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A Fixed Point Approach to Stationary Heat Transfer in Electric Cables

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Abstract. In this paper we present a novel procedure for the determination of temperature in electric conductors. A Helmholtz-to-Poisson estimate is proved, that justifies to restrict the temperature dependence of the electrical resistivity to the conductor boundary. Hence we obtain a nonlinear potential problem for the relevant boundary temperatures, where the temperature dependence of the heat transfer coefficient is fully regarded. Using boundary integral operators, we represent the unknowns as the fixed point of a contraction. Finally a benchmark example is given in the rotationally symmetric case.

Keywords: Heat transfer problem, nonlinear boundary integral equations, contractive mapping, iterative method, electric cables.

AMS Subject Classification: 35J05; 45G05; 80A20.

1 Introduction

At present, the use of electric conductors grows steadily in modern technology and an economic choice of the conductor parameters becomes mandatory. Especially in systems with limited weight and size - e.g. aircrafts, cars or other mobile systems - it is necessary to find optimal geometric and material parameters of the electric cable. For this reason, it is important to develop effective procedures that permit the direct determination of temperature at characteristic positions of the conductor.

The purpose of this article is the set up of a mathematical model which starts from the physical background of the heat transfer problem and arrives at a novel procedure for the determination of the relevant temperatures. In contrast to numerical methods that employ finite element and finite volume methods for discretization and use iterative schemes solving the discretized nonlinear problems, see e.g. [10, 11, 24], we present a fixed point approach for the continuous problem. One advantage of this approach is the comparatively late involvement of a discretization method to obtain numerical data. Thus we are able to treat such problems without a restriction to a specific geometry.

We consider an insulated electric cable (main for short) of infinite length. The given current flow induces a heat flow which - after attaining the thermodynamical equilibrium - leads to a steady temperature distribution in the cross-section of the conductor. Supported by experimental observations, we assume constant heat conductivities in the conductor and in the insulator of the main. By comparatively large heat conductivity, small differences in temperature in the conductor material can be expected. This motivates a restriction of the dependence to a mean value boundary temperature. Here we prove a Helmholtz-to-Poisson estimate which, to our knowledge, is a new result. The corresponding error is minimized by an orthogonal projection and an optimal Poincaré constant for convex domains [27]. Due to its generality, this estimate applies also to other elliptic boundary value problems. Then we formulate the heat transfer problem on the insulator domain only. Here the maximum principle for elliptic equations implies that the temperatures at the boundary of the insulator domain are the extremal and thus relevant unknowns.

These arguments give rise to treat the heat transfer problem by boundary integral equations on the boundary of the multiply connected insulator domain. The outer boundary condition is set up by the nonlinearly temperature dependent heat transfer coefficient. For a known qualitative temperature dependence, the quantitative analytical determination of such a coefficient is investigated as an inverse problem in [23].

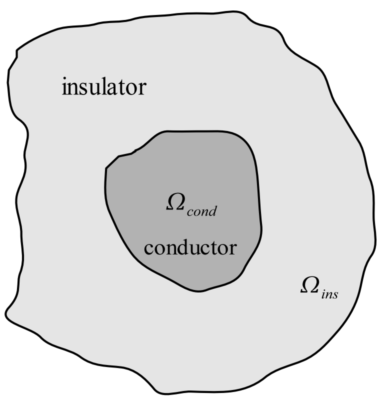
We truncate the heat transfer coefficient function at the boundaries of a physically consistent temperature interval by constant values. This, and a scaling condition for the insulator domain gives us an existence and uniqueness result for the boundary temperatures extending the analysis by Ruotsalainen and Wendland in [31] from simply to doubly connected domains. Then we transform the nonlinear boundary integral equation to a fixed point equation on an appropriate Sobolev space and compute the solution via an iterative method presented for abstract Hilbert spaces by Brézis and Sibony in [5]. The related error estimate provides a linear order of convergence for the approximating iterative sequence.

Next we extend these investigations to multiply connected domains. Here, as in the doubly connected case, the strong monotonicity of the Poincaré–Steklov operator of the underlying boundary value problem is essential. In this context we introduce an abstract property for boundaries of multiply connected domains - the damping property. This property enables us to verify the strong monotonicity of the Poincaré–Steklov operator independently from the conductor parameters, i.e. just using the outer boundary condition.

Finally we deal with the case of rotational symmetry. Here the boundary integral operators reduce to matrices which can be computed explicitly. Thus we obtain the solution as the limit of an iterative sequence of vectors. These calculations show clearly that the scaling condition is essential for the positivity of the single layer operator. The performed calculations serve as a benchmark example for numerical computations of heat transfer problems with similar geometries.

For the occurring physical entities we use the following notation: I is the electric current, ρ denotes the electric resistivity of the conductor material, λ_1 is the heat conductivity of the conductor material, λ_2 is the heat conductivity of the insulator material, u_{env} denotes the temperature of the environment, α is the heat transfer coefficient on the insulator surface. For the temperature dependence of α in general we refer to [6, 9, 28, 34].

2 Modelling of the Heat Transfer Problem



Using standard notation for partial differential equations [16] we describe the cross-section of the main by the simply connected and open union $\Omega_0 = \overline{\Omega}_{cond} \cup \Omega_{ins} \subset \mathbb{R}^2$ with Lipschitz boundaries $\partial\Omega_0$, $\partial\Omega_{cond}$. The temperature distribution $u : \Omega_0 \to \mathbb{R}$ has to satisfy the following boundary value problem

$$
\begin{aligned}
-\operatorname{div}(\lambda_1(u)\nabla u) &= f(u) \quad \text{in } \Omega_{cond}, && (2.1)\\
-\operatorname{div}(\lambda_2(u)\nabla u) &= 0 \quad \text{in } \Omega_{ins}, && (2.2)\\
u_{cond} &= u_{ins} \quad \text{on } \partial\Omega_{cond},\\
\lambda_1(u)\frac{\partial u_{cond}}{\partial n} &= \lambda_2(u)\frac{\partial u_{ins}}{\partial n} \quad \text{on } \partial\Omega_{cond},\\
-\lambda_2(u)\frac{\partial u}{\partial n} &= \alpha(u)(u-u_{env}) \quad \text{on } \partial\Omega_0.
\end{aligned}
$$

Here u_{ins} and u_{cond} denote the restrictions $u_{ins} = u|_{\Omega_{ins}}$, $u_{cond} = u|_{\Omega_{cond}}$ and n the outer normal w.r.t. the considered domain.

Thus the two equations on the boundary $\partial\Omega_{cond}$ describe the transmission conditions, i.e. the continuity of temperature and the equality of the heat flows. In the following we will use the abbreviations $\Omega_1 = \Omega_{cond}$, $\Omega_2 = \Omega_{ins}$. The right-hand side

$$f(u) := \rho(u)\frac{I^2}{|\Omega_1|^2}$$

stands for the temperature dependent heat power density in the conductor.

