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# Some Multiplicity Results to the Existence of Three Solutions for a Dirichlet Boundary Value Problem Involving the $p$-Laplacian*

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**Abstract.** In this paper we prove the existence of two intervals of positive real parameters $\lambda$ for a Dirichlet boundary value problem involving the $p$-Laplacian which admit three weak solutions, whose norms are uniformly bounded with respect to $\lambda$ belonging to one of the two intervals. Our main tool is a three critical points theorem due to G. Bonanno [A critical points theorem and nonlinear differential problems, J. Global Optim., **28**:249–258, 2004].

**Keywords:** three solutions, critical point, variational methods, multiplicity results, Dirichlet problem.

**AMS Subject Classification:** 35J20; 34A15.

## 1 Introduction

The purpose of this paper is to establish the existence of two intervals of positive real parameters $\lambda$ for which the problem

$$\begin{cases} \Delta_p u + \lambda f(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian operator, $\Omega \subset R^N$ $(N \geqslant 1)$ is a non-empty bounded open set with smooth boundary $\partial\Omega$, $p > N$, $\lambda$ is a positive parameter and $f : \Omega \times R \to R$ is an $L^1$- Carathéodory function,

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admits three weak solutions, whose norms are uniformly bounded in respect to $\lambda$ belonging to one of the two intervals.

We recall that a function $f : \Omega \times R \to R$ is said to be $L^1$-Carathéodory if

($\delta_1$) $x \to f(x,t)$ is measurable for every $t \in R$;

($\delta_2$) $t \to f(x,t)$ is continuous for almost every $x \in \Omega$;

($\delta_3$) for every $\varrho > 0$ there exists a function $l_\varrho \in L^1(\Omega)$ such that

$$\sup_{|t|\leqslant\varrho} |f(x,t)| \leqslant l_\varrho(x)$$

for almost every $x \in \Omega$.

We say that $u$ is a weak solution to the problem (1.1) if $u \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,dx - \lambda\int_\Omega f(x,u(x))v(x)\,dx = 0$$

for every $v \in W_0^{1,p}(\Omega)$.

In recent years, many publications [1, 7, 8, 9, 10, 11, 12, 14] have appeared about elliptic problems with Dirichlet boundary conditions which have been used in a great variety of application. For example, Ramaswamy and Shivaji in [14] established the existence of three positive solutions for classes of non-decreasing, $p$-sublinear functions $f$ belonging to $C^1([0,\infty))$ for a $p$-Laplacian version of [3], i.e., the problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $p > 1$, $\lambda > 0$ is a parameter and $\Omega$ is a bounded domain in $R^N$; $N \geq 2$ with $\partial\Omega$ of class $C^2$ and connected. Uniqueness of positive solutions to the problem (1.2) when $p > 1$ and $f(u)/u^{p-1}$ is decreasing on $(0,+\infty)$ was obtained in Guo and Webb [11] and Drabek and Hernandez [9]. A natural question is that, whether uniqueness holds under the weaker condition than $f(u)/u^{p-1}$ is decreasing for large u. When $\Omega$ is a ball, Hai and Shivaji [12] showed that the answer is affirmative. However, the approach used in [12] depends on ordinary differential equations techniques and cannot be applied to the case of a general domain. In [7], Ricceri's three critical points theorem [15] has been successfully used to obtain existence of at least three weak solutions to the problem (1.1) in $W_0^{1,p}(\Omega)$. In [1], based on Ricceri's three critical points theorem [15] we obtained the existence of an interval $\Lambda \subseteq [0,+\infty[$ and a positive real number $q$ such that for each $\lambda \in \Lambda$ problem

$$\begin{cases} \Delta_p u + \lambda f(x,u) = a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset R^N$ ($N \geqslant 2$) is non-empty bounded open set with smooth boundary $\partial\Omega$, $p > N$, $\lambda > 0$, $f : \Omega \times R \to R$ is a continuous function and positive weight function $a(x) \in C(\overline{\Omega})$, admits at least three weak solutions whose norms in

$W_0^{1,p}(\Omega)$ are less than $q$ that we extended the main result of [4] by using of the results of [7] to the general case. In [8], the authors employing Ricceri's three critical points theorem [16] obtained multiple weak solutions for the following BVP

$$\begin{cases} -\Delta_p u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset R^N$ is a non-empty bounded open set with smooth boundary $\partial\Omega$, $p > N$, $f, g : \Omega \times R \to R$ are two Carathéodory functions and $\lambda$, $\mu$ are two positive parameters.

