formulated in highly heterogeneous media is the homogenization theory. It enables us to compute the macroscopic (or large scale) properties from the knowledge of microstructure. Although the homogenization provides quite easy and powerful tool and many interesting results have been achieved so far, its practical use is still restricted to the case of periodic structures. Unfortunately, real materials have rather almost periodic or even stochastic structures. Also material properties like the modulus of elasticity, heat conductivity, etc. are not known precisely, since each measurement is corrupted by an error. From this point of view, we talk about

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uncertain input data and a natural question arises, how to incorporate them into a mathematical model.

We shall deal with the homogenization of a nonlinear boundary value problem in the form

$$\begin{aligned} A(u) \equiv -\operatorname{div}(a(x, \nabla u)) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

This kind of problem represents a nonlinear conservation law. The coefficients of the operator A are considered to be uncertain, known in some bounds only, but still satisfying certain continuity and monotonicity conditions.

We adopt a deterministic approach to the problem with uncertainties, the worst scenario method introduced by Hlaváček, for a comprehensive guide we refer to [9]. The main idea consists in considering a functional defined, in general, on a suitable set of admissible data and a space, where the solution is looked for. This functional is a criterion evaluating a state/physical quantity characterized by the solution of the model problem from a certain point of view. In particular, it says which data are "bad" or "good". The maximization of this functional yields the "worst case". In other words, strategy of the method is to stay on the safe side – it searches for "dangerous" data. Although the method is currently well known, its usage in the homogenization as well as in the case of monotone operators seems to be new. In [12], the method was applied to the homogenization problem of the monotone type in dimension one, here an analogous problem in higher dimension is discussed. Linear problems were studied in [11].

The paper is organized into 5 sections. The model problem is introduced in Section 2. Section 3 contains the related homogenized problem. In these two sections some known results are recalled, hence the proofs are outlined only. Section 4 is devoted to the worst scenario method including the main result on solvability of the corresponding maximization problem. Concluding remarks in Section 5 close the paper.

2 Model Problem

Throughout the paper, Ω will be a domain in $\mathbb{R}^d$ with the Lipschitz boundary, $(\cdot,\cdot)$ the scalar product and $|\cdot| = \sqrt{(\cdot,\cdot)}$ the Euclidean norm in $\mathbb{R}^d$. For a set $S \subset \mathbb{R}^d$, the symbol $|S|$ means the d-dimensional Lebesgue measure. The Lebesgue space $L^2(\Omega)$ and its vector-valued analogue $L^2(\Omega;\mathbb{R}^d)$ are equipped with the standard norms. The Sobolev space $H^1(\Omega)$ of square-integrable functions with square-integrable derivatives is equipped with the norm $\|u\|_{H^1(\Omega)} = [\int_\Omega (u^2 + |\nabla u|^2)\,\mathrm{d}x]^{1/2}$. Its subspace of functions with zero traces on $\partial\Omega$ is denoted by $H_0^1(\Omega)$. Due to the Friedrichs inequality the seminorm $\|\nabla u\|_{L^2(\Omega;\mathbb{R}^d)}$ serves as an equivalent norm on $H_0^1(\Omega)$. The symbol $C_0^\infty(\Omega)$ denotes the space of infinitely smooth functions with compact support in Ω.

Let $Y = [0,1)^d$ be the unit cube. A function u defined on $\mathbb{R}^d$ is said to be Y-periodic, if $u(y+k) = u(y)$ holds $\forall y \in Y$, $\forall k \in \mathbb{Z}^d$. Banach spaces of Y-periodic functions are denoted by $X_\#(Y)$. Let us remind that a function

v is in $X_{\#}(Y)$, if it is Y-periodic and $v \in X_{\mathrm{loc}}(\mathbb{R}^d)$. In particular, functions of Sobolev space $H^1_{\#}(Y)$ have the same traces on the opposite faces of Y. Further, $H^1_{\#0}(Y)$ denotes the subspace of functions $u \in H^1_{\#}(Y)$ having the zero mean value over Y, i.e. $\int_Y u \,\mathrm{d}y = 0$. The norm of $H^1_{\#0}(Y)$ is introduced as $\|u\|_{H^1_{\#0}(Y)} = \|u\|_{H^1(Y)}$. We note that $\|u\|_{H^1_{\#0}(Y)}$ is equivalent to the seminorm $\|\nabla u\|_{L^2(Y;\mathbb{R}^d)}$ due to the Poincaré–Wirtinger inequality.