Specification of $\rho(u)$ and λ_1. We postulate the standard model of a linear-affine temperature dependence of $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\rho(u) = \rho_0\left(1 + \alpha_\rho(u-u_0)\right); \qquad u = u(x), \quad x \in \Omega_1.$$

$\rho_0 > 0$ denotes the resistivity value to a reference temperature u_0, $\alpha_\rho \in \mathbb{R}$ identifies the linear temperature coefficient of ρ. Assume moreover that the heat conductivity $\lambda_1 > 0$ is constant. These assumptions provide accurate approximations to experimental data of many conductor materials. In this modelling assumption we can approximate the solution u_{cond} of (2.1) by the solution of a related Poisson-equation where the temperature dependence of ρ is restricted to a constant (conductor-)temperature.

Comparison with a constant right-hand side

Thus we are led to consider the following boundary value problems. For given $g \in H^{\frac{1}{2}}(\partial\Omega_1)$ we have the Dirichlet problem for the Helmholtz equation for given $\varsigma \in \mathbb{R}$, $f_h \in L^2(\Omega_1)$,

$$-\Delta u_\varsigma = \varsigma u_\varsigma + f_h \quad \text{in } \Omega_1; \qquad u_\varsigma = g \quad \text{on } \partial\Omega_1 \tag{2.3}$$

and the Dirichlet problem for the Poisson equation for given $f_p \in L^2(\Omega_1)$,

$$-\Delta u = f_p \quad \text{in } \Omega_1; \qquad u = g \quad \text{on } \partial\Omega_1. \tag{2.4}$$

Now we estimate the difference $u_\varsigma - u$ with respect to ς, f_h and f_p in the following result.

Lemma 1 [Helmholtz-to-Poisson estimate]. *Let u_ς and u solve* (2.3) *and* (2.4)*, respectively and suppose that $|\varsigma| < \Lambda^2$, where*

$$\Lambda = \inf_{v \in H_0^1(\Omega_1)} \left\{ \|\nabla v\|_{L^2(\Omega_1)} : \|v\|_{L^2(\Omega_1)} = 1 \right\}$$

denotes the Poincaré constant of Ω_1. Then

$$\|u_\varsigma - u\|_{H_0^1(\Omega_1)} := \|\nabla(u_\varsigma - u)\|_{L^2(\Omega_1)} \le \frac{\Lambda}{\Lambda^2 - |\varsigma|} \|\varsigma u + f_h - f_p\|_{L^2(\Omega_1)}.$$

Proof. Observe that by the Lemma of Lax–Milgram the solutions u_ς and u of the associated weak formulations of (2.3) and (2.4) exist uniquely in $H^1(\Omega_1)$. Hence the difference $u_\varsigma - u =: \psi \in H_0^1(\Omega_1)$ fulfills in the weak sense

$$-\Delta\psi = \varsigma u_\varsigma + f_h - f_p \quad \text{in } \Omega_1; \qquad \psi = 0 \quad \text{on } \partial\Omega_1. \tag{2.5}$$

With $\varsigma u_\varsigma = \varsigma\psi + \varsigma u$ the variational form of (2.5) reads as

$$\langle\psi, \varphi\rangle_{H_0^1(\Omega_1)} = \varsigma \langle\psi, \varphi\rangle_{L^2(\Omega_1)} + \langle\varsigma u + f_h - f_p, \varphi\rangle_{L^2(\Omega_1)}, \quad \forall\varphi \in H_0^1(\Omega_1).$$

$\varphi = \psi$ and the Cauchy–Schwarz inequality provides

$$\|\psi\|^2_{H_0^1(\Omega_1)} \le \|\psi\|_{L^2(\Omega_1)} \left(|\varsigma| \, \|\psi\|_{L^2(\Omega_1)} + \|\varsigma u + f_h - f_p\|_{L^2(\Omega_1)} \right).$$

The Poincaré inequality yields

$$\|\psi\|_{H_0^1(\Omega_1)} \le \frac{1}{\Lambda} \left(\frac{|\varsigma|}{\Lambda} \|\psi\|_{H_0^1(\Omega_1)} + \|\varsigma u + f_h - f_p\|_{L^2(\Omega_1)} \right)$$

and hence we get for $|\varsigma| < \Lambda^2$ the assertion of Lemma 1. $\square$

Using previous notations we treat now the equations

$$-\lambda_1 \Delta u_\varsigma = f(u_\varsigma) \quad \text{in } \Omega_1 \quad \text{and} \quad -\lambda_1 \Delta u = f(\bar{u}) \quad \text{in } \Omega_1 \tag{2.6}$$

where for the moment $\bar{u} \in \mathbb{R}$ denotes an arbitrary constant temperature and $u_\varsigma|_{\partial\Omega_1} = u|_{\partial\Omega_1} = g \in H^{1/2}(\partial\Omega_1)$. With the settings

$$f_p = \frac{f(\bar{u})}{\lambda_1}; \qquad f_h = f_p + \varsigma\bar{u} \quad \text{and} \quad \frac{\Lambda|\varsigma|}{\Lambda^2 - |\varsigma|} =: \mathcal{C}(\varsigma)$$

the right-hand side of the estimate in Lemma 1 reads as $\mathcal{C}(\varsigma)\,\|u-\bar{u}\|_{L^2(\Omega_1)}$. We minimise this term with respect to $\bar{u} \in \mathbb{R}$ using the orthogonal projection in $L^2(\Omega_1)$.

Corollary 1 [Best approximation]. Let u_ς, u solve the Helmholtz- respectively the Poisson equation in (2.6). Suppose that the Poisson datum $f(\bar{u})$ is given by the mean value $\bar{u} = \frac{1}{|\Omega_1|}\int_{\Omega_1} u\,\mathrm{d}x \in \mathbb{R}$ and $|\varsigma| < \Lambda^2$ where Λ denotes the Poincaré constant of Ω_1. Then $\bar{u}$ yields the unique best approximation u of u_ς i.e.

$$\|u_\varsigma - u\|_{H_0^1(\Omega_1)} \le \mathcal{C}(\varsigma)\,\|u-\bar{u}\|_{L^2(\Omega_1)} = \mathcal{C}(\varsigma)\sqrt{\|u\|^2_{L^2(\Omega_1)} - |\Omega_1|\bar{u}^2}.$$

Proof. The first estimate follows directly from Lemma 1. The identities and the minimal property of the mean value $\bar{u}$ follow from the perpendicular principle applied to the subspace $\{v \in L^2(\Omega_1) : v \equiv \mathit{const}\} = \mathbb{R}$ of $L^2(\Omega_1)$: The unique minimiser $a \in \mathbb{R}$ is given by the orthogonal projection of u on $\mathbb{R}$, i.e. by the solution of $\langle a, a-u\rangle_{L^2(\Omega_1)} = 0$. This implies

$$a = \frac{1}{|\Omega_1|}\int_{\Omega_1} u\,\mathrm{d}x \quad \text{and} \quad \inf_{a\in\mathbb{R}} \|u-a\|^2_{L^2(\Omega_1)} = \|u\|^2_{L^2(\Omega_1)} - |\Omega_1|\bar{u}^2. \quad \square$$