Bonanno in [6] established the existence of two intervals of positive real parameters $\lambda$ for which the functional $\Phi + \lambda\Psi$ has three critical points, whose norms are uniformly bounded with respect to $\lambda$ belonging to one of the two intervals. He illustrated the result for a two point boundary value problem, and here we are interested to illustrate this result to the problem (1.1). Our main result is Theorem 1 that ensures the existence of two intervals $\Lambda_1'$ and $\Lambda_2'$ such that, for each $\lambda \in \Lambda_1' \cup \Lambda_2'$, the problem (1.1) admits at least three weak solutions whose norms are uniformly bounded with respect to $\lambda \in \Lambda_2'$. The technique used in our proof has been introduced in [7].

As an immediate consequences of Theorem 1, we obtain Corollary 1, in which the function $f$ has separated variables. The applicability of the result is illustrated by Example 1. Finally, we present the application of Theorem 1 in the ordinary case with $p = 2$, that Example 2 illustrates the result.

## 2 Main Results

First we recall for the reader's convenience Theorem 3.1 of [6] (see also [2, 5, 13, 15, 16] for related results) to transfer the existence of three solutions of the problem (1.1) into the existence of critical points of the Euler functional:

**Theorem A ([6, Theorem 3.1])** *Let $X$ be a separable and reflexive real Banach space; $\Phi : X \longrightarrow R$ a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on $X^*$; $J : X \longrightarrow R$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = J(x_0) = 0$ and that*

(i) $\lim_{\|x\| \to +\infty}(\Phi(x) - \lambda J(x)) = +\infty$ *for all* $\lambda \in [0, +\infty[$.

*Further, assume that there are $r > 0$, $x_1 \in X$ such that:*

(ii) $r < \Phi(x_1)$,

(iii) $\sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x) < \frac{r}{r+\Phi(x_1)} J(x_1)$.

*Then, for each*

$$\lambda \in \Lambda_1 = \left] \frac{\Phi(x_1)}{J(x_1) - \sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x)}, \frac{r}{\sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x)} \right[,$$

*the equation $\Phi'(u) - \lambda J'(u) = 0$ has at least three solutions in $X$ and, moreover, for each $h > 1$, there exist an open interval*

$$\Lambda_2 \subseteq \left[0, hr/\left(rJ(x_1)/\Phi(x_1) - \sup_{x \in \overline{\Phi^{-1}(-\infty, r[)}^w} J(x)\right)\right]$$

*and a positive real number $\sigma$ such that, for each $\lambda \in \Lambda_2$, the equation given above has at least three solutions in $X$ whose norms are less than $\sigma$.*

Here and in the sequel, X will denote the Sobolev space $W_0^{1,p}(\Omega)$ with the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p \, dx\right)^{1/p}.$$

Put $F(x,t) = \int_0^t f(x,\xi)\, d\xi$ for each $(x,t) \in \Omega \times R$, and

$$c = \sup_{u \in X \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u(x)|}{\|u\|}.$$

Since $p > N$, $X$ is compactly embedded in $C^0(\overline{\Omega})$, one has $c < +\infty$. In addition, it is known [18, formula (6b)] that

$$c \leqslant \frac{N^{-1/p}}{\sqrt{\pi}} \left[\Gamma\left(1 + \frac{N}{2}\right)\right]^{1/N} \left(\frac{p-1}{p-N}\right)^{1-1/p} [m(\Omega)]^{1/N-1/p},$$

where $\Gamma$ denotes the Gamma function and $m(\Omega)$ is the Lebesgue measure of the set $\Omega$, and equality occurs when $\Omega$ is a ball.