The duality pairing between a space and its dual is denoted by $\langle\cdot,\cdot\rangle$. The convergence in norm is denoted by "$\to$", the weak convergence by "$\rightharpoonup$" and the uniform convergence by "$\rightrightarrows$".

The homogenization approach considers a sequence of problems of the same type with diminishing period coefficients, where one term in this sequence is supposed to be the original problem, see e.g. [4, Chap. 5] for a detailed explanation. This sequence is controlled by a sequence of positive parameters $\varepsilon_n \to 0$ as $n \to \infty$ (as usual, we omit the subscript n). We shall study the following sequence of weak formulations corresponding to the problem (1.1):

$$\begin{cases} \displaystyle\int_\Omega (a_\varepsilon(x, \nabla u^a_\varepsilon), \nabla v) \,\mathrm{d}x = \langle f, v\rangle, \quad \forall v \in H^1_0(\Omega), \\ u^a_\varepsilon \in H^1_0(\Omega), \end{cases} \tag{2.1}$$

where $a_\varepsilon(x,\xi) \equiv a(y,\xi)|_{y=x/\varepsilon}$ and $a(y,\xi)$ is an uncertain function from a set of admissible data U^{ad}.

Let us describe U^{ad} in details. Let Y consist of a finite number of subdomains Y_k occupied by different components of a composite, i.e. $\overline{Y} = \bigcup_{k=1}^m \overline{Y}_k$, $Y_j \cap Y_k = \emptyset$, $\forall j \neq k$, and $|Y_k| > 0$. Each coefficient $a_i(y,\xi)$, $i = 1,\dots,d$, is supposed to be a function Y-periodic in y, constant in y on each Y_k and in the variable ξ dependent on ξ_i only, i.e. $a_i(y,\xi) = a^k_i(\xi_i)$ for $y \in Y_k$, where $a^k_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous and strongly monotone inside a fixed interval I_i and linear outside of it. More precisely, let $I_i = [\xi^\ell_i, \xi^r_i]$, $i = 1,\dots,d$, be fixed closed intervals and let each function a^k_i satisfy for all $k = 1,\dots,m$:

$$\begin{aligned} &|a^k_i(\xi_i) - a^k_i(\eta_i)| \leqslant L^k_i|\xi_i - \eta_i|, \quad \forall \xi_i, \eta_i \in I_i, \\ &(a^k_i(\xi_i) - a^k_i(\eta_i)) \cdot (\xi_i - \eta_i) \geqslant \alpha^k_i(\xi_i - \eta_i)^2, \quad \forall \xi_i, \eta_i \in I_i, \\ &a^k_i(\xi_i) = a^k_i(\xi^\ell_i) - c^k_i(\xi^\ell_i - \xi_i), \quad \forall \xi_i < \xi^\ell_i, \\ &a^k_i(\xi_i) = a^k_i(\xi^r_i) + c^k_i(\xi_i - \xi^r_i), \quad \forall \xi_i > \xi^r_i, \end{aligned}$$

where α^k_i, L^k_i, c^k_i are fixed positive constants. Let $S_i(\alpha^k_i, L^k_i, c^k_i)$ denote the set of all functions $a_i(y,\xi)$ satisfying the conditions listed above. Now we can define the admissible set U^{ad}_i for the i-th coefficient $a_i(y,\xi)$, $i = 1,\dots,d$:

$$U^{ad}_i = \bigl\{a_i \in S_i(\alpha^k_i, L^k_i, c^k_i) \colon a^{\min}_i(y,\xi) \leqslant a_i(y,\xi) \leqslant a^{\max}_i(y,\xi)\bigr\},$$

where $a^{\min}_i$, $a^{\max}_i$ are given functions from $S_i(\alpha^k_i, L^k_i, c^k_i)$. The entire set U^{ad} is defined as $U^{ad} = U^{ad}_1 \times \cdots \times U^{ad}_d$. The solvability of the problem (2.1) results from the following abstract theorem known from the theory of monotone operators:

Theorem 1. *Let V be a Hilbert space and $A : V \to V'$ an operator satisfying for some $\beta_1, \beta_2 > 0$ and for all $u_1, u_2 \in V$:*

$$\|A(u_1) - A(u_2)\|_{V'} \leqslant \beta_1 \|u_1 - u_2\|_V \quad (\textit{Lipschitz continuity}), \tag{2.2}$$

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geqslant \beta_2 \|u_1 - u_2\|_V^2 \quad (\textit{strong monotonicity}). \tag{2.3}$$

Then the operator equation $A(u) = f$ has a unique solution for each $f \in V'$.

The theorem can be proved by means of the Banach fixed point theorem. The function u is a solution of the equation $A(u) = f$ iff it is the fixed point of the mapping $T_\theta(u) = u - \theta J^{-1}(A(u) - f)$, where $J : V \to V'$ is the duality map of V and $\theta > 0$. It can be shown that for $0 < \theta < 2\alpha/L^2$ the mapping $T_\theta : V \to V$ is contractive and thus there exists a fixed point. Details can be found e.g. in [7, Sect. 4], [15, Sect. 25.4].

In our problem, the construction of U^{ad} implies existence of positive constants α and L such that every function $a \in U^{ad}$ satisfies the estimates

$$|a(y, \xi) - a(y, \eta)| \leqslant L|\xi - \eta|, \quad \forall y, \xi, \eta \in \mathbb{R}^d, \tag{2.4}$$

$$(a(y, \xi) - a(y, \eta), \xi - \eta) \geqslant \alpha|\xi - \eta|^2, \quad \forall y, \xi, \eta \in \mathbb{R}^d. \tag{2.5}$$

Then, taking $V = H_0^1(\Omega)$, it is an easy exercise to verify that the operator $A_\varepsilon(u) \equiv -\mathrm{div}(a_\varepsilon(x, \nabla u))$ from (2.1) satisfies (2.2) and (2.3) with $\beta_1 = L$, $\beta_2 = \alpha K$, where K is the constant from the Friedrichs inequality.

To summarize the above considerations we can state:

Theorem 2. *Let $a \in U^{ad}$. Then there exists a unique solution u_ε^a of the problem* (2.1) *for every $f \in (H_0^1(\Omega))'$ and every $\varepsilon > 0$ fixed.*

Although existence and uniqueness of the solution can be obtained also under weaker monotonicity and continuity assumptions, see e.g. [7], we shall need the introduced properties in the following sections.

3 Homogenization Problem

The mathematical homogenization theory deals with the asymptotic behaviour of solutions to partial differential equations with rapidly oscillating coefficients. Its main development dates to seventies of the 20th century. The original methods for the periodic case are based on the assumption (a priori not verifiable) that the solution u_ε of a model problem can be written in the form of an asymptotic expansion in ε containing functions of multiple scales (i.e. a "slow" and "fast" variable are introduced), see e.g. [4]. To study more abstract problems (possibly beyond the periodic settings), the notions of G-convergence and H-convergence were introduced, see e.g. [6, 10]. For variational problems, the Γ-convergence of functionals was developed, see e.g. [5]. At the end of 80's the concept of two-scale convergence method was presented, see [1, 13], which appeared to be the most powerful tool. In the non-periodic homogenization the two-scale convergence was replaced by the Σ-convergence introduced by G. Nguetseng, see [14].

