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# Investigation of Characteristic Curve for Sturm–Liouville Problem with Nonlocal Boundary Conditions on Torus

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**Abstract.** In this paper, we investigate the second-order Sturm–Liouville problem with two additional Nonlocal Boundary Conditions. Nonlocal boundary conditions depends on two parameters. We find condition for existence of zero eigenvalue in the parameters space and classified Characteristic Curves in the plane and extended plane is described as torus. The Characteristic Curve on torus may be of three types only. Some new conclusions about existence and uniqueness domain of solution are presented.

**Keywords:** Sturm–Liouville problem, Nonlocal Boundary Conditions, Torus.

**AMS Subject Classification:** 34B24; 34B10.

## 1 Introduction

During the last decade there has been a growing interest in the investigation various problems with nonlocal boundary conditions and numerical methods for them. Problems with nonlocal integral conditions were investigated by Cannon [2, 1963] and Kamynin [17, 1964], Bitsadze and Samarskii have investigated elliptic boundary value problem [1, 1969]. Il'in obtained necessary and sufficient conditions for the existence of a subsystem of eigenfunctions and adjoint functions (as a basis) for Keldysh's bundle of ordinary differential operators [11].

Initially, these conditions were nameless, but later called nonclassical. For the first time the concept of nonlocal boundary conditions probably was used in [15, 1977], where one-dimensional parabolic equation with integral condition

$$\int_0^1 u(x,t)\mathrm{d}x = \mu(t) \tag{1.1}$$

was considered. This term is used already in Samarskii and Nikolaev book [22, 1978 (Russian Ed.)]. Il'in and Moiseev [13, 14], Sapagovas and Čiegis [24] investigated the existence and uniqueness for second order ordinary differential equations with various NBC. Survey of other results in theory and applications for problems with NBC is presented in [9, 10, 12, 18].

The investigation of existence and properties of solution for stationary problems with NBC is connected with Sturm–Liouville problem with NBC [27] and Green's function properties for stationary problems with NBC [19, 28]. Properties of spectrum and particularly conditions when zero eigenvalue exists are useful for investigation multi-dimensional and non-stationary problems or stability numerical methods and convergence of iterative methods [3, 16, 23, 25].

In paper [6] the second order linear stationary equation with nonlocal boundary conditions was investigated

$$lu := -\frac{\mathrm{d}}{\mathrm{d}x}\Big(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\Big) + q(x)u = f,\ x \in (0,1), \tag{1.2}$$

$$u(0) = \gamma_0\langle k_0, u\rangle + f_0, \quad \gamma_0 \geqslant 0, \tag{1.3}$$

$$u(1) = \gamma_1\langle k_1, u\rangle + f_1, \quad \gamma_1 \geqslant 0, \tag{1.4}$$

where $k_0$ and $k_1$ are linear functionals:

$$\langle k_0, u\rangle := \alpha_0 u(a_0) + \int_0^1 \beta_0(x)u(x)\,\mathrm{d}x, \tag{1.5}$$

$$\langle k_1, u\rangle := \alpha_1 u(a_1) + \int_0^1 \beta_1(x)u(x)\,\mathrm{d}x. \tag{1.6}$$

and $u \in C^2(0,1) \cap C^1[0,1]$ for $0 < p_0 \leqslant p \in C^1[0,1]$, $0 \leqslant q \in C[0,1]$ and weights $\beta_i \in L_1(0,1)$. The existence, uniqueness and stability of the solution for this problem were proved.

In the domain $[0,1]$ we introduce meshes

$$\bar\omega^h = \{0 = x_0 < x_1 < \cdots < x_n = 1\}, \quad \omega^h = \bar\omega^h \setminus \{x_0, x_n\}$$

with step sizes $h_i = x_i - x_{i-1}, 1 \le i \le n$, $h_0 = h_{n+1} = 0$ and semi-integer mesh

$$\omega^h_{1/2} = \{x_{i+\frac12} | x_{i+\frac12} = (x_i + x_{i+1})/2,\ 0 \le i \le n-1\}$$

with step sizes $h_{i+\frac12} = (h_i + h_{i+1})/2$, $0 \le i \le n$. A mesh $\omega$ will denote one of the meshes $\bar\omega^h$, $\omega^h$, $\bar\omega^h_{1/2}$. We denote $h = \max\limits_{1\le i\le n} h_i$. Let $\mathcal{F}(\omega)$ be a space of real valued functions which are defined on the mesh $\omega$ and $\mathcal{F}[0,1]$ is a space of real valued functions on $[0,1]$.

We define the following mesh operators

$$(\delta Z)_{i+\frac12} = \frac{Z_{i+1} - Z_i}{h_{i+1}},\ Z \in \mathcal{F}(\bar\omega^h), \quad (\delta Z)_i = \frac{Z_{i+\frac12} - Z_{i-\frac12}}{h_{i+\frac12}},\ Z \in \mathcal{F}(\omega^h_{1/2}).$$

We can use the same notation for continuous and discrete norms, inner products and linear functionals, e.g.:

$$(u,v) := \int_0^1 u(x)v(x)\,\mathrm{d}x, \quad (U,V) := \sum_{i=0}^{n} U_i V_i h_{i+\frac12},$$

where $u, v \in \mathcal{F}[0,1]$, $U, V \in \mathcal{F}(\bar{\omega}^h)$.