Now, fix $x^0 \in \Omega$ and pick $r_1$, $r_2$ with $0 < r_1 < r_2$ such that

$$S(x^0, r_1) \subset S(x^0, r_2) \subseteq \Omega$$

where $S(x^0, r_i)$ denotes the ball with center at $x^0$ and radius of $r_i$ for $i = 1, 2$. Put

$$k_1 = k_1(N, p, r_1, r_2) = \frac{c}{r_2 - r_1}\left((r_2^N - r_1^N)\frac{\pi^{N/2}}{\Gamma(1 + N/2)}\right)^{1/p}. \tag{2.1}$$

We formulate our main result as follows:

**Theorem 1.** *Let $f : \Omega \times R \to R$ be an $L^1$-Carathéodory function, and denote $F(x,t) = \int_0^t f(x,\xi)\, d\xi$ for each $(x,t) \in \Omega \times R$. Assume that there exist three positive constants $\theta$, $\tau$ and $\gamma$ with $k_1\tau > \theta$, $\gamma < p$ and a function $\mu \in L^1(\Omega)_+$ such that*

($\alpha_1$) *$F(x,t) \geqslant 0$ for each $(x,t) \in (\Omega \setminus S(x^0, r_1)) \times [0, \tau]$,*

($\alpha_2$) *$\int_\Omega \sup_{t \in [-\theta, \theta]} F(x,t)\, dx < \frac{1}{2}(\frac{\theta}{k_1 \tau})^p \int_{S(x^0, r_1)} F(x, \tau)\, dx$,*

($\alpha_3$) *$F(x,t) \leqslant \mu(x)(1 + |t|^\gamma)$ for almost every $x \in \Omega$ and for all $t \in R$,* *where $k_1$ is given in* (2.1)*. Then, for each*

$$\lambda \in \Lambda_1' = \left]\frac{\frac{1}{p}(\frac{k_1\tau}{c})^p}{\int_{S(x^0,r_1)} F(x,\tau)\,dx - \int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx}, \frac{\frac{1}{p}(\frac{\theta}{c})^p}{\int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx}\right[,$$

*the problem* (1.1) *admits at least three weak solutions in $X$ and, moreover, for each $h > 1$, there exist an open interval*

$$\Lambda_2' \subseteq \left[0, \frac{\frac{h}{p}(\frac{\theta}{c})^p}{(\frac{\theta}{k_1\tau})^p \int_{S(x^0,r_1)} F(x,\tau)\,dx - \int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx}\right]$$

*and a positive real number $\sigma$ such that, for each $\lambda \in \Lambda_2'$, the problem* (1.1) *admits at least three weak solutions in $X$ whose norms are less than $\sigma$.*

*Proof.* In order to apply Theorem A, we begin by setting

$$\Phi(u) = \frac{\|u\|^p}{p}, \quad J(u) = \int_\Omega F(x, u(x))\,dx$$

for each $u \in X$. It is well known that $J$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $J'(u) \in X^*$, given by

$$J'(u)(v) = \int_\Omega f(x, u(x))v(x)\,dx$$

for every $v \in X$. We claim that $J' : X \to X^*$ is a compact operator. To this end, it is enough to show that $J'$ is strongly continuous on $X$. For this, for fixed $u \in X$ let $u_n \to u$ weakly in $X$ as $n \to +\infty$, then we have $u_n$ converges uniformly to $u$ on $\Omega$ as $n \to +\infty$ (see [17]). Since $F(x,\cdot)$ is $C^1$ in $R$ for every $x \in \Omega$, so it is continuous in $R$ for every $x \in \Omega$, and we get that $F(x, u_n) \to F(x, u)$ strongly as $n \to +\infty$ which follows $J'(u_n) \to J'(u)$ strongly as $n \to +\infty$. Thus we proved that $J'$ is strongly continuous on $X$, which implies that $J'$ is a compact operator by Proposition 26.2 of [19]. Hence the claim is true.