Let us recall the homogenization result related to the problem (2.1).

Theorem 3. *Let $a \in U^{ad}$, u^a_ε be the solution of (2.1) with $f \in (H^1_0(\Omega))'$ and $\varepsilon \to 0+$. Then*

$$u^a_\varepsilon \rightharpoonup u^a \quad \text{in } H^1_0(\Omega), \quad a_\varepsilon(x, \nabla u^a_\varepsilon) \rightharpoonup b_a(\nabla u^a) \quad \text{in } L^2(\Omega; \mathbb{R}^d),$$

where u^a is the unique solution of the so-called homogenized problem

$$\begin{cases} \displaystyle\int_\Omega (b_a(\nabla u^a), \nabla v)\,\mathrm{d}x = \langle f, v\rangle, \quad \forall v \in H^1_0(\Omega), \\ u^a \in H^1_0(\Omega). \end{cases} \tag{3.1}$$

The coefficient $b_a : \mathbb{R}^d \to \mathbb{R}^d$ is defined as

$$b_a(\xi) = \int_Y a(y, \xi + \nabla w^a_\xi)\,\mathrm{d}y, \tag{3.2}$$

where w^a_ξ is the unique solution of the so-called local problem

$$\begin{cases} \displaystyle\int_Y (a(y, \xi + \nabla w^a_\xi), \nabla \phi)\,\mathrm{d}x = 0, \quad \forall \phi \in H^1_{\#0}(Y), \\ w^a_\xi \in H^1_{\#0}(Y). \end{cases} \tag{3.3}$$

Moreover, $b_a : \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following estimates

$$|b_a(\xi) - b_a(\eta)| \leqslant \widetilde{L}|\xi - \eta|, \quad \forall \xi, \eta \in \mathbb{R}^d, \tag{3.4}$$

$$(b_a(\xi) - b_a(\eta), \xi - \eta) \geqslant \alpha|\xi - \eta|^2, \quad \forall \xi, \eta \in \mathbb{R}^d, \tag{3.5}$$

where the constant $\widetilde{L}$ depends on α, L and the bound on the coefficient $a(y, \xi)$ at the point $\xi = 0$.

A detailed proof of this theorem can be found in [6, Thm. 5.3]. Let us sketch the procedure. For $v = u_\varepsilon$ in (2.1) and using (2.5), (2.4) we obtain the boundedness of the sequences $\{u^a_\varepsilon\}$ and $\{a_\varepsilon(x, \nabla u^a_\varepsilon)\}$ in $H^1_0(\Omega)$ and $L^2(\Omega; \mathbb{R}^d)$, respectively. Thus there exists a subsequence (still denoted by ε) such that

$$u^a_\varepsilon \rightharpoonup u_* \quad \text{in } H^1_0(\Omega), \quad a_\varepsilon(x, \nabla u^a_\varepsilon) \rightharpoonup a_* \quad \text{in } L^2(\Omega; \mathbb{R}^d).$$

Passing to the limit in (2.1), the last convergence implies that the equation

$$\int_\Omega (a_*, \nabla v)\,\mathrm{d}x = \langle f, v\rangle.$$

is satisfied. The crucial step of the proof is to show that $a_* = b_a(\nabla u_*)$ a.e. in Ω, where b_a is given by (3.2). To this end we need to introduce a sequence of the functions $v^\xi_\varepsilon(x) = (\xi, x) + \varepsilon w^\xi(x/\varepsilon)$, where $w^\xi(y)$ is the Y-periodic extension of the solution w^a_ξ to the local problem (3.3). The periodicity of v^ξ_ε and a_ε yield

$$\begin{aligned} &v^\xi_\varepsilon \rightharpoonup (\xi, x) \quad \text{in } H^1(\Omega), \quad \nabla v^\xi_\varepsilon \rightharpoonup \xi \quad \text{in } L^2(\Omega; \mathbb{R}^d), \\ &a_\varepsilon(x, \nabla v^\xi_\varepsilon) \rightharpoonup b_a(\xi) \quad \text{in } L^2(\Omega; \mathbb{R}^d). \end{aligned}$$