In papers [4, 6] analogous results for finite-difference scheme

$$LU := -\delta(P\delta U) + QU = F \quad \text{in } \omega^h, \tag{1.7}$$

$$U|_{i=0} = \gamma_0\langle K_0, U\rangle + f_0, \quad \gamma_0 \geqslant 0, \tag{1.8}$$

$$U|_{i=n} = \gamma_1\langle K_1, U\rangle + f_1, \quad \gamma_1 \geqslant 0 \tag{1.9}$$

were established. In discrete case linear functionals are defined by

$$\langle K_0, U\rangle := \alpha_0\widetilde{U}(a_0) + (B_0, U), \tag{1.10}$$

$$\langle K_1, U\rangle := \alpha_1\widetilde{U}(a_1) + (B_1, U), \tag{1.11}$$

where $U, B_0, B_1 \in \mathcal{F}(\bar{\omega}^h)$, $0 < p_0 \le P \in \mathcal{F}(\omega^h_{1/2})$, $0 \le Q \in \mathcal{F}(\omega^h)$, $F \in \mathcal{F}(\omega^h)$ and $\widetilde{Z}$ be a linear interpolant

$$\widetilde{Z}(x) = \frac{x_i - x}{h_i} Z_{i-1} + \frac{x - x_{i-1}}{h_i} Z_i \quad \text{for} \quad x \in [x_{i-1}, x_i].$$

In these two papers [4, 6] the dependence of solution on parameters $\gamma_0$ and $\gamma_1$ was investigated. The main result of these papers is that the solution for differential problem with NBC (1.2)–(1.4) or solution for FDS with NBC (1.7)–(1.9) exists and is unique for all $\gamma_0 \geqslant 0$ and $\gamma_1 \geqslant 0$ except points on hyperbola or line(s). So, sufficient and necessary conditions for existence of unique solution for stationary problem were found. In [5] the one dimensional parabolic equation with three types of integral nonlocal boundary conditions is approximated by the implicit Euler finite difference scheme. Stability analysis is done in the maximum norm and it is proved that the radius of the stability region depends on the signs of coefficients in the nonlocal boundary condition.

The analysis of characteristic function for stationary problem with one classical boundary condition ($\gamma_0 = 0$) and another NBC [26, 27, 29] and investigation of auxiliary stationary problems [16, 21, 23, 25] show that restrictions $\gamma_0 \geqslant 0$ and $\gamma_1 \geqslant 0$ are not necessary, and in general case we can take $\gamma_0, \gamma_1 \in \bar{\mathbb{C}}$. In this paper we assume that $\gamma_0, \gamma_1 \in \bar{\mathbb{R}}$.

The structure of the paper is as follows. We introduce some notations in Section 2. In Section 3 we investigate second–order linear equation with two additional functional conditions. Then we reformulate results of paper [6] for problem (1.2)–(1.3) (problem (1.7)–(1.9)) in the case $\gamma_0, \gamma_1 \in \mathbb{R}$ in Section 4. We investigate Characteristic Curve for equation (1.2) with NBC on torus in Section 5 and formulate main result of this article in Lemma 3. Finally, we give some conclusions.

## 2 Notation

Now we introduce a few notations related to linear functionals. Let $F(X) := \{u \mid w\colon X \to \mathbb{R}\}$ be a linear space of real functions, where $X$ can be any set. If we have vector-function $\boldsymbol{w} = [w_1, \ldots, w_n] \in F^n(X) := \prod_{i=1}^{n} F(X)$, and

$\boldsymbol{x} = (x_1, \dots, x_n) \in X^n$, then we consider matrix-function $[\boldsymbol{w}]\colon X^n \to M_{n\times n}(\mathbb{R})$ and its functional determinant $\det[\boldsymbol{w}]\colon X^n \to \mathbb{R}$:

$$[\boldsymbol{w}](\boldsymbol{x}) = [w_1, \dots, w_n](x_1, \dots, x_n) := \begin{pmatrix} w_1(x_1) & \dots & w_1(x_n) \\ \dots & \dots & \dots \\ w_n(x_1) & \dots & w_n(x_n) \end{pmatrix},$$

$$D[\boldsymbol{w}](\boldsymbol{x}) = \det[\boldsymbol{w}](\boldsymbol{x}) = \det[w_1, \dots, w_n](x_1, \dots, x_n) := \begin{vmatrix} w_1(x_1) & \dots & w_1(x_n) \\ \dots & \dots & \dots \\ w_n(x_1) & \dots & w_n(x_n) \end{vmatrix}.$$

We consider the space $F^*(X)$ of linear functionals in the space $F(X)$, and we use the notations $\langle f, w\rangle$, $\langle f(\cdot), w(\cdot)\rangle$, $\langle f(x), w(x)\rangle$ for the functional $f$ value of the function $w$. For example, if $f$ is a regular functional, then $\langle f, w\rangle = \int_0^l f(x)w(x)\,\mathrm{d}x$; $\delta_x$ is a functional if $\langle \delta_x, w\rangle = w(x)$. If $f \in F^*(X)$, $g \in F^*(Y)$, then we can define the linear functional (direct product) $f \cdot g \in F^*(X \times Y)$:

$$\langle f(x) \cdot g(y), w(x, y)\rangle := \langle f(x), \langle g(y), w(x, y)\rangle\rangle, \quad w(x, y) \in F(X \times Y).$$

Analogously, we can define the direct product for $\boldsymbol{f} = (f_1, \dots, f_n)$, $f_i \in F^*(X_i)$

$$\dot{\boldsymbol{f}} = f_1 \cdot f_2 \cdot \ldots \cdot f_n = f_1 \cdot (f_2 \cdot \ldots \cdot f_n) \in F^*(\prod_{i=1}^{n} X_i).$$