Moreover, the functional $\Phi$ is a continuously Gâteaux differentiable whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,dx.$$

$\Phi'$ admits a continuous inverse on $X^*$. Indeed, owing to (2.2) of [17], for every $u,\ v \in X$ there exists a positive constant $c_p$ such that

$$\langle |\nabla u(x)|^{p-2}\nabla u(x) - |\nabla v(x)|^{p-2}\nabla v(x), \nabla u(x) - \nabla v(x)\rangle \geqslant c_p|\nabla u(x) - \nabla v(x)|^p$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $R$. So, we have

$$(\Phi'(u) - \Phi'(v))(u - v) \geqslant c_p\|u - v\|^p$$

for every $u, v \in X$, namely $\Phi'$ is an uniformly monotone operator in $X$, and since $\Phi$ is coercive and hemicontinuous in $X$, by applying Theorem 26.A. [19], we have that $\Phi'$ admits a continuous inverse on $X^*$. Using again that $\Phi'$ is monotone, we obtain that $\Phi$ is sequentially weakly lower semi continuous (see [19, Proposition 25.20]).

Thanks to $(\alpha_3)$, for each $\lambda > 0$ one has that

$$\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda J(u)) = +\infty.$$

Now, set

$$u^*(x) = \begin{cases} 0, & x \in \Omega \setminus S(x^0, r_2) \\ \frac{\tau}{r_2 - r_1}[r_2 - \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}], & x \in S(x^0, r_2) \setminus S(x^0, r_1) \\ \tau, & x \in S(x^0, r_1) \end{cases}$$

and $r = \frac{1}{p}(\frac{\theta}{c})^p$. It is easy to see that $u^* \in X$ and, in particular, one has

$$\Phi(u^*) = \frac{1}{p}(r_2^N - r_1^N)\frac{\pi^{N/2}}{\Gamma(1 + N/2)}\left(\frac{\tau}{r_2 - r_1}\right)^p.$$

So, since $k_1\tau > \theta$, we have $\Phi(u^*) > r$. Moreover, since

$$\sup_{x\in\Omega}|u(x)| \leqslant c\|u\|$$

for each $u \in X$, one has

$$\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^w} J(u) = \sup_{u\in\Phi^{-1}(]-\infty,r])} J(u) \leqslant \int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx,$$

and since $0 \leqslant u^*(x) \leqslant \tau$ for each $x \in \Omega$, the condition $(\alpha_1)$ ensures that

$$\int_{\Omega\setminus S(x^0,r_2)} F(x, u^*(x))\,dx + \int_{S(x^0,r_2)\setminus S(x^0,r_1)} F(x, u^*(x))\,dx \geqslant 0.$$

Therefore, owing to our assumptions, we have

$$\begin{aligned} \sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^w} J(u) &= \sup_{\|u\|^p\leqslant pr}\int_\Omega F(x,u(x))\,dx \\ &\leqslant \int_\Omega \sup_{|t|\leqslant\theta} F(x,t)\,dx < \frac{1}{2}\left(\frac{\theta}{k_1\tau}\right)^p \int_{S(x^0,r_1)} F(x,\tau)\,dx \\ &\leqslant \frac{\frac{1}{p}(\frac{\theta}{c})^p}{\frac{1}{p}(\frac{\theta}{c})^p + \frac{1}{p}(\frac{k_1\tau}{c})^p}\int_{S(x^0,r_1)} F(x,\tau)\,dx \leqslant \frac{r}{r + \Phi(u^*)}J(u^*). \end{aligned}$$

Now, we can apply Theorem A. Taking into account that

$$\begin{aligned} &\Phi(u^*)/\big(J(u^*) - \sup_{x\in\overline{\Phi^{-1}(]-\infty,r[)}^w} J(u^*)\big) \\ &\qquad \leqslant \frac{\frac{1}{p}(\frac{k_1\tau}{c})^p}{\int_{S(x^0,r_1)} F(x,\tau)\,dx - \int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx}; \end{aligned}$$

$$\frac{r}{\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^w} J(u)} \geqslant \frac{\frac{1}{p}(\frac{\theta}{c})^p}{\int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx};$$