Thus u_ς converges asymptotically linear to u with the error bound

$$\limsup_{\varsigma\to 0} \frac{\|u_\varsigma - u\|_{H_0^1(\Omega_1)}}{|\varsigma|} \le \frac{1}{\Lambda}\sqrt{\|u\|^2_{L^2(\Omega_1)} - |\Omega_1|\bar{u}^2}$$

using the minimiser $\bar{u}$ and the general error bound depending on $\bar{u} \in \mathbb{R}$ in the Poisson-datum,

$$\limsup_{\varsigma\to 0} \frac{\|u_\varsigma - u\|_{H_0^1(\Omega_1)}}{|\varsigma|} \le \frac{1}{\Lambda}\,\|u-\bar{u}\|_{L^2(\Omega_1)}\,. \tag{2.7}$$

With $\varsigma = \frac{\rho_0\alpha_\rho I^2}{\lambda_1|\Omega_1|^2}$, this asymptotic result means in particular that the solution of the Helmholtz equation converges to the solution of the Poisson equation if the heat conductivity λ_1 becomes large (constant temperature profile in the conductor) or $\alpha_\rho \to 0$ (electrical resistivity ρ becomes temperature independent). An even more explicit estimate can be given for convex domains Ω_1, where by [27] the optimal Poincaré constant is known as $\Lambda = \frac{\pi}{diam(\Omega_1)}$. By Corollary 1 we are able to control the error if instead of the fully temperature dependent ρ, we only consider resistivities $\rho(\bar{u})$ that depend on constant temperatures. We shall give an explicit example for the error bound when we treat the case of rotational symmetry.

The outer domain formulation. Now we formulate the boundary value problem (2.1), (2.2) in the insulator domain only. We choose the constant mean value boundary temperature (m.v.b.t.) $\bar{u} := \frac{1}{|\partial\Omega_1|}\int_{\partial\Omega_1} u\,\mathrm{d}s_x$ in the Poisson datum of (2.6). This is not the error minimising choice; nevertheless, as we shall see, it is the appropriate one for the forthcoming boundary integral formulation. The error is bounded by (2.7).

Consider now the heat flow density $q = q(u)$ over the boundary $\partial\Omega_1$ which enters in the inner boundary condition $-\lambda_1 \frac{\partial u_{cond}}{\partial n} = q$. Using the equality of heat flows $\lambda_1 \frac{\partial u_{cond}}{\partial n} = \lambda_2 \frac{\partial u_{ins}}{\partial n}$ this condition becomes $-\lambda_2 \frac{\partial u}{\partial n} = q$ on $\partial\Omega_1$ for $u = u_{ins}$. Assume for the moment the heat flow density q as given. Note that by the Gauß divergence theorem, q has to fulfill

$$\int_{\partial\Omega_1} q\,\mathrm{d}s_x = |\Omega_1| f(\bar{u}) = \rho(\bar{u})\frac{I^2}{|\Omega_1|}. \tag{2.8}$$

The simplified form of the right-hand side is justified by the Helmholtz-to-Poisson-estimate above. Thus we consider the following boundary value problem

$$-\Delta u = 0 \quad \text{in } \Omega_{ins} =: \Omega,$$
$$\lambda_2 \frac{\partial u}{\partial n} = q(u) \quad \text{on } \partial\Omega_1, \tag{2.9}$$
$$-\lambda_2 \frac{\partial u}{\partial n} = \alpha(u)(u - u_{env}) \quad \text{on } \partial\Omega \setminus \partial\Omega_1 =: \partial\Omega_2, \tag{2.10}$$

where now n denotes the outer normal w.r.t. Ω.

Specification of the heat flow density. For the computation of $q = q(u)$ one has to regard the specific geometry of the boundary $\partial\Omega_1$ and the source term $f = f(u)$. The general situation can be treated as an inverse problem. We refer to [4, 14, 15, 36].

In our case the source term is given by the m.v.b.t. approximation discussed above and reads as $f = f(\bar{u}) = \frac{\rho(\bar{u})I^2}{|\Omega_1|^2}$. Now we suggest an explicit form of the heat flow density for the following considerations. Since conductor cross sections of electric cables are nearly rotationally symmetric, let us assume $q = q(\bar{u})$. I.e. q does not depend on $x \in \partial\Omega_1$. Then (2.8) yields $q = q(\bar{u}) = \frac{\rho(\bar{u})I^2}{|\partial\Omega_1||\Omega_1|}$. Now if we drop the assumption that u is constant then again by (2.8), q and f have locally the same monotonicity behaviour w.r.t. to the boundary temperature. Thus we approximate a temperature dependent heat flux by

$$\tilde{q}(u) := \frac{\rho(u)I^2}{|\partial\Omega_1||\Omega_1|}. \tag{2.11}$$

We observe that by the weak maximum principle (see e.g. [16]) the extremal values of u are attained at the boundary of Ω. In applications, these values are the most interesting ones which motivates the following method.

3 Treatment by Nonlinear Boundary Integral Equations

In the following we are concerned with the temperatures on the boundary of the insulator domain only. Using Green's representation formula we derive an equivalent nonlinear boundary integral equation for the doubly connected domain Ω with $\partial\Omega_1 =: \Gamma_1$, $\partial\Omega_2 =: \Gamma_2$ that includes the boundary conditions (2.9), (2.10). Starting from $-\Delta u = 0$ in Ω the representation formula for harmonic functions and the jump relations of potential theory yield for the boundary values of u:

$$u(x) = 2\int_\Gamma \left(u(y)\frac{\partial}{\partial n_y}F(x-y) - \frac{\partial u(y)}{\partial n_y}F(x-y)\right) \mathrm{d}s_y, \quad x \in \Gamma := \partial\Omega, \tag{3.1}$$

where $F(z) := \frac{1}{2\pi}\ln(|z|)$ denotes the fundamental solution of the Laplace-equation in $\mathbb{R}^2 \setminus \{0\}$.

Let Φ be defined componentwise by $(\varphi_1, \varphi_2)^T$ and $u_i := u|_{\Gamma_i}$; $i = 1, 2$ then (2.9) and (2.10) can be written in vector notation as $-\frac{\partial u}{\partial n} = \Phi(u)$ on Γ. Moreover we need the Poincaré–Steklov-operator $\mathcal{P} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, $u \mapsto \varphi$ defined by

$$\mathcal{P}u = \varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \Phi(u) = \frac{1}{\lambda_2}\begin{pmatrix}-q(u_1)\\ \alpha(u_2)(u_2 - u_{env})\end{pmatrix}. \tag{3.2}$$

We emphasise that we consider a heat flux $q = q(u_1)$ that may fully depend on the boundary temperature which can be obtained by an inverse treatment or experimental data. Note that the nonlinearity of Φ appears in the second component, due to the heat transfer coefficient $\alpha = \alpha(u_2)$, that enters in the outer boundary condition.

The function spaces for the boundary Γ of the doubly connected domain Ω are given by $H^s(\Gamma) := H^s(\Gamma_1) \times H^s(\Gamma_2)$, $\|\cdot\|^2_{H^s(\Gamma)} := \|\cdot\|^2_{H^s(\Gamma_1)} + \|\cdot\|^2_{H^s(\Gamma_2)}$ for $s \in \{-1/2, 1/2\}$; see e.g. [3, 17, 19] for various approaches in multiply connected domains.