We define a matrix

$$M(\boldsymbol{f})[\boldsymbol{w}] := \begin{pmatrix} \langle f_1, w_1\rangle & \dots & \langle f_n, w_1\rangle \\ \dots & \dots & \dots \\ \langle f_1, w_n\rangle & \dots & \langle f_n, w_n\rangle \end{pmatrix}$$

for $\boldsymbol{f} = (f_1, \dots, f_n)$, $\boldsymbol{w} = [w_1, \dots, w_n]$, and determinant

$$D(\boldsymbol{f})[\boldsymbol{w}] := \langle \dot{\boldsymbol{f}}(\boldsymbol{x}), D[\boldsymbol{w}](\boldsymbol{x})\rangle = \begin{vmatrix} \langle f_1, w_1\rangle & \dots & \langle f_n, w_1\rangle \\ \dots & \dots & \dots \\ \langle f_1, w_n\rangle & \dots & \langle f_n, w_n\rangle \end{vmatrix} = \det M(\boldsymbol{f})[\boldsymbol{w}].$$

For example, $D(\boldsymbol{\delta_x})[\boldsymbol{w}] = \langle \dot{\boldsymbol{\delta}}_{\boldsymbol{x}}(\boldsymbol{y}), D[\boldsymbol{w}](\boldsymbol{y})\rangle = D[\boldsymbol{w}](\boldsymbol{x})$.

## 3 Problem with Two Additional Conditions

In papers [4, 6] the maximum principle was used for investigation of stability. So, the condition $q \geqslant 0$ was proposed. In this paper we require $q \in C[0, l]$ only.

We consider the second-order ordinary linear differential equation (LDE)

$$\mathcal{L}u := u'' + r(x)u' + \bar{q}(x)u = \bar{f}(x), \tag{3.1}$$

where $r, q, f \in C[0, l]$, too. This equation can be rewritten in self-adjoint form (1.2) [8], where $p(x) = \exp(\int_0^x r(t)\,\mathrm{d}t) \geq p_0 > 0$, $p \in C^1[0, l]$, $q = -\bar{q}p \in C[0, l]$, $f = -\bar{f}p \in C[0, l]$. But in this paper we do not distinguish these two forms.

General solution for equation (3.1) has a form $u = c_0u_0 + c_1u_1 + u_f$, where $\{u_0, u_1\}$ is a fundamental system of the corresponding homogeneous equation

$$u'' + r(x)u' + \bar{q}(x)u = 0, \tag{3.2}$$

and $u_f$ is a particular solution of the nonhomogeneous equation (3.1) and $c_0, c_1$ are constants of integration. Function $\bar{f}$ on the right hand side influences only properties of solution $u_f$.

If $r, q \in C[0, l]$ then fundamental system $\{u_0, u_1\}$ exists [8], i.e. we have linearly independent solutions $u_0, u_1 \in C^2[0, l]$. On the other hand, if we know fundamental system $\{u_0, u_1\}$ of the homogeneous equation (3.2) then Wronskian

$$W[u_0, u_1](x) := \begin{vmatrix} u_0(x) & u_1(x) \\ u_0'(x) & u_1'(x) \end{vmatrix} \neq 0 \text{ for } x \in [0, l].$$

Let us write equation

$$\begin{vmatrix} u_0 & u_1 & u \\ u_0' & u_1' & u' \\ u_0'' & u_1'' & u'' \end{vmatrix} = W[u_0, u_1]\, u'' - \begin{vmatrix} u_0 & u_1 \\ u_0'' & u_1'' \end{vmatrix} u' + \begin{vmatrix} u_0' & u_1' \\ u_0'' & u_1'' \end{vmatrix} u = 0. \tag{3.3}$$

This equation is equivalent to differential equation (3.2) with

$$r[u_0, u_1] = - \begin{vmatrix} u_0 & u_1 \\ u_0'' & u_1'' \end{vmatrix} / W[u_0, u_1], \quad \bar{q}[u_0, u_1] = \begin{vmatrix} u_0' & u_1' \\ u_0'' & u_1'' \end{vmatrix} / W[u_0, u_1],$$

and $r, q \in C[0, l]$. If $\{\bar{u}_0, \bar{u}_1\}$ is another fundamental system, and $[\bar{u}_0, \bar{u}_1] = \mathbf{P}[u_0, u_1]$, where $\mathbf{P} \in GL_2(\mathbb{R})$, then $r[\bar{u}_0, \bar{u}_1] = r[u_0, u_1]$, $\bar{q}[\bar{u}_0, \bar{u}_1] = \bar{q}[u_0, u_1]$.

Finally, the coefficients $r, \bar{q} \in C[0, l]$ define all the solutions, i.e., the two dimensional linear space $S := \{u \in C^2[0, l] : \mathcal{L}u = 0\}$, which can be described by the fundamental system $\{u_0, u_1\}$, and conversely, the two dimensional linear subspace in $C^2[0, l]$ fully determines $r, \bar{q} \in C[0, 1]$. So, properties of fundamental system determine properties of coefficients, and conversely.