$$\frac{hr}{r\frac{J(u^*)}{\Phi(u^*)} - \sup_{u\in\overline{\Phi^{-1}(-\infty,r[)}^w} J(u)}$$
$$\leqslant \frac{\frac{h}{p}(\frac{\theta}{c})^p}{(\frac{\theta}{k_1\tau})^p \int_{S(x^0,r_1)} F(x,\tau)\,dx - \int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx} = \rho;$$

and with $x_0 = 0$, $x_1 = u^*$, and see $\Lambda_1' \subseteq \Lambda_1$, $\Lambda_2 \subseteq \Lambda_2'$, and also taking into account that the weak solutions of the problem (1.1) are exactly the solutions of the equation

$$\Phi'(u) - \lambda J'(u) = 0,$$

from Theorem A it follows that, for each $\lambda \in \Lambda_1'$, the problem (1.1) admits at least three weak solutions, and there exist an open interval $\Lambda_2' \subseteq [0,\rho]$ and a real positive number $\sigma$ such that, for each $\lambda \in \Lambda_2'$, the problem (1.1) admits at least three weak solutions that whose norms in X are less than $\sigma$. Hence, we have the conclusion. □

*Remark 1.* In Theorem 1,

$$\frac{\frac{1}{p}(\frac{k_1\tau}{c})^p}{\int_{S(x^0,r_1)} F(x,\tau)\,dx - \int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx} < \frac{\frac{1}{p}(\frac{\theta}{c})^p}{\int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx}.$$

Because, from $(\alpha_2)$ we have

$$2(k_1\tau)^p \int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx < \theta^p \int_{S(x^0,r_1)} F(x,\tau)\,dx,$$

and since $k_1\tau > \theta$, we get

$$(\theta^p + (k_1\tau)^p) \int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx < \theta^p \int_{S(x^0,r_1)} F(x,\tau)\,dx,$$

and so

$$(k_1\tau)^p \int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx < \theta^p \Big( \int_{S(x^0,r_1)} F(x,\tau)\,dx - \int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx \Big).$$

Hence, multiplying by $\frac{1}{pc^p}$ we obtain

$$\frac{1}{p}\Big(\frac{k_1\tau}{c}\Big)^p \int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx$$
$$< \frac{1}{p}(\frac{\theta}{c})^p \Big( \int_{S(x^0,r_1)} F(x,\tau)\,dx - \int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx \Big),$$

which follows

$$\frac{\frac{1}{p}(\frac{k_1\tau}{c})^p}{\int_{S(x^0,r_1)} F(x,\tau)\,dx - \int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx} < \frac{\frac{1}{p}(\frac{\theta}{c})^p}{\int_\Omega \sup_{t\in[-\theta,\theta]} F(x,t)\,dx}.$$

*Remark 2.* In applying Theorem 1, it is enough to know as explicit upper bound of the constant $c$. To be precise, we can use formula (2.1) as constant $c$ the right-hand term of the formula in page 393, so that the constant $k_1$ in Theorem 1 is numerically well determined.

We now present a particular case of Theorem 1, in which the function $f$ has separated variables.

*Corollary 1.* Let $f_1 \in L^1(\Omega)$ and $f_2 \in C(R)$ be two functions. Put $\widetilde{F}(t) = \int_0^t f_2(\xi)\,d\xi$ for all $t \in R$, and assume that there exist four positive constants $\theta$, $\tau$, $\eta$ and $\gamma$ with $k_1\tau > \theta$, $\gamma < p$ such that

($\alpha_1'$) $f_1(x) \geqslant 0$ for each $x \in \Omega \setminus S(x^0,r_1)$ and $f_2(t) \geqslant 0$ for each $t \in [0,\tau]$,

($\alpha_2'$) $\max_{t\in[-\theta,\theta]} \widetilde{F}(t)(\int_\Omega f_1(x)\,dx) < \frac{\widetilde{F}(\tau)}{2}(\frac{\theta}{k_1\tau})^p \int_{S(x^0,r_1)} f_1(x)\,dx$,