3.1 Representation by single and double layer potential operators in a doubly connected domain

Assume $\Gamma \in C^2$. Following singular boundary integral operator theory [19, 22, 26] we define the continuous single layer operator $\tilde{\mathcal{S}} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ by

$$\tilde{\mathcal{S}}(\varphi)(x) = -\int_\Gamma \varphi(y)F(x-y)\,\mathrm{d}s_y = \begin{pmatrix}\tilde{\mathcal{S}}_{11} & \tilde{\mathcal{S}}_{12}\\ \tilde{\mathcal{S}}_{21} & \tilde{\mathcal{S}}_{22}\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \tag{3.3}$$

where $\tilde{\mathcal{S}}_{ij}(\varphi_j) = -\int_{\Gamma_j} \varphi_j(y)F(x-y)\,\mathrm{d}s_y$, $x \in \Gamma_i$; $i, j = 1, 2$ and the compact double layer potential operator $\tilde{\mathcal{K}} : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ with

$$\tilde{\mathcal{K}}(u)(x) = \int_\Gamma u(y)\frac{\partial}{\partial n_y}F(x-y)\,\mathrm{d}s_y = \begin{pmatrix}\tilde{\mathcal{K}}_{11} & \tilde{\mathcal{K}}_{12}\\ \tilde{\mathcal{K}}_{21} & \tilde{\mathcal{K}}_{22}\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}, \tag{3.4}$$

where $\tilde{\mathcal{K}}_{ij}(u_j)(x) = \int_{\Gamma_j} u_j(y) \frac{\partial}{\partial n_y} F(x-y)\, \mathrm{d}s_y$, $x \in \Gamma_i$; $i, j = 1, 2$.

Our aim is to obtain a strongly elliptic operator $\mathcal{S}$ on a boundary of a multiply connected domain i.e. $\exists c > 0$: $\langle \mathcal{S}(\varphi), \varphi \rangle_{L^2(\Gamma)} \geq c \, \|\varphi\|^2_{H^{-1/2}(\Gamma)}$. Therefore we introduce the diagonalized form:

$$\mathcal{S}_{ij}(\varphi_j) := \begin{cases} \tilde{\mathcal{S}}_{ij}(\varphi_j), & i = j \\ 0, & i \neq j, \end{cases} \qquad \mathcal{K}_{ij}(u_j, \varphi_j) := \begin{cases} \tilde{\mathcal{K}}_{ij}(u_j), & i = j, \\ \tilde{\mathcal{K}}_{ij}(u_j) + \tilde{\mathcal{S}}_{ij}(\varphi_j), & i \neq j. \end{cases}$$

These definitions and (3.1) provide the following boundary integral equation

$$\frac{u}{2} - \mathcal{K}(u, \varphi) + \mathcal{S}(\Phi(u)) = 0, \quad \varphi = \Phi(u) = \frac{1}{\lambda_2} \begin{pmatrix} -q(u_1) \\ \alpha(u_2)(u_2 - u_{env}) \end{pmatrix}. \tag{3.5}$$

The compactness of $\mathcal{K}_{ij}$ for $i = j$, $\Gamma \in C^2$ is a classical result [19, 22] and it is immediate for $i \neq j$ since the kernel of $\mathcal{K}_{ij}$ is continuous in this case. Thus $\mathcal{K}$ is a compact operator in $H^{1/2}(\Gamma)$.

3.2 Existence and uniqueness of a solution of the nonlinear boundary integral equation

We assume the following conditions:

(A1) *Scaling*: $diam(\Omega) < 1$.

As we shall see in Section 6 this assumption can be arranged without loss of generality. It implies that $\mathcal{S} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is an isomorphism and strongly elliptic ([18, 35]).

(A2) *Asymptotic linearity*: Setting $h_1(u) := -\frac{q(u)}{\lambda_2}$ and $h_2(u) := \frac{\alpha(u)}{\lambda_2}(u - u_{env})$ we require that the derivatives satisfy

$$\min_{1 \leq i \leq 2} \left(\inf_{u \in \mathbb{R}} |h_i'(u)| \right) \geq c_{min} > 0 \quad \text{and} \quad \max_{1 \leq i \leq 2} \left(\sup_{u \in \mathbb{R}} |h_i'(u)| \right) \leq c_{max} < \infty.$$

The assumption provides Lipschitz continuity and strong monotonicity of the operator $\Phi : L^2(\Gamma) \to L^2(\Gamma)$.

Theorem 1. *Assume that* (A1) *and* (A2) *are satisfied. Then there exists a unique solution* $u \in H^{1/2}(\Gamma)$ *of* (3.5).

Proof. The proof given in [31] easily extends from the simply connected case to the doubly connected case considered here. □

Remark. Let us a give an example for a suitable Φ that satisfies the condition (A2) in both components, i.e. for $x \in \Gamma_1$ and $x \in \Gamma_2$. (A2) holds true in the first component with the heat flow density $\tilde{q} = \tilde{q}(u)$ in view of the linear-affine resistivity $\rho(u) := \rho_0 \left(1 + \alpha_\rho(u - u_0)\right)$, $\alpha_\rho > 0$. In the second component (A2) is satisfied e.g. for the following truncation and extension of the monotone and continuous heat transfer coefficient α,

$$\tilde{\alpha}(u) := \begin{cases} \alpha_l & \text{for } u < u_l, \\ \alpha_h & \text{for } u > u_h, \\ \alpha(u) & \text{in } [u_l, u_h], \end{cases} \tag{3.6}$$

where $0 < \alpha(u_l) = \alpha_l < \alpha(u_h) = \alpha_h$ for $u_l < u_h$. With these settings (A2) is satisfied with

$$c_{min} = \frac{\min(\alpha_l, c_0)}{\lambda_2}, \quad c_{max} = \frac{\max(\alpha_h, c_0)}{\lambda_2}, \quad \text{where } c_0 = \frac{\rho_0 \alpha_\rho I^2}{|\Gamma_1||\Omega_1|}. \tag{3.7}$$

For the strong monotonicity condition of Φ namely $\min(\alpha_l, c_0) > 0$ we require $I > 0$. This is no restriction since $I = 0$ implies $u \equiv u_{env}$.

We note that condition (A2) can be relaxed to a polynomial growth condition on Φ [29]. In view of applications it makes sense to consider bounded temperature intervals. The truncation outside of the interval $[u_l, u_h]$ does not change the heat transfer in the relevant temperature range. Thus it suffices to consider asymptotically linear operators Φ in (3.2).

4 Iterative Determination of the Boundary Temperature as a Fixed Point

Consider $\mathcal{T} : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ with $\mathcal{T}(u) := u/2 - \mathcal{K}(u, \varphi) + \mathcal{S}(\Phi(u))$, $\varphi = \Phi(u)$. The equation (3.5) for the boundary temperature u is satisfied iff the fixed point relation

$$\mathcal{G}_\gamma(u) := u - \gamma \mathcal{T}(u) = u \tag{4.1}$$

holds for at least one $\gamma \in \mathbb{R} \setminus \{0\}$. By previous considerations there exists a unique fixed point $u \in H^{1/2}(\Gamma)$ for (4.1). In this section we will determine a γ which ensures that $\mathcal{G}_\gamma$ is a contraction in $H^{1/2}(\Gamma)$. For this purpose we need Lipschitz-continuity of $\mathcal{T}$.