Due to the strong monotonicity of a_ε we have

$$\int_\Omega (a_\varepsilon(x,\nabla u^a_\varepsilon) - a_\varepsilon(x,\nabla v^\xi_\varepsilon), \nabla u^a_\varepsilon - \nabla v^\xi_\varepsilon)\varphi(x)\,\mathrm{d}x \geqslant 0, \;\; \varphi \in C_0^\infty(\Omega),\; \varphi \geqslant 0.$$

The left-hand side contains a product of two weakly converging sequences, nevertheless the assumptions of the compensated compactness lemma (see e.g. [10, Lemma 1]) are fulfilled so that we can pass to the limit

$$\int_\Omega (a_*(x) - b_a(\xi), \nabla u_*(x) - \xi)\varphi(x)\,\mathrm{d}x \geqslant 0, \quad \varphi \in C_0^\infty(\Omega),\; \varphi \geqslant 0. \tag{3.6}$$

It is not difficult to verify that b_a is monotone and satisfies (3.4). It implies that b_a is also maximal monotone and thus by (3.6) we have $a_*(x) = b_a(\nabla u_*(x))$. In the sequel, it is possible to prove (3.5) with help of the sequence of v^ξ_ε. Finally, by uniqueness of the solution of (3.1) we have $u_* = u^a$ and the entire sequences converge.

In the literature, the additional assumption $a(y,0) = 0$ is usually used which slightly simplifies some steps in the proof. In that case, the estimate (3.4) holds with $\widetilde{L} = L^2/\alpha$, see [8, Prop. 2.7 and Lemma 2.4].

Note, that the auxiliary sequence $\{v^\xi_\varepsilon\}$ is constructed via the solution of the local problem (3.3) – the problem which, in particular, defines the homogenized problem. The form of (3.1)—(3.3) itself can be derived using the asymptotic expansion method (the procedure for problems of the type (2.1) can be found e.g. in [2, Chap. 3, §7]).

The same homogenization result for the case of Sobolev space $H_0^{1,p}(\Omega)$, $p \neq 2$, under analogous hypothesis on $a(y,\xi)$ was first presented in [8]. Let us note that some other variants of monotonicity and continuity assumptions have also been studied. The most general result on homogenization of monotone operators was formulated in [3] covering also the case of multivalued mappings.

4 Worst Scenario Method

In this section we introduce the criterion functional over the set U^{ad} and formulate the corresponding worst scenario problem. We have fairly enough freedom with the choice of this functional based on the aim of interest and expert decisions, however certain continuity assumptions must be satisfied, for details see [9, Chap. II]. For our purposes the following definition is satisfactory:

Definition 1. The criterion functional is a functional $\Phi : U^{ad} \times H_0^1(\Omega) \to \mathbb{R}$ satisfying: if $a_n, a \in U^{ad}$, $a_n \rightrightarrows a$ and $v_n \to v$ in $H_0^1(\Omega)$ as $n \to \infty$, then $\Phi(a_n, v_n) \to \Phi(a, v)$.

In our problem, Φ can be given by $\Phi(a,u^a) = |\widetilde{\Omega}|^{-1}\int_{\widetilde{\Omega}} u^a\,\mathrm{d}x$, where u^a is the solution of (3.1) and $\widetilde{\Omega}$ a subset of Ω of a positive measure. This choice is motivated by having interest in finding some critical values of the homogenized solution (representing e.g. temperature) in some crucial places of the material (e.g. where measurements take place). Since the solution need not be continuous and thus the maximum need not exist, the integral mean value is used. Similarly, the solution u^a in Φ can be replaced e.g. by its gradient or the generalized gradient. Once the set of admissible data and the criterion functional are given, we can formulate the worst scenario problem:

Problem 1. Find $\widehat{a} \in U^{ad}$ such that $\Phi(a, u^a) \leqslant \Phi(\widehat{a}, u^{\widehat{a}})$, $\forall a \in U^{ad}$, where u^a is the solution of the homogenized problem (3.1) and Φ is the criterion functional from Definition 1.