Suppose, we know the fundamental system $\{u_0, u_1\}$ for the homogeneous differential equation (3.2). Then general solution of this equations has form

$$u(x) = c_0u_0(x) + c_1u_1(x). \tag{3.4}$$

In paper [6] functions $u_0$ and $u_1$ were defined as solutions with classical boundary conditions:

$$a) \begin{cases} u_0(0) = 1, \\ u_0(1) = 0; \end{cases} \qquad b) \begin{cases} u_1(0) = 0, \\ u_1(1) = 1. \end{cases} \tag{3.5}$$

Such solutions may not exist for non positive $q(x)$. So, in this article we use fundamental system defined by initial conditions:

$$a) \begin{cases} u_0(0) = 1, \\ u_0'(0) = 0; \end{cases} \qquad b) \begin{cases} u_1(0) = 0, \\ u_1'(0) = 1. \end{cases} \tag{3.6}$$

The unique solution for problems (3.1),(3.5) and (3.1),(3.6) exists if homogeneous problems

$$\begin{cases} \mathcal{L}u = 0, \\ u(0) = 0, \quad u(1) = 0; \end{cases} \qquad \begin{cases} \mathcal{L}u = 0, \\ u(0) = 0, \quad u'(0) = 0 \end{cases} \tag{3.7}$$

have unique solution, respectively. The second problem always has the unique solution, but the first one may have non unique solution. These two examples of boundary conditions show that investigation of such homogeneous problems with homogeneous boundary conditions is important task for investigation of nonhomogeneous problem, too.

In general case $\{u_0, u_1\}$ can be any fundamental system. We have two unknown constants of integration $c_0$, $c_1$. So, we use two additional conditions:

$$\langle L_0, u\rangle = 0, \quad \langle L_1, u\rangle = 0, \tag{3.8}$$

where $L_0$, $L_1 \in S^*$ are linearly independent linear functionals and $\boldsymbol{L} = (L_0, L_1)$. We introduce new functions

$$v_0(x) := \frac{D(\delta_x, L_1)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]}, \quad v_1(x) := \frac{D(L_0, \delta_x)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]}. \tag{3.9}$$

These two solutions are well-defined, because the following lemma is valid [28].

**Lemma 1.** *Let $\{u_0, u_1\}$ be the basis of the linear space $S$. Then the following are equivalent:*

1. *Functionals $L_0$ and $L_1$ are linearly independent;*
2. *Functions $v_0(x)$ and $v_1(x)$ are linearly independent;*
3. *$D(\boldsymbol{L})[\boldsymbol{u}] \neq 0$.*

The two basis $\{v_0, v_1\}$ and $\{L_0, L_1\}$ are biorthogonal:

$$\langle L_i, v_j\rangle = \delta_{ij}, \quad i, j = 0, 1, \tag{3.10}$$

and Wronskian (see, [28]) is equal to

$$W[\boldsymbol{v}](x) = \frac{W[\boldsymbol{u}](x)}{D(\boldsymbol{L})[\boldsymbol{u}]}. \tag{3.11}$$

*Remark 1.* If $\{\bar{u}_0, \bar{u}_1\}$ is another fundamental system, and $[\bar{u}_0, \bar{u}_1] = \mathbf{P}[u_0, u_1]$, where $\mathbf{P} \in GL_2(\mathbb{R})$, then

$$\frac{D(\delta_x, L_1)[\bar{\boldsymbol{u}}]}{D(\boldsymbol{L})[\bar{\boldsymbol{u}}]} = \frac{D(\delta_x, L_1)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]}, \quad \frac{D(L_0, \delta_x)[\bar{\boldsymbol{u}}]}{D(\boldsymbol{L})[\bar{\boldsymbol{u}}]} = \frac{D(L_0, \delta_x)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]}.$$

So, definition of $\boldsymbol{v}(x) := [v_0(x), v_1(x)]$ is invariant with regard to basis $\{u_0, u_1\}$.

*Corollary 1.* If $D(\boldsymbol{L})[\boldsymbol{u}] \neq 0$, then homogeneous problem (3.2) with two additional conditions (3.8) has only trivial solution.

So, in this case zero is not eigenvalue for problem (3.2) with additional conditions (3.8) and we can write solution of nonhomogeneous problems using Green's function [20, 28].

*Remark 2.* The case $D(\boldsymbol{L})[\boldsymbol{u}] = 0$ will be realized if one of these functionals can be expressed by the other one. So, in this case we have only one functional equation, for example, $\langle L_0, u\rangle = 0$. If $\operatorname{Ker} L_0 = S$ ($\dim \operatorname{Ker} L_0 = 2$), then we have a trivial case $L_0 = 0$, i.e. $u = c_0 u_1 + c_1 u_1$ is solution for all $c_0$, $c_1$. In the case $\operatorname{Ker} L_0 \neq S$ ($\dim \operatorname{Ker} L_0 = 1$) we have a nontrivial solution $\tilde{u}(x)$ and all solutions are $u = c\tilde{u}(x)$.

*Example 1.* Let us consider differential equation ($\lambda$ is parameter)

$$u'' + \lambda u = 0$$

with two types of conditions which correspond to boundary conditions (3.5) and (3.6):

$$\bar{L}_0 = \delta(x), \quad \bar{L}_1 = \delta(x-1), \tag{3.12}$$

$$L_0 = \delta(x), \quad L_1 = -\delta'(x), \tag{3.13}$$

where $\langle \delta(x), u\rangle = u(0)$, $\langle \delta(x-1), u\rangle = u(1)$, $\langle \delta'(x), u\rangle = -u'(0)$, and denote $q = \sqrt{\lambda}$ for $\lambda \geqslant 0$, $q = \sqrt{-\lambda}$ for $\lambda < 0$. Fundamental system