($\alpha_3'$) $|\widetilde{F}(t)| \leqslant \eta(1+|t|^\gamma)$ for all $t \in R$,

where $k_1$ is given in (2.1). Then, for each

$$\lambda \in \Lambda_1' = \left]\frac{\frac{1}{p}(\frac{k_1\tau}{c})^p}{\widetilde{F}(\tau)\int_{S(x^0,r_1)} f_1(x)\,dx - \max_{|t|\leqslant\theta} \widetilde{F}(t)(\int_\Omega f_1(x)\,dx)}, \frac{\frac{1}{p}(\frac{\theta}{c})^p}{\max_{|t|\leqslant\theta} \widetilde{F}(t)(\int_\Omega f_1(x)\,dx)}\right[,$$

the problem

$$\begin{cases} \Delta_p u + \lambda f_1(x) f_2(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

admits at least three weak solutions in $X$ and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda_2' \subseteq \left[0, \frac{\frac{h}{p}(\frac{\theta}{c})^p}{(\frac{\theta}{k_1\tau})^p \widetilde{F}(\tau)\int_{S(x^0,r_1)} f_1(x)\,dx - \max_{|t|\leqslant\theta}\widetilde{F}(t)(\int_\Omega f_1(x)\,dx)}\right]$$

and a positive real number $\sigma$ such that, for each $\lambda \in \Lambda_2'$, the problem (2.2) admits at least three weak solutions in $X$ whose norms are less than $\sigma$.

*Proof.* Set $f(x,u) = f_1(x)f_2(u)$ for each $(x,u) \in \Omega \times R$. Since

$$F(x,t) = f_1(x)\widetilde{F}(t), \tag{2.3}$$

from ($\alpha_1'$) and ($\alpha_2'$) we obtain ($\alpha_1$) and ($\alpha_2$), respectively. From (2.3) and ($\alpha_3'$) we have

$$F(x,t) \leqslant |f_1(x)\widetilde{F}(t)| \leqslant \eta|f_1(x)|(1+|t|^\gamma)$$

for each $(x,t) \in \Omega \times R$, so condition ($\alpha_3$) follows with $\mu(x) = \eta|f_1(x)|$. Then, Theorem 1 yields the conclusion. □

*Example 1.* Consider the problem

$$\begin{cases} \operatorname{div}(|\nabla u|\nabla u) + \lambda(e^{-u}u^{10}(11-u)) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $\Omega = \{(x,y) \in R^2;\ x^2 + y^2 < 9\}$. Taking into account $c = \sqrt[6]{36/\pi^2}$, choosing $x^0 = (0,0)$, $r_1 = 1$, $r_2 = 2$, $f_1(x) = 1$ for all $x \in \Omega$ and

$$f_2(u) = e^{-u}u^{10}(11-u)$$

for each $u \in R$, so that $k_1 = \sqrt[6]{324}$, all the assumptions of Corollary 1, with $p = 3$, are satisfied by choosing, for instance $\theta = 1$, $\tau = 3$, $\gamma = 2$ and $\eta$ sufficiently large. So for each $\lambda \in \left]\frac{3e}{3^9 e^{-2}-1}, \frac{e}{162}\right[$, the problem (2.4) admits at least three non-trivial weak solutions in $W_0^{1,3}(\Omega)$ and, moreover, for each $h > 1$, there exist an open interval $\Lambda \subseteq ]0, \frac{he}{9(729e^{-2}-18)}[$ and a positive real number $\sigma$ such that, for each $\lambda \in \Lambda$, the problem (2.4) admits at least three weak solutions in $W_0^{1,3}(\Omega)$ whose norms are less than $\sigma$.

Finally, we want to point out a simple consequence of Theorem 1 in the ordinary case with $p = 2$, and then we present an example of application.