Now, let us state the main result of the paper:

Theorem 4. *There exists a solution (generally non-unique) of Problem* 1.

The proof of this theorem relies on the following lemmas.

Lemma 1. *The set U^{ad} is compact in the following sense: each sequence of functions $\{a_n\} \subset U^{ad}$ contains a uniformly convergent subsequence $\{a_{n'}\}$ of $\{a_n\}$, i.e. there exists an element $a \in U^{ad}$ such that $a_{n'} \rightrightarrows a$ on $Y \times \mathbb{R}^d$.*

Proof. Let $U^{ad}_{I_i}$ denote the set of admissible functions from U^{ad}_i restricted to the set $Y \times \mathbb{R}^{i-1} \times I_i \times \mathbb{R}^{d-i}$, $i = 1, \ldots, d$. Since every function $a_i \in U^{ad}_{I_i}$ can be identified with m one-variable functions a^k_i that are Lipschitz continuous and bounded on I_i with the same constants, Arselà-Ascoli theorem yields that every sequence $\{a^n_i\}$ of $U^{ad}_{I_i}$ contains a subsequence $\{a^{n'}_i\}$ converging uniformly to an element $a_i \in U^{ad}_{I_i}$ (this limit belongs to $U^{ad}_{I_i}$ as it is a closed set). Further, since outside the interval I_i the function a^k_i is a continuous extension by lines with the same slope, the uniform convergence on the whole $Y \times \mathbb{R}^d$ follows. □

Lemma 2. *Let $a_n, a \in U^{ad}$ be such that $a_n \rightrightarrows a$ on $Y \times \mathbb{R}^d$ as $n \to \infty$. Then $b_{a_n} \rightrightarrows b_a$ on $\mathbb{R}^d$, where b_{a_n} and b_a are defined by* (3.2) *with the integrand a_n and a, respectively.*

Proof. Let us set $h^i_n = \sup_{(y,\xi)} |a^i(y,\xi) - a^i_n(y,\xi)|$, $h_n = (h^1_n, \ldots, h^d_n)$, where a^i and a^i_n is the i-th component of the vector a and a_n, respectively. Note that these suprema are finite due to the definition of the set U^{ad}. Then by (2.5)

$$\begin{aligned}
&\alpha \|\nabla w^{a_n}_\xi - \nabla w^a_\xi\|^2_{L^2(Y;\mathbb{R}^d)} \\
&\quad \leqslant \int_Y (a_n(y, \xi + \nabla w^{a_n}_\xi) - a_n(y, \xi + \nabla w^a_\xi), \nabla w^{a_n}_\xi - \nabla w^a_\xi)\,\mathrm{d}y \\
&\quad = \int_Y (a_n(y, \xi + \nabla w^{a_n}_\xi), \nabla w^{a_n}_\xi - \nabla w^a_\xi)\,\mathrm{d}y \\
&\qquad + \int_Y (a(y, \xi + \nabla w^a_\xi) - a_n(y, \xi + \nabla w^a_\xi), \nabla w^{a_n}_\xi - \nabla w^a_\xi)\,\mathrm{d}y \\
&\qquad - \int_Y (a(y, \xi + \nabla w^a_\xi), \nabla w^{a_n}_\xi - \nabla w^a_\xi)\,\mathrm{d}y.
\end{aligned}$$