$$\{\bar{u}_0, \bar{u}_1\} = \begin{cases} \left\{\frac{\sin((1-x)q)}{\sin q}, \frac{\sin(xq)}{\sin q}\right\} & \text{for } \lambda > 0,\ \lambda \neq \pi^2 k^2,\ k \in \mathbb{N}, \\ \{1-x, x\} & \text{for } \lambda = 0, \\ \left\{\frac{\sinh((1-x)q)}{\sinh q}, \frac{\sinh(xq)}{\sinh q}\right\} & \text{for } \lambda < 0 \end{cases}$$

corresponds to functionals $\bar{L}_0, \bar{L}_1$, and fundamental system

$$\{u_0, u_1\} = \begin{cases} \left\{\frac{\sinh((1-x)q)}{\sinh q}, \frac{\sinh(xq)}{\sinh q}\right\} & \text{for } \lambda > 0, \\ \{1, x\} & \text{for } \lambda = 0, \\ \left\{\frac{\sinh((1-x)q)}{\sinh q}, \frac{\sinh(xq)}{\sinh q}\right\} & \text{for } \lambda < 0 \end{cases}$$

corresponds to functionals $L_0, L_1$. We have

$$D(\bar{\boldsymbol{L}})[\boldsymbol{u}] = \begin{cases} \sin q/q & \text{for } \lambda > 0, \\ 1 & \text{for } \lambda = 0, \\ \sinh q/q & \text{for } \lambda < 0; \end{cases} \qquad D(\boldsymbol{L})[\boldsymbol{u}] = W[\boldsymbol{u}](0) = 1$$

for (3.12) and (3.13), respectively. If $\lambda = \pi^2 k^2$, $k \in \mathbb{N}$, then $D(\bar{\boldsymbol{L}})[\boldsymbol{u}] = 0$. In this case the first problem (3.7) has nontrivial solutions (i.e., zero eigenvalue exists) and fundamental system do not exists.

If $\lambda \neq \pi^2 k^2$, $k \in \mathbb{N}$ then formulae (3.9) give relations between two fundamental systems

$$\bar{u}_0(x) = \frac{D(\delta_x, \bar{L}_1)[\boldsymbol{u}]}{D(\bar{\boldsymbol{L}})[\boldsymbol{u}]}, \quad \bar{u}_1(x) = \frac{D(\bar{L}_0, \delta_x)[\boldsymbol{u}]}{D(\bar{\boldsymbol{L}})[\boldsymbol{u}]}$$

and
$$u_0(x)=\frac{D(\delta_x,L_1)[\bar{\boldsymbol{u}}]}{D(\boldsymbol{L})[\bar{\boldsymbol{u}}]},\quad u_1(x)=\frac{D(L_0,\delta_x)[\bar{\boldsymbol{u}}]}{D(\boldsymbol{L})[\bar{\boldsymbol{u}}]}.$$
So, there is no difference which fundamental system is used.

If $\lambda=\pi^2k^2$, $k\in\mathbb{N}$ then linear functionals $\bar{L}_0$, $\bar{L}_1$ are not linearly independent. In this case fundamental system exists for the case of functionals (3.13), but there is no fundamental system for the case of the functionals (3.12).

## 4 Sturm–Liouville Problems with NBC

The main aim of this paper is to investigate existence condition for zero eigenvalue for problems with two nonlocal boundary conditions. So, we formulate Sturm–Liouville problem with two nonlocal boundary conditions (as functional condition):

$$\mathcal{L}u:=-(p(x)u')'+q(x)u=\lambda u, \tag{4.1}$$
$$\langle k_0,u\rangle=\gamma_0\langle n_0,u\rangle,\quad \gamma_0\in\mathbb{R}, \tag{4.2}$$
$$\langle k_1,u\rangle=\gamma_1\langle n_1,u\rangle,\quad \gamma_1\in\mathbb{R}, \tag{4.3}$$

where $p(x)\geq p_0>0$, $p\in C^1[0,1]$, $q\in C[0,1]$. We can write many problems with NBC in this form, where $\langle k_i,u\rangle:=\langle k_i(x),u(x)\rangle$ is a classical part, and $\langle n_i,u\rangle:=\langle n_i(x),u(x)\rangle$, $i=0,1$, is a nonlocal part of boundary conditions. For example, the functionals $n_i$, $i=0,1$, can describe the multi-point ($\xi_j\in[0,1]$, $j=1,\dots,m$) or integral NBCs

$$\langle n,u(t)\rangle=\sum_{j=1}^{m}\bigl(\varkappa_j u(\xi_j)+\kappa_j u'(\xi_j)\bigr),\quad \langle n,u(t)\rangle=\int_0^1\varkappa(t)u(t)\,\mathrm{d}t,$$

and the functional $k_i$, $i=0,1$, can describe the local (classical) boundary conditions

$$\langle k_0,u(t)\rangle=\alpha_0u(0)+\beta_0u'(0),\quad \langle k_1,u(t)\rangle=\alpha_1u(1)+\beta_1u'(1),$$

where the parameters $|\alpha_i|+|\beta_i|>0$, $i=0,1$. If $\gamma_0,\gamma_1=0$, then problem (4.1)–(4.3) becomes classical.

*Remark 3.* Equation (4.1) can be rewritten in the form

$$-(p(x)u')'+(q(x)-\lambda)u=0. \tag{4.4}$$

So, we can restrict to investigation only of the case $\lambda=0$.