For simplicity, we fix $\Omega = (a,b)$ for $a,b \in R$ and $x^0 \in \Omega$. Taking into account that, in this situation, $c = \frac{(b-a)^{\frac{1}{2}}}{2}$, $k_1 = (\frac{b-a}{2(r_2-r_1)})^{\frac{1}{2}}$ and $k_2 = \frac{1}{2}(\frac{b-a}{r_1(r_2-r_1)})^{\frac{1}{2}}$, we have the following result:

*Corollary 2.* Let $f : [a,b] \times R \to R$ be a continuous function and put $F(x,t) = \int_0^t f(x,\xi)\,d\xi$ for each $(x,t) \in [a,b] \times R$. Assume that there exist three positive constants $\theta$, $\tau$ and $\gamma$ with $(\frac{b-a}{2(r_2-r_1)})^{\frac{1}{2}}\tau > \theta$, $\gamma < 2$ and a function $\mu \in L^1([a,b])_+$ such that

($\alpha_1''$) $F(x,t) \geqslant 0$ for each $(x,t) \in ((a,b) \setminus (x^0 - r_1, x^0 + r_1)) \times [0,\tau]$,

($\alpha_2''$) $\int_a^b \sup_{t\in[-\theta,\theta]} F(x,t)\,dx < \frac{r_2-r_1}{b-a}(\frac{\theta}{\tau})^2 \int_{x^0-r_1}^{x^0+r_1} F(x,\tau)\,dx$,

($\alpha_3''$) $F(x,t) \leqslant \mu(x)(1+|t|^\gamma)$ for almost every $x \in (a,b)$ and for all $t \in R$.

Then, for each

$$\lambda \in \Lambda_1' = \left] \frac{\tau^2/(r_2-r_1)}{\int_{x^0-r_1}^{x^0+r_1} F(x,\tau)\,dx - \int_a^b \sup_{t\in[-\theta,\theta]} F(x,t)\,dx}, \right. \left. \frac{2\theta^2}{(b-a)\int_a^b \sup_{t\in[-\theta,\theta]} F(x,t)\,dx} \right[,$$

the problem

$$\begin{cases} u'' + \lambda f(x,u) = 0 & \text{in } (a,b), \\ u(a) = u(b) = 0, \end{cases} \tag{2.5}$$

admits at least three weak solutions in $X$ and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda_2' \subseteq \left[0, \frac{2h\theta^2}{2(r_2-r_1)(\frac{\theta}{\tau})^2 \int_{x^0-r_1}^{x^0+r_1} F(x,\tau)\,dx - (b-a)\int_a^b \sup_{t\in[-\theta,\theta]} F(x,t)\,dx}\right]$$

and a positive real number $\sigma$ such that, for each $\lambda \in \Lambda_2'$, the problem (2.5) admits at least three classical solutions in $X$ whose norms are less than $\sigma$.

*Example 2.* Put

$$f(x,u) = e^{-(x+u)}u^6(7-u)$$

for each $(x,u) \in (-3,3) \times R$, and choose $x^0 = 0$, $r_1 = 1$, $r_2 = 2$. It is easy to verify that with $\theta = 1$, $\tau = 3$, $\gamma = 1$ and $\mu(x)$ for each $x \in (-3,3)$ sufficiently large, all the assumptions of Corollary 2, are satisfied. So for each $\lambda \in \left]\frac{9}{2187(e^{-2}-e^{-4})+e^{-4}-e^{2}}, \frac{1}{3(e^{2}-e^{-4})}\right[$, the problem

$$\begin{cases} u'' + \lambda(e^{-(x+u)}u^6(7-u)) = 0 & \text{in } (-3,3), \\ u(-3) = u(3) = 0. \end{cases} \tag{2.6}$$

admits at least three non-trivial classical solutions in $W_0^{1,2}([-3,3])$ and, moreover, for each $h > 1$, there exist an open interval $\Lambda \subseteq \left]0, \frac{h}{32(e^{-2}-e^{-4})-3(e^{2}-e^{-4})}\right[$ and a positive real number $\sigma$ such that, for each $\lambda \in \Lambda$, the problem (2.6) admits at least three classical solutions in $W_0^{1,2}([-3,3])$ whose norms are less than $\sigma$.

*Remark 3.* The weak solutions of the problem (1.1) where $f$ is a continuous function, in the ordinary case with $\Omega = (a,b)$, $a,b \in R$ and $p = 2$, by using standard methods, belong to $C^2([a,b])$ and are classical solutions for the problem (1.1). Namely, in this case, the classical and the weak solutions of the problem (1.1) coincide.