Lemma 2. *Suppose* (A2). *Then there exists* $\tilde{L} > 0$ *such that*

$$\|\mathcal{T}(u) - \mathcal{T}(v)\|_{H^{1/2}(\Gamma)} \le \tilde{L} \|u - v\|_{H^{1/2}(\Gamma)} \quad \text{for } u, v \in H^{1/2}(\Gamma).$$

Proof. $\mathcal{S} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is a bounded linear operator, i.e.

$$\|\mathcal{S}(\varphi)\|_{H^{1/2}(\Gamma)} \le L_\mathcal{S} \|\varphi\|_{H^{-1/2}(\Gamma)} \quad \text{for } \varphi \in H^{-1/2}(\Gamma)$$
$$\Rightarrow \quad \|\mathcal{S}(\Phi(u)) - \mathcal{S}(\Phi(v))\|_{H^{1/2}(\Gamma)} \le L_\mathcal{S} \|\Phi(u) - \Phi(v)\|_{L^2(\Gamma)}.$$

Φ is Lipschitz continuous in $L^2(\Gamma)$ by (A2) and hence

$$\|\mathcal{S}(\Phi(u)) - \mathcal{S}(\Phi(v))\|_{H^{1/2}(\Gamma)} \le L_\mathcal{S} L_\Phi \|u - v\|_{L^2(\Gamma)} \le L_\mathcal{S} L_\Phi \|u - v\|_{H^{1/2}(\Gamma)}.$$

By (A2) $\mathcal{K}$ is a bounded operator in $H^{1/2}(\Gamma)$. Hence

$$\|(I/2 - \mathcal{K})(u) - (I/2 - \mathcal{K})(v)\|_{H^{1/2}(\Gamma)} \le L_\mathcal{K} \|u - v\|_{H^{1/2}(\Gamma)}$$

and the assertion of the lemma follows with $\tilde{L} = L_\mathcal{S} L_\Phi + L_\mathcal{K}$. □

On the other hand, we need strong monotonicity of $\mathcal{T}$ with respect to an appropriate Hilbert space norm. For this we follow [18, 35] and introduce a norm on $H^{1/2}(\Gamma)$ induced by the inverse of the single layer operator

$$\|u\|^2_{\mathcal{S}^{-1}(\Gamma)} := \langle u, \mathcal{S}^{-1}(u) \rangle_{L^2(\Gamma)}, \quad u \in H^{1/2}(\Gamma),$$

which is well defined by assumption (A1). This norm is equivalent to the Sobolev–Slobodetskii-norm on $H^{1/2}(\Gamma)$, i.e.

$$\exists c_\Gamma > 0 : \frac{1}{c_\Gamma} \|u\|_{\mathcal{S}^{-1}(\Gamma)} \leq \|u\|_{H^{1/2}(\Gamma)} \leq c_\Gamma \|u\|_{\mathcal{S}^{-1}(\Gamma)} .$$

Lemma 3. *Suppose* (A1) *and* (A2). *For* $u, v \in H^{1/2}(\Gamma)$ *there exists* $\tilde{m} > 0$ *with*

$$\left\langle \mathcal{T}(u) - \mathcal{T}(v), \mathcal{S}^{-1}(u-v) \right\rangle_{L^2(\Gamma)} \geq \tilde{m} \|u - v\|^2_{H^{1/2}(\Gamma)} .$$

Proof. Setting $\langle \cdot,\cdot \rangle_{L^2(\Gamma)} := \sum_{i=1}^2 \langle \cdot,\cdot \rangle_{L^2(\Gamma_i)}$, the proof follows directly from [30, 31] applied componentwise. □

Construction of the iterative sequence. Now we can establish an iterative sequence $(u^{(n)})_{n\in\mathbb{N}} \subset H^{1/2}(\Gamma)$ which converges to the solution of (3.7) for an arbitrary initial function $u^{(1)} \in H^{1/2}(\Gamma)$. Before doing so we observe that the Lipschitz and monotonicity estimates in Lemmas 2 and 3 also hold w.r.t. the $\mathcal{S}^{-1}$-norm on $H^{1/2}(\Gamma)$.

Moreover by (A1), $\mathcal{S} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is a strongly elliptic, self-adjoint operator and so is $\mathcal{S}^{-1} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$. Thus the bilinear form $\langle u, v\rangle_{\mathcal{S}^{-1}(\Gamma)} := \left\langle u, \mathcal{S}^{-1}(v) \right\rangle_{L^2(\Gamma)}$ is symmetric and we obtain the following result.

Theorem 2. *Let the assumptions* (A1) *and* (A2) *hold. Define the iterative sequence* $(u^{(n)})_{n\in\mathbb{N}} \subset H^{1/2}(\Gamma)$ *by* $u^{(n+1)} := \mathcal{G}_\gamma\left(u^{(n)}\right)$, $\gamma = m/L^2$ *where* $L = c_\Gamma^2 \tilde{L}$ *denotes the* $\mathcal{S}^{-1}$ *- Lipschitz constant and* $m = \tilde{m}/c_\Gamma^2$ *the* $\mathcal{S}^{-1}$*-monotonicity constant of* $\mathcal{T}$. *The Lipschitz constant is chosen sufficiently large such that* $L > m$. *Then, for every initial function* $u^{(1)} \in H^{1/2}(\Gamma)$, $(u^{(n)})_{n\in\mathbb{N}}$ *converges to the solution* u *of* (3.7) *with respect to* $\|\cdot\|_{\mathcal{S}^{-1}}$ *with the a priori error estimate*

$$\|u^{(n)} - u\|_{\mathcal{S}^{-1}} \leq \frac{k^n}{1-k} \|u^{(2)} - u^{(1)}\|_{\mathcal{S}^{-1}}, \quad k = \sqrt{1 - \frac{m^2}{L^2}}.$$

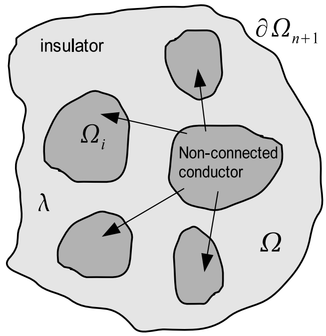
Proof. It suffices to verify that $\mathcal{G}_\gamma : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is contractive w.r.t. $\|\cdot\|_{\mathcal{S}^{-1}}$. The assertions of Theorem 2 then follow by Banach's fixed point theorem. Here

$$\begin{aligned} \|\mathcal{G}_\gamma(u) - \mathcal{G}_\gamma(v)\|^2_{\mathcal{S}^{-1}} &= \|u - v\|^2_{\mathcal{S}^{-1}} - 2\gamma \left\langle \mathcal{T}(u) - \mathcal{T}(v), \mathcal{S}^{-1}(u-v) \right\rangle_{L^2(\Gamma)} \\ &\quad + \gamma^2 \|\mathcal{T}(u) - \mathcal{T}(v)\|^2_{\mathcal{S}^{-1}} \\ &\leq \left(1 - 2m\gamma + L^2\gamma^2\right) \|u - v\|^2_{\mathcal{S}^{-1}} \end{aligned}$$

where we used the self-adjointness of $\mathcal{S}^{-1}$. The estimate is provided by the Lipschitz continuity and strong monotonicity of $\mathcal{T}$. The minimum of $1 - 2m\gamma + L^2\gamma^2$ is attained at $\gamma = \frac{m}{L^2}$ and amounts to $1 - \frac{m^2}{L^2}$. Hence we get $k = \sqrt{1 - \frac{m^2}{L^2}}$ as the constant of contraction. □