If we take $L_i=\gamma_in_i-k_i$, $i=0,1$, then we can rewrite the condition $D(\boldsymbol{L})[\boldsymbol{u}]=0$ as

$$D(n_0,n_1)[\boldsymbol{u}]\gamma_0\gamma_1-D(n_0,k_1)[\boldsymbol{u}]\gamma_0-D(k_0,n_1)[\boldsymbol{u}]\gamma_1+D(k_0,k_1)[\boldsymbol{u}]=0. \tag{4.5}$$

We call the solution of equation (4.5) a *Characteristic Curve* for problem (4.1)–(4.3) and denote a set of it's points in plane $\mathbb{R}^2_{\gamma_0,\gamma_1}$ by the letter $\mathcal{C}$.

*Remark 4.* If $\{\bar{u}_0, \bar{u}_1\}$ is another fundamental system, and $[\bar{u}_0, \bar{u}_1] = \mathbf{P}[u_0, u_1]$, where $\mathbf{P} \in GL_2(\mathbb{R})$, then

$$D(n_0,n_1)[\bar{\boldsymbol{u}}] = D(n_0,n_1)[\boldsymbol{u}]\det\mathbf{P}, \quad D(n_0,k_1)[\bar{\boldsymbol{u}}] = D(n_0,k_1)[\boldsymbol{u}]\det\mathbf{P},$$
$$D(k_0,n_1)[\bar{\boldsymbol{u}}] = D(k_0,n_1)[\boldsymbol{u}]\det\mathbf{P}, \quad D(k_0,k_1)[\bar{\boldsymbol{u}}] = D(k_0,k_1)[\boldsymbol{u}]\det\mathbf{P}.$$

Thus, equation (4.5) (and equations (4.8)–(4.10)) is invariant with regard to fundamental system $\{u_0, u_1\}$ and notation "Characteristic Curve" corresponds to this point of view.

The plane algebraic curve of the second degree (conic section)

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{4.6}$$

may be classified by the discriminant $B^2 - 4AC$. If the conic is non-degenerate, then equation (4.6):

1. if $B^2 - 4AC < 0$, represents an ellipse;
2. if $B^2 - 4AC = 0$, represents a parabola;
3. if $B^2 - 4AC > 0$, represents a hyperbola.

There are seven degenerate cases: two parallel lines (corresponding to an ellipse with one axis infinite and the other axis real and non-zero, the distance between the lines); a point (corresponding to degeneration of an ellipse); a pair of intersecting lines (corresponding to degeneration of an hyperbola); a straight line (a line with multiplicity 2, corresponding to degeneration of a parabola or two parallel lines); empty set ($A = B = C = D = E = 0,\ F \neq 0$); a plane ($A = B = C = D = E = F = 0$); a line ($A = B = C = 0,\ D \neq 0$ or $E \neq 0$).

It is easy to prove (see, [28]):

$$\begin{vmatrix} D(n_0,n_1)[\boldsymbol{u}] & D(n_0,k_1)[\boldsymbol{u}] \\ D(k_0,n_1)[\boldsymbol{u}] & D(k_0,k_1)[\boldsymbol{u}] \end{vmatrix} = D(n_0,k_0)[\boldsymbol{u}]D(n_1,k_1)[\boldsymbol{u}]. \tag{4.7}$$

So, we can rewrite the equation (4.5):

$$\Big(\gamma_0 - \tfrac{D(n_0,k_1)}{D(n_0,n_1)}\Big)\Big(\gamma_1 - \tfrac{D(k_0,n_1)}{D(n_0,n_1)}\Big) = \tfrac{D(k_0,n_0)}{D(n_0,n_1)}\,\tfrac{D(n_1,k_1)}{D(n_0,n_1)},\ \text{if } D(n_0,n_1) \neq 0; \tag{4.8}$$

$$\tfrac{D(n_0,k_1)}{D(k_0,k_1)}\gamma_0 + \tfrac{D(k_0,n_1)}{D(k_0,k_1)}\gamma_1 = 1,\ \text{if } D(n_0,n_1) = 0,\ D(k_0,k_1) \neq 0; \tag{4.9}$$

$$D(n_0,k_1)\gamma_0 + D(k_0,n_1)\gamma_1 = 0,\ \text{if } D(k_0,k_1) = D(n_0,n_1) = 0. \tag{4.10}$$

*Corollary 2.* Problem (4.1)–(4.3) in the classical case ($\gamma_0 = 0$ and $\gamma_1 = 0$) has zero eigenvalue if and only if $D(k_0, k_1) = 0$.

*Corollary 3.* If $D(n_0, n_1) \neq 0$ and $D(k_0, n_0)D(n_1, k_1) \neq 0$ then $\lambda = 0$ is eigenvalue of problem (4.1)–(4.3) if and only if the point $(\gamma_0, \gamma_1)$ lies on hyperbola (4.8). The hyperbola has vertical and horizontal asymptotes: $\gamma_0 = \frac{D(n_0,k_1)}{D(n_0,n_1)}$, $\gamma_1 = \frac{D(k_0,n_1)}{D(n_0,n_1)}$.