Due to the weak formulation (3.3) for the solutions $w^{a_n}_\xi$ and w^a_ξ, the first and the third term are zero. Using the Cauchy–Schwarz inequality, the second term can be estimated by

$$\begin{aligned}
&\int_Y |a(y,\xi+\nabla w_\xi^a) - a_n(y,\xi+\nabla w_\xi^a)|\,|\nabla w_\xi^{a_n} - \nabla w_\xi^a|\,\mathrm{d}y \\
&\quad \leqslant |h_n| \int_Y |\nabla w_\xi^{a_n} - \nabla w_\xi^a|\,\mathrm{d}y \leqslant |h_n|\,|Y|^{1/2} \|\nabla w_\xi^{a_n} - \nabla w_\xi^a\|_{L^2(Y;\mathbb{R}^d)}.
\end{aligned}$$

Since $|Y| = 1$, we have

$$\|\nabla w_\xi^{a_n} - \nabla w_\xi^a\|_{L^2(Y;\mathbb{R}^d)} \leqslant \frac{|h_n|}{\alpha}. \tag{4.1}$$

Similarly, using the definition (3.2) we have for the i-th component of $b(\xi)$

$$\begin{aligned}
|b_{a_n}^i(\xi) - b_a^i(\xi)| &= \Big| \int_Y \Big[a_n^i(y,\xi+\nabla w_\xi^{a_n}) - a^i(y,\xi+\nabla w_\xi^a) \Big]\,\mathrm{d}y \Big| \\
&\leqslant \int_Y |a_n^i(y,\xi+\nabla w_\xi^{a_n}) - a_n^i(y,\xi+\nabla w_\xi^a)|\,\mathrm{d}y \\
&\quad + \int_Y \big| a_n^i(y,\xi+\nabla w_\xi^a) - a^i(y,\xi+\nabla w_\xi^a) \big|\,\mathrm{d}y \\
&\leqslant L \int_Y |\nabla w_\xi^{a_n} - \nabla w_\xi^a|\,\mathrm{d}y + h_n^i \\
&\leqslant L \|\nabla v_\xi^{a_n} - \nabla v_\xi^a\|_{L^2(\Omega;\mathbb{R}^d)} + h_n^i \leqslant \frac{L}{\alpha}|h_n| + h_n^i,
\end{aligned}$$

where we have used (2.4), the Cauchy–Schwarz inequality, $|Y| = 1$ and (4.1). Since the estimate of the left-hand side does not depend on ξ, the uniform convergence is a consequence of $h_n \to 0$ due to the assumption $a_n \rightrightarrows a$. □

Lemma 3. *Let $a_n, a \in U^{ad}$ be such that $a_n \rightrightarrows a$ on $Y \times \mathbb{R}^d$ as $n \to \infty$. Then $u^{a_n} \to u^a$ in $H_0^1(\Omega)$, where u^{a_n} and u^a are the solutions of* (3.1) *with the coefficient b_{a_n} and b_a, respectively.*

Proof. The proof utilizes the similar line of arguments as in the proof of Lemma 2. Due to the Friedrichs inequality it is enough to show the convergence of the gradient $\nabla u^{a_n} \to \nabla u^a$ in $L^2(\Omega;\mathbb{R}^d)$. Denoting $h_n^i = \sup_\xi |b^i(\xi) - b_n^i(\xi)|$, $h_n = (h_n^1, \dots, h_n^d)$, then by (3.5) we have

$$\begin{aligned}
&\alpha \|\nabla u^{a_n} - \nabla u^a\|^2_{L^2(\Omega;\mathbb{R}^d)} \leqslant \int_\Omega (b_{a_n}(\nabla u^{a_n}) - b_{a_n}(\nabla u^a), \nabla u^{a_n} - \nabla u^a)\,\mathrm{d}x \\
&\quad = \int_\Omega b_{a_n}(\nabla u^{a_n}), \nabla u^{a_n} - \nabla u^a)\,\mathrm{d}x + \int_\Omega (b_a(\nabla u^a) - b_{a_n}(\nabla u^a), \nabla u^{a_n} - \nabla u^a)\,\mathrm{d}x \\
&\quad - \int_\Omega (b_a(\nabla u^a), \nabla u^{a_n} - \nabla u^a)\,\mathrm{d}x.
\end{aligned}$$