5 The Case of a Multiply Connected Domain

In this section we extend our previous considerations from a doubly connected (insulator) domain to a multiply connected one. Hence we can treat electrical cables with an ensemble of conductors with possibly different current loads. We use the following notation: N denotes the quantity of conductor cross-sections, Ω_i are the conductor cross-sections $i = 1, \dots, N$, Ω is the insulator cross-section, Γ_{N+1} denotes the (outer-) insulator boundary, $u_j \in H^{1/2}(\Gamma_j)$ denote the boundary temperatures $j = 1, \dots, N+1$, $q_i = q(u_i)$ is the heat flux over Γ_i, λ denotes the heat conductivity of the insulator.



For $\Gamma = \partial\Omega = \bigcup_{j=1}^{N+1} \Gamma_j$ the corresponding function spaces $H^s(\Gamma)$, $s \in \{-1/2, 1/2\}$ are given by $H^s(\Gamma) = \prod_{j=1}^{N+1} H^s(\Gamma_j)$ and $\|\cdot\|^2_{H^s(\Gamma)} = \sum_{j=1}^{N+1} \|\cdot\|^2_{H^s(\Gamma_j)}$.

The boundary value problem

$$-\Delta u = 0 \quad \text{in } \Omega;$$

$$\lambda \frac{\partial u}{\partial n} = q_i(u) \quad \text{on } \Gamma_i; \quad -\lambda \frac{\partial u}{\partial n} = \alpha(u)(u - u_{env}) \quad \text{on } \Gamma_{N+1}$$

leads again to a boundary integral equation for $u = (u_1, \dots, u_{N+1})^T \in H^{1/2}(\Gamma)$:

$$\frac{u}{2} - \mathcal{K}(u, \varphi) + \mathcal{S}(\Phi(u)) = 0$$

with $\varphi = \Phi(u) = \frac{1}{\lambda}\left(-q_1(u_1), \dots, -q_N(u_N), \alpha(u_{N+1})(u_{N+1} - u_{env})\right)^T$. The single and double layer potential operators $\mathcal{S}$ and $\mathcal{K}$ are defined and diagonalised in the same way as in (3.3), (3.4) for $i, j = 1, \dots, N+1$. With these settings Theorem 1 applies to the multiply connected domain case.

Application to multiwire cables. Now we will see how the crucial assumption (A2) of Theorem 1 is satisfied and how the iterative determination is realised in applications.

If the material out of the conductor cross sections is inhomogeneous (e.g. air gaps between the insulator material), then the constant heat conductivity λ of the insulator material, can be replaced by a homogenised heat conductivity $\bar{\lambda}$. Here we refer to [21, 25, 33].

The Helmholtz-to-Poisson estimate from Section 2 can be applied for each conductor cross-section separately. Thus again, we use the approximate heat flow densities over the boundary of the conductor cross section for $i = 1, \dots, N$ as

$$\tilde{q}_i = \tilde{q}_i(u_i) = \frac{\rho_i(u_i) I_i^2}{|\Gamma_i||\Omega_i|}$$

with $\rho_i(u_i) = (\rho_0)_i\,(1 + (\alpha_\rho)_i(u_i - u_0))$. The indexed quantities have the same meaning as before.

Moreover we use the truncated heat transfer coefficient $\tilde{\alpha}$ from the Remark in Section 3. Thus the associated functions $h_j : \mathbb{R} \to \mathbb{R};\ j = 1, \ldots, N+1$ with

$$h_i(u) := \frac{\tilde{q}_i(u)}{\lambda}; \quad i = 1, \ldots, N \quad \text{and} \quad h_{N+1}(u) := \frac{\tilde{\alpha}(u)}{\lambda}(u - u_{env})$$

fulfill the assumption (A2) with the following bounds

$$c_{min} = \tfrac{\min(\alpha_l, b_{min})}{\lambda} \quad \text{and} \quad c_{max} = \tfrac{\max(\alpha_h, b_{max})}{\lambda} \quad \text{where} \tag{5.1}$$
$$b_{min} = \min_{1\le i\le n}\left\{\tfrac{(\rho_0)_i(\alpha_\rho)_i I_i^2}{|\Gamma_i||\Omega_i|}\right\} \quad \text{and} \quad b_{max} = \max_{1\le i\le n}\left\{\tfrac{(\rho_0)_i(\alpha_\rho)_i I_i^2}{|\Gamma_i||\Omega_i|}\right\}.$$

For the strong monotonicity of Φ we assume w.l.o.g. $I_i > 0;\ i = 1, \ldots, N$. Otherwise, we only consider cross-sections with $I_j > 0$; the currentless cross-sections are included in the insulator domain Ω and are taken into account when the homogenised heat conductivity $\bar{\lambda}$ is computed.

Damping property. Let $u = (u_1, \ldots, u_{N+1})^T \in H^{1/2}(\Gamma)$ and function $\varphi = (\varphi_1, \ldots, \varphi_{N+1})^T \in H^{-1/2}(\Gamma)$ denotes a solution of $\frac{u}{2} = \mathcal{K}(u,\varphi) - \mathcal{S}(\varphi)$. Hereto we extract the diagonal components of the associated Poincaré–Steklov operator $\mathcal{P} : u \mapsto \varphi$ by $\mathcal{P}_j : H^{1/2}(\Gamma_j) \to H^{-1/2}(\Gamma_j), \mathcal{P}_j : u_j \mapsto \varphi_j$ with $\mathcal{P}_j = \mathcal{S}_{jj}^{-1} \circ (\mathcal{K}_{jj} - \frac{1}{2});\ j = 1, \ldots, N+1$.

Definition. Γ has the damping property if

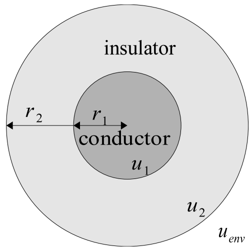
$$\min_{1\le i\le N} m_i \ge m_{N+1}, \quad \text{where } m_j = \inf_{v\in H^{1/2}(\Gamma_j)\setminus\{0\}} \frac{\|\mathcal{P}_j(v)\|_{H^{-1/2}(\Gamma_j)}}{\|v\|_{\mathcal{S}^{-1}(\Gamma_j)}}. \tag{5.2}$$

This property means that the change of the boundary temperature changes the inner normal derivatives more than the outer normal derivative.