**Table 1.** Classification of the Characteristic Curves.

| Case | curve in plane | $\mathbb{R}^2$ | matrix $\mathbf{A}$ | $\mathbb{R}^2_1$ | $\mathbb{R}^2_2$ | $\mathbb{R}^2_3$ | Case curve on $\mathbb{T}^2$ | |
|---|---|---|---|---|---|---|---|---|
| 1 | whole plane | 1 | $\mathbf{A}=\mathbf{O}$ | 1 | 1 | 1 | 3 | whole torus |
| 2 | empty set | 2 | $\begin{pmatrix}0 & 0\\ 0 & a_{11}\end{pmatrix}$ | 4a | 3b | 3a | 2 | two circles |
| 3 | line | 3a | $\begin{pmatrix}0 & a_{01}\\ 0 & 0\end{pmatrix}$ | 3b | 4a | 2 | 2 | two circles |
| | | 3b | $\begin{pmatrix}0 & 0\\ a_{10} & 0\end{pmatrix}$ | 3a | 2 | 4a | | |
| | | 3c | $\begin{pmatrix}0 & a_{01}\\ 0 & a_{11}\end{pmatrix}$ | 4c | 4c | 3c | | |
| | | 3d | $\begin{pmatrix}0 & 0\\ a_{10} & a_{11}\end{pmatrix}$ | 4b | 3d | 4b | | |
| | | 3e | $\begin{pmatrix}0 & a_{01}\\ a_{10} & 0\end{pmatrix}$ | 3e | 5b | 5b | 1 | circle |
| | | 3f | $\begin{pmatrix}0 & a_{01}\\ a_{10} & a_{11}\end{pmatrix}$ | 5e | 5d | 5c | | |
| 4 | two lines | 4a | $\begin{pmatrix}a_{00} & 0\\ 0 & 0\end{pmatrix}$ | 2 | 3a | 3b | 2 | two circles |
| | | 4b | $\begin{pmatrix}a_{00} & a_{01}\\ 0 & 0\end{pmatrix}$ | 3d | 4b | 3d | | |
| | | 4c | $\begin{pmatrix}a_{00} & 0\\ a_{10} & 0\end{pmatrix}$ | 3c | 3c | 4c | | |
| | | 4d | $\begin{pmatrix}a_{00} & a_{01}\\ a_{10} & a_{11}\end{pmatrix}$, $\det\mathbf{A}=0$ | 4d | 4d | 4d | | |
| 5 | hyperbola | 5a | $\begin{pmatrix}a_{00} & a_{01}\\ a_{10} & a_{11}\end{pmatrix}$, $\det\mathbf{A}\neq 0$, | 5a | 5a | 5a | 1 | circle |
| | | 5b | $\begin{pmatrix}a_{00} & 0\\ 0 & a_{11}\end{pmatrix}$ | 5b | 3e | 3e | | |
| | | 5c | $\begin{pmatrix}a_{00} & a_{01}\\ 0 & a_{11}\end{pmatrix}$ | 5d | 5e | 3f | | |
| | | 5d | $\begin{pmatrix}a_{00} & 0\\ a_{10} & a_{11}\end{pmatrix}$ | 5c | 3f | 5e | | |
| | | 5e | $\begin{pmatrix}a_{00} & a_{01}\\ a_{10} & 0\end{pmatrix}$ | 3f | 5c | 5d | | |

*Corollary 4.* If $D(n_0,n_1)\neq 0$ and $D(k_0,n_0)D(n_1,k_1)=0$ then $\lambda=0$ is eigenvalue of problem (4.1)–(4.3) if and only if the point $(\gamma_0,\gamma_1)$ lies on the union of vertical and horizontal lines (4.8): $\gamma_0=\frac{D(n_0,k_1)}{D(n_0,n_1)}$, $\gamma_1=\frac{D(k_0,n_1)}{D(n_0,n_1)}$.

*Corollary 5.* If $D(n_0,n_1)=0$, $D(k_0,k_1)\neq 0$ and $D(n_0,k_1)\neq 0$ or $D(k_0,n_1)\neq 0$ then $\lambda=0$ is eigenvalue of problem (4.1)–(4.3) if and only if the point $(\gamma_0,\gamma_1)$ lies on the line (4.9).

*Corollary 6.* If $D(n_0,n_1)=0$ and $D(k_0,k_1)\neq 0$ and $D(n_0,k_1)=D(k_0,n_1)=0$ then $\lambda=0$ is not eigenvalue of problem (4.1)–(4.3) (the case of empty set).

*Corollary 7.* If $D(n_0,n_1)=D(k_0,k_1)=0$ and $D(n_0,k_1)\neq 0$ or $D(k_0,n_1)\neq 0$ then $\lambda=0$ is eigenvalue of problem (4.1)–(4.3) if and only if the point $(\gamma_0,\gamma_1)$ lies on the line (4.10). The point $(0,0)$ (the classical case) is on this line.

*Corollary 8.* If $D(n_0, n_1) = D(k_0, k_1) = D(n_0, k_1) = D(k_0, n_1) = 0$ then $\lambda = 0$ is eigenvalue of problem (4.1)–(4.3) for all $\gamma_0, \gamma_1$ (whole plane).

Let denote matrix

$$\mathbf{A} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} := \begin{pmatrix} D(n_0, n_1) & D(n_0, k_1) \\ D(k_0, n_1) & D(k_0, k_1) \end{pmatrix}. \tag{4.11}$$

From Corollaries 2–8 we get classification of Characteristic Curves in the plane $\mathbb{R}^2 := \mathbb{R}^2_{\gamma_0,\gamma_1}$. Each matrix $\mathbf{A}$ corresponds to one of the five types of Characteristic Curves. More detailed classification is shown in Table 1 (in this table $a_{ij}$, $i, j = 0, 1$ are nonzero elements). We have 16 types of matrices overall and one type is split into two cases( $\det \mathbf{A} = 0$ and $\det \mathbf{A} \neq 0$). So, the next lemma is valid.