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## References

[1] G.A. Afrouzi and S. Heidarkhani. Three solutions for a Dirichlet boundary value problem involving the $p$-Laplacian. *Nonlinear Anal.*, **66**:2281–2288, 2007. Doi:10.1016/j.na.2006.03.019.

[2] D. Averna and G. Bonanno. A three critical points theorem and its applications to the ordinary Dirichlet problem. *Topol. Methods Nonlinear Anal.*, **22**:93–104, 2003.

[3] R.I. Avery and J. Henderson. Three symmetric positive solutions for a second-order boundary value problem. *Appl. Math. Lett.*, **13**:1–7, 2000. Doi:10.1016/S0893-9659(99)00177-9.

[4] G. Bonanno. Existence of three solutions for a two point boundary value problem. *Appl. Math. Lett.*, **13**:53–57, 2000. Doi:10.1016/S0893-9659(00)00033-1.

[5] G. Bonanno. Some remarks on a three critical points theorem. *Nonlinear Anal.*, **54**:651–665, 2003. Doi:10.1016/S0362-546X(03)00092-0.

[6] G. Bonanno. A critical points theorem and nonlinear differential problems. *J. Global Optim.*, **28**:249–258, 2004. Doi:10.1023/B:JOGO.0000026447.51988.f6.

[7] G. Bonanno and R. Livrea. Multiplicity theorems for the Dirichlet problem involving the $p$-Laplacian. *Nonlinear Anal.*, **54**:1–7, 2003. Doi:10.1016/S0362-546X(03)00027-0.

[8] F. Cammaroto, A. Chinni‘ and B. Di Bella. Multiple solutions for a Dirichlet problem involving the $p$-Laplacian. *Dynam. Systems Appl.*, **16**:673–680, 2007.

[9] P. Drabek and J. Hernandez. Existence and uniqueness of positive solutions for some quasilinear elliptic problems. *Nonlinear Anal.*, **44**:189–204, 2001. Doi:10.1016/S0362-546X(99)00258-8.

[10] Z.M. Guo. On the number of positive solutions for quasilinear elliptic equations when a parameter is large. *Nonlinear Anal.*, **27**:229–247, 1996. Doi:10.1016/0362-546X(94)00352-I.

[11] Z.M. Guo and J.R.L. Webb. Uniqueness of positive solutions for quasilinear elliptic equations when a parameter is large. *Proc. Roy. Soc. Edinburgh*, **124**:189–198, 1994.

[12] D.D. Hai and R. Shivaji. Existence and uniqueness for a class of quasilinear elliptic boundary value problems. *J. Differential Equations*, **193**:500–510, 2003. Doi:10.1016/S0022-0396(03)00028-7.

[13] S.A. Marano and D. Motreanu. On a three critical points theorem for non-differentiable functions and applications nonlinear boundary value problems. *Nonlinear Anal.*, **48**:37–52, 2002. Doi:10.1016/S0362-546X(00)00171-1.

[14] M. Ramaswamy and R. Shivaji. Multiple positive solutions for classes of $p$-Laplacian equations. *Differ. Integral Equ.*, **17**:1255–1261, 2004.

[15] B. Ricceri. On a three critical points theorem. *Arch. Math. (Basel)*, **75**:220–226, 2000.

[16] B. Ricceri. A three critical points theorem revisited. *Nonlinear Anal.*, **70**:3084–3089, 2009. Doi:10.1016/j.na.2008.04.010.

[17] J. Simon. *Regularitè de la Solution d'une Equation Non lineaire dans $R^N$. LMN 665.* P. Benilan ed., Berlin, Heidelberg, New York, 1978.

[18] G. Talenti. Some inequalities of Sobolev type on two-dimensional spheres. *in: W. Walter (Ed.), General Inequalities, Vol. 5, Internat. Ser Numer. Math.*, **8**:401–408, 1987.

[19] E. Zeidler. *Nonlinear Functional Analysis and Its Applications.* Vol. II/B. Berlin, Heidelberg, New York, 1985.