For domains with the damping property the lower bounds in (3.7) and (5.1) read as $c_{min} = \alpha_l/\lambda_2$. Moreover, if (5.2) is verified by an a priori estimate, there is no need to exclude the case $I = 0$. Now the Lipschitz- and monotonicity estimates from the previous section and Theorem 2 can be applied analogously to the doubly connected domain case.

6 The Case of Rotational Symmetry

Finally we treat the outer domain boundary value problem (2.9), (2.10) with a rotationally symmetric cross section. This case can be used as a benchmark example for the iteration in Theorem 2 or boundary element methods solving (3.5). In addition to the previous notation we introduce



r_1 radius of the conductor

r_2 radius of the main

u_1 inner boundary temperature at $\Gamma_1 = \partial B_{r_1}$

u_2 outer boundary temperature at $\Gamma_2 = \partial B_{r_2}$.

Without loss of generality we can choose a suitable unit for the radius such that the relation $0 < r_1 < r_2 < 1/2$ is fulfilled. Due to the rotational symmetry of the system, the boundary temperatures u_1 and u_2 are constant.

Helmholtz-to-Poisson estimate. The source term f in the conductor reads as $f(u) = \rho_0(1 + \alpha_\rho(u - u_0))I^2/(\pi^2 r_1^4)$. We compare the solutions of

$$\begin{aligned} -\lambda_1 \Delta u_\varsigma &= f(u_\varsigma) \quad \text{in } B_{r_1}, \\ -\lambda_1 \Delta u &= f(u_1) \quad \text{in } B_{r_1}, \\ u_\varsigma &= u|_{\partial B_{r_1}} = u_1 \quad \text{on } \partial B_{r_1}. \end{aligned} \tag{6.1}$$

Here the Helmholtz coefficient equals $\varsigma = \frac{\rho_0 \alpha_\rho I^2}{\lambda_1 \pi^2 r_1^4}$. The solution u of (6.1) is given by $u(r) = \frac{f(u_1)}{4\lambda_1}(r_1^2 - r^2) + u_1, \quad r \in [0, r_1]$. Then for $\varsigma < \Lambda^2 = \frac{\pi^2}{4r_1^2}$ i.e. $4\rho_0\alpha_\rho I^2 \le \pi^4 r_1^2 \lambda_1$ we have by Corollary 1

$$\|u_\varsigma - u\|_{H_0^1(B_{r_1})} \le \frac{\varsigma \Lambda}{\Lambda^2 - \varsigma} \sqrt{\frac{\pi}{3}} \frac{f(u_1) r_1^3}{4\lambda_1}. \tag{6.2}$$

Specification of q and α. By (2.8) the heat flow density $q = q(u_1)$ becomes

$$q(u_1) = \frac{|\Omega_1|}{|\Gamma_1|} f(u_1) = \frac{\rho_0(1 + \alpha_\rho(u_1 - u_0))I^2}{2\pi^2 r_1^3}.$$

We follow fluid mechanical considerations in [2, 7, 8, 20] concerning the heat transfer coefficient on cylindrical surfaces $\alpha = \alpha(u_2)$. Accordingly we have

$$\alpha(u_2) = \underbrace{\left(\frac{\alpha_d}{\sqrt{r_2}} + \alpha_u \sqrt[6]{u_2 - u_{env}} \right)^2}_{=\alpha_c} + \underbrace{\epsilon\sigma(u_2^2 + u_{env}^2)(u_2 + u_{env})}_{=\alpha_r}.$$

Thus α decomposes in a convection part α_c and a radiation part α_r. Here σ and ϵ denote the Stefan–Boltzmann constant, respectively the degree of thermic emission of the insulator surface. The parameters α_d and α_u describe the dependence of the convection part on the radius r_2 and the difference in temperature, respectively.

Application of Theorem 2

We follow the Remark in Section 3 and truncate α such that (A2) is fulfilled. The scaling condition (A1) is fulfilled for the choice of $r_2 < \frac{1}{2}$. We can specify $\mathcal{T} = \frac{Id}{2} - \tilde{\mathcal{K}} + \tilde{\mathcal{S}} \circ \Phi$ for constant boundary temperatures $u = (u_1, u_2)^T \in H^{1/2}(\Gamma) \cap \mathbb{R}^2$ and for constant heat flux $\varphi = (\varphi_1, \varphi_2)^T \in H^{-1/2}(\Gamma) \cap \mathbb{R}^2$ by

$$\mathcal{K}(u,\varphi) = \begin{pmatrix} (u_1+u_2)/2 - r_2 \ln r_2 \varphi_2 \\ (u_1+u_2)/2 - r_1 \ln r_2 \varphi_1 \end{pmatrix},$$

$$\mathcal{S}(\varphi) = -\begin{pmatrix} r_1 \ln r_1 & 0 \\ 0 & r_2 \ln r_2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},$$

where $\left(\begin{smallmatrix} \varphi_1 \\ \varphi_2 \end{smallmatrix}\right) = \frac{1}{\lambda_2}\left(\begin{smallmatrix} -q(u_1) \\ \alpha(u_2)(u_2 - u_{env}) \end{smallmatrix}\right)$.

Lemma 4 [Verification of the damping property]. *Suppose that* u *satisfies* (3.5), *i.e.* $\mathcal{T}(u) = 0$ *is specified as above. Then*

$$\left|\frac{\partial}{\partial u_1}\varphi_1\right| = \frac{r_2}{r_1}\left|\frac{\partial}{\partial u_2}\varphi_2\right| \geq \left|\frac{\partial}{\partial u_2}\varphi_2\right| \geq \frac{\alpha_l}{\lambda_2}. \tag{6.3}$$

Proof. The equality in (6.3) follows by differentiation of the first and the second row of $\frac{u}{2} = \mathcal{K}(u,\varphi) - \mathcal{S}(\varphi)$ w.r.t. u_1 and u_2. Using $r_2 > r_1$, the outer boundary condition $\varphi_2 = \frac{\alpha(u_2)}{\lambda_2}(u_2 - u_{env})$ and the truncation of α in (3.6), we see the estimates. □

With the estimates for Φ in (3.7) and the damping property we can estimate the Lipschitz and the monotonicity constants of $\mathcal{T}$. This is essential for the error estimate in the iterative scheme of Theorem 2.

We use the norm $\|\cdot\|^2_{\mathcal{S}^{-1}(\Gamma)} := \|\cdot\|^2_{\mathcal{S}^{-1}(\partial B_{r_1})} + \|\cdot\|^2_{\mathcal{S}^{-1}(\partial B_{r_2})}$.