Due to (3.1), the first and the third term equal to $\langle f, u^{a_n} - u^a \rangle$ with the opposite signs and thus they vanish. The middle term can be estimated by $|h_n|\,|\Omega|^{1/2}\,\|\nabla u^{a_n} - \nabla u^a\|_{L^2(\Omega;\mathbb{R}^d)}$. Lemma 2 implies $h_n \to 0$ so that the demanded convergence follows. □

Proof of Theorem 4. Since the set of values of $\Phi(a, u^a)$ on U^{ad} is a subset of the reals, there exists a maximizing sequence $\{a_n\} \subset U^{ad}$ satisfying

$$\lim_{n\to\infty} \Phi(a_n, u^{a_n}) = \sup_{a\in U^{ad}} \Phi(a, u^a). \tag{4.2}$$

By Lemma 1 there exists an element $\widetilde{a} \in U^{ad}$ and a subsequence $\{a_{n'}\} \subset \{a_n\}$ such that $a_{n'} \rightrightarrows \widetilde{a}$ on $Y \times \mathbb{R}^d$. Lemma 3 yields $u^{a_{n'}} \to u^{\widetilde{a}}$ in $H_0^1(\Omega)$ and due to Definition 1 we have also the convergence

$$\Phi(a_{n'}, u^{a_{n'}}) \to \Phi(\widetilde{a}, u^{\widetilde{a}}) \quad \text{as } n' \to \infty. \tag{4.3}$$

Combining (4.2) and (4.3) we obtain

$$\lim_{n'\to\infty} \Phi(a_{n'}, u^{a_{n'}}) = \Phi(\widetilde{a}, u^{\widetilde{a}}) = \sup_{a\in U^{ad}} \Phi(a, u^a).$$

Since $\Phi(\widetilde{a}, u^{\widetilde{a}}) < \infty$, the element $\widetilde{a}$ maximizes Φ which proves the result. □

5 Concluding Remarks

The worst scenario method searches for the danger situations caused by uncertainties in the input data. Knowledge of the worst states (and of the data under which these states arise) can serve as a feedback to make some adjustments in the model/technological process. The method is sometimes too pessimistic, especially in cases when the probability of occurrence of the "bad" data is small. On the other hand, compared to stochastic methods, it does not require any probabilistic information (distribution) of the inputs.

We have applied the worst scenario method to the homogenization of nonlinear monotone type boundary value problem with uncertain coefficients in the equation. We restricted uncertainties of the coefficients to their values, the partition of the period and the periodicity were considered to be exact. This approach reflects the fact that these values are obtained by experimental measurements, tabular (laboratory) parameters can differ from those of commonly manufactured materials, they can change in time, etc. Obviously, all mentioned aspects contain a "noise" that can be superimposed in the case of highly heterogeneous materials. Using the worst scenario method for study of influence of the spatial distribution uncertainty of the material components is an open question.

One of the keystones of the worst scenario method is the compactness of the set of admissible functions U^{ad} in a suitable topology. Here, we were successful due to the two restrictions. First, the i-th component of the coefficient $a(y, \xi)$ was considered to be constant in the variable ξ except the i-th component ξ_i, i.e. the problem is not treated in its full generality. Second, the uncertain coefficients were restricted to intervals of finite lengths so that the Arzelà-Ascoli theorem could be applied (it is not a significant limitation in practical problems, since these intervals can be arbitrarily large). A possible generalization and weakening the introduced properties are subjects of further research.

We have not discussed the finite dimensional approximations of the studied problems and their convergence analysis. We only note that such procedure requires a discretization of both the set of admissible data and spaces, where the solutions are looked for. This is left for further research.

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