Lipschitz estimate. As $\|1\|_{\mathcal{S}^{-1}(\partial B_r)} = \sqrt{-2\pi/\ln r}$, $r < 1/2$, we have

$$\|u-v\|^2_{\mathcal{S}^{-1}(\Gamma)} = -2\pi \sum_{i=1}^{2} \frac{(u_i - v_i)^2}{\ln r_i} \geq \frac{-2\pi}{\ln r_1}|u-v|^2.$$

On the other hand, the Lipschitz continuity of Φ yields for $c_{max} = \dfrac{\max(\alpha_h, c_0)}{\lambda_2}$

$$\|\mathcal{T}(u) - \mathcal{T}(v)\|^2_{\mathcal{S}^{-1}(\Gamma)} = \left\langle \mathcal{T}(u) - \mathcal{T}(v), \mathcal{S}^{-1}(\mathcal{T}(u) - \mathcal{T}(v)) \right\rangle_{L^2(\Gamma)}$$

$$\leq -2\pi(u-v)^T \underbrace{\left(\frac{Id}{2} - \tilde{\mathcal{K}} + c_{max}\tilde{\mathcal{S}}\right)^T \begin{pmatrix} \frac{1}{\ln r_1} & 0 \\ 0 & \frac{1}{\ln r_2} \end{pmatrix} \left(\frac{Id}{2} - \tilde{\mathcal{K}} + c_{max}\tilde{\mathcal{S}}\right)}_{=:-A_L}(u-v)$$

Let λ_{max} denote the maximal eigenvalue of A_L, then there holds

$$\|\mathcal{T}(u) - \mathcal{T}(v)\|^2_{\mathcal{S}^{-1}(\Gamma)} \leq 2\pi\lambda_{max}|u-v|^2.$$

Thus we obtain the Lipschitz constant $L = \sqrt{-\lambda_{max}\ln r_1}$.

Monotonicity estimate. With the damping property the monotonicity of Φ yields for $c_{min} = \alpha_l/\lambda_2$

$$\begin{aligned}&\left\langle \mathcal{T}(u) - \mathcal{T}(v), \mathcal{S}^{-1}(u-v)\right\rangle_{L^2(\Gamma)}\\ &\quad \geq -2\pi(u-v)^T \underbrace{\left(\frac{Id}{2} - \tilde{\mathcal{K}} + c_{min}\tilde{\mathcal{S}}\right)^T \begin{pmatrix} \frac{1}{\ln r_1} & 0 \\ 0 & \frac{1}{\ln r_2}\end{pmatrix}}_{=:-A_m}(u-v).\end{aligned}$$

A_m is positive definite for every $c_{min} > 0$ and $0 < r_1 < r_2 < 1/2$. Let λ_{min} denote the minimal eigenvalue of the symmetric part of A_m then

$$\left\langle \mathcal{T}(u) - \mathcal{T}(v), \mathcal{S}^{-1}(u-v)\right\rangle_{L^2(\Gamma)} \geq 2\pi\lambda_{min}|u-v|^2.$$

Analogously we get

$$\|u-v\|^2_{\mathcal{S}^{-1}(\Gamma)} = -2\pi\sum_{i=1}^{2}\frac{(u_i-v_i)^2}{\ln r_i} \leq \frac{-2\pi}{\ln r_2}|u-v|^2.$$

Hence by Lemma 3 we arrive at the monotonicity constant $m = -\lambda_{min}\ln r_2$.

7 Conclusions

We present an application to physical data. Let us fix some physical data with

Temperatures: $u_0 = 20,\ u_{env} = 50$

Conductor parameters: $\lambda_1 = 400,\ \rho_0 = 1.72\cdot 10^{-8},\ \alpha_\rho = 3.83\cdot 10^{-3}$

Insulator parameters: $\lambda_2 = 0.17,\ \epsilon = 0.93,\ r_1 = 7\cdot 10^{-4},\ r_2 = 1\cdot 10^{-3}$.

For the given material and geometric parameters (in SI units) we firstly evaluate the Helmholtz to Poisson estimate provided that $\varsigma < \Lambda^2$, i.e. $I < 8500$. Moreover, for applicational reasons, we fix an upper bound for the conductor boundary temperature with $u_{1max} = 130$ and obtain by (6.2)

$$\|u_\varsigma - u\|_{H^1_0(B_{r_1})} \leq f(u_{max})\frac{3,5\cdot 10^{-11}\, I^2}{5\cdot 10^6 - 0,1\, I^2} \leq \frac{3,7\cdot 10^{-7}\, I^4}{5\cdot 10^6 - 0,1\, I^2} \overset{I=20}{=} 1,18\cdot 10^{-8}.$$

In view of this estimate we fix the values $u_l = u_{env} = 50$, $u_h = u_{1max} = 130$ and consequently $\alpha_l = 12,2$, $\alpha_h = 30,8$ for the truncation of α. Considering the case $I \leq 30$, we obtain the $\mathcal{S}^{-1}$-Lipschitz- and the $\mathcal{S}^{-1}$-monotonicity constant of $\mathcal{T} = \frac{Id}{2} - \tilde{\mathcal{K}} + \tilde{\mathcal{S}}\circ\Phi$ (estimated above for the case of rot. symmetry) with $L = 1,71$ and $m = 0,34$.

Thus the fixed point mapping $\mathcal{G}_\gamma$ of Theorem 2 is given by $\gamma := \frac{m}{L^2} = 0.117$ and is contractive with $k = 0.9797$. For $u^{(1)} \equiv u_{env}$ the a priori error estimate of the corresponding iteration reads for $n \geq 800$ as

$$\begin{aligned}\|u^{(n)} - u\|_{\mathcal{S}^{-1}} &\leq \frac{k^n}{1-k}\,\|\mathcal{G}_\gamma(u_{env}) - u_{env}\|_{\mathcal{S}^{-1}}\\ &\leq \frac{\sqrt{2}\gamma k^n}{1-k}\,\|1\|_{S^{-1}(\partial B_{r_2})}\sqrt{\left|-u_{env}/2 + r_1\ln r_1\frac{q(u_{env})}{\lambda_2}\right|} \leq 5,9\cdot 10^{-6}.\end{aligned}$$

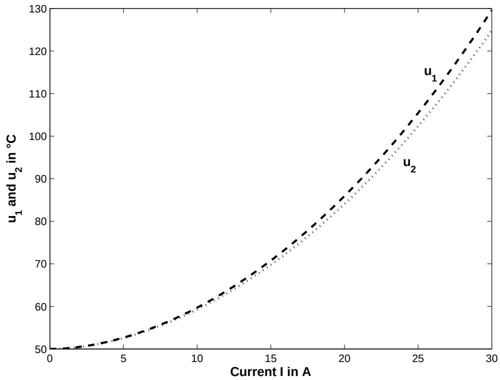


Figure 1. Iteration sequence $\mathcal{G}_\gamma(u^{(n)})$.

We iterate the sequence $\mathcal{G}_\gamma(u^{(n)})$ and the computational results are shown in Fig. 1. A very good agreement between these calculated temperatures and experimental results is obtained. Our fixed point approach can be applied to non-symmetric domains and result in numerical methods, provided the fixed point iteration is combined with numerical quadrature for the occurring singular integrals. We refer to [1, 12, 13, 32].

On the other hand the approximative error of the heat flux $\tilde{q}$ from Section 2 should be estimated with respect to the considered non-symmetric geometries. Note that the damping property of domains introduced in Section 5, is essential for obtaining a numerically acceptable monotonicity constant in Theorem 2, especially for low currents. This property can be seen as a natural property of the insulator, i.e. of harmonic functions w.r.t. the considered boundary conditions. However, it has to be verified separately in applications, which motivates a study of this property for general situations.

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