**Lemma 2.** *A Characteristic Curve for problem* (4.1)–(4.3) *in the plane* $\mathbb{R}^2$ *can be one of the following five types:*

1. *If* $D(n_0, n_1) = D(k_0, k_1) = D(n_0, k_1) = D(k_0, n_1) = 0$ *then the curve is* whole plane*;*
2. *If* $D(n_0, n_1) = D(n_0, k_1) = D(k_0, n_1) = 0$*,* $D(k_0, k_1) \neq 0$ *then the curve is* empty set*;*
3. *If* $D(n_0, n_1) = 0$*,* $D(n_0, k_1) \neq 0$ *or* $D(n_0, n_1) = 0$*,* $D(k_0, n_1) \neq 0$ *then the curve is* line*;*
4. *If* $D(n_0, n_1) \neq 0$ *and* $\det \mathbf{A} = 0$ *then the curve is* union of vertical and horizontal lines*;*
5. *If* $D(n_0, n_1) \neq 0$ *and* $\det \mathbf{A} \neq 0$ *then the curve is* hyperbola.

*Remark 5.* We see, that Characteristic Curve in the plane $\mathbb{R}^2$ cannot be algebraic curve such as ellipse, parabola, point, parallel lines, double line.

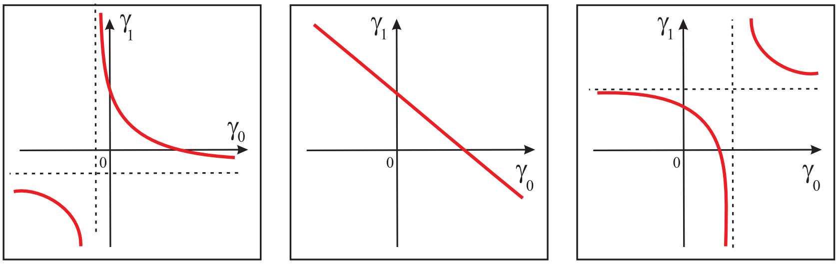
*Remark 6.* If $\det \mathbf{A} \neq 0$ then the line (Case 3) is neither vertical nor horizontal (see Cases 3e,f in Table 1), otherwise we have single vertical or single horizontal line (see Cases 3a-d in Table 1).

*Example 2* [see [6]]. In this example we analyze Characteristic Curve for the nonlocal boundary conditions given in [6]: $l = 1$, $k_0 = \delta(x)$ and $k_1 = \delta(x - 1)$, i.e. $\langle k_0, u\rangle = u(0)$ and $\langle k_1, u\rangle = u(1)$. We use fundamental system (3.5). In this case we have:

$$D(k_0, k_1)[\boldsymbol{u}] = \begin{vmatrix} u_0(0) & u_1(0) \\ u_0(1) & u_1(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1;$$

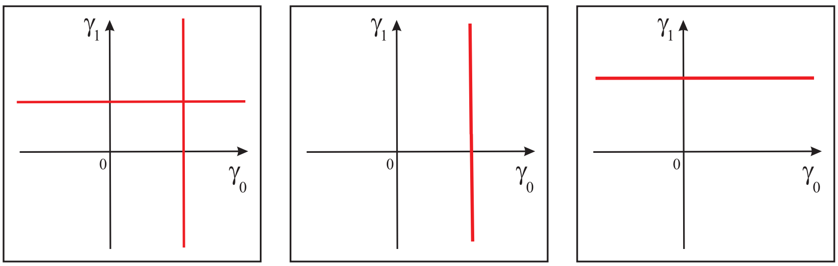
$$D(n_0, k_1)[\boldsymbol{u}] = \begin{vmatrix} \langle n_0, u_0\rangle & \langle n_0, u_1\rangle \\ u_0(1) & u_1(1) \end{vmatrix} = \begin{vmatrix} \langle n_0, u_0\rangle & \langle n_0, u_1\rangle \\ 0 & 1 \end{vmatrix} = \langle n_0, u_0\rangle;$$

$$D(k_0, n_1)[\boldsymbol{u}] = \begin{vmatrix} u_0(0) & u_1(0) \\ \langle n_1, u_0\rangle & \langle n_1, u_1\rangle \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \langle n_1, u_0\rangle & \langle n_1, u_1\rangle \end{vmatrix} = \langle n_1, u_1\rangle.$$




(a) $D(n_0, n_1) < 0$ (b) $D(n_0, n_1) = 0$ (c) $D(n_0, n_1) > 0$

**Figure 1.** Characteristic Curve for Bitsadze-Samarskii type nonlocal boundary conditions in the case $0 < \xi_0, \xi_1 < l$.




(a) $D(n_0, n_1) \neq 0$ (b) $D(n_0, n_1) = 0$, $\langle n_0, u_0\rangle \neq 0$ (c) $D(n_0, n_1) = 0$, $\langle n_1, u_1\rangle \neq 0$

**Figure 2.** Characteristic Curve for Bitsadze-Samarskii type nonlocal boundary conditions in the case $\det \mathbf{A} = 0$.

So, we have the same equation for Characteristic Curve as in [6]:

$$D(n_0, n_1)[\boldsymbol{u}]\gamma_0\gamma_1 - \langle n_0, u_0\rangle\gamma_0 - \langle n_1, u_1\rangle\gamma_1 + 1 = 0 \tag{4.12}$$

and

$$\mathbf{A} = \begin{pmatrix} D(n_0, n_1)[\boldsymbol{u}] & \langle n_0, u_0\rangle \\ \langle n_1, u_1\rangle & 1 \end{pmatrix}, \quad \det \mathbf{A} = -\langle n_0, u_1\rangle \cdot \langle n_1, u_0\rangle.$$

In this example the Characteristic Curve can be (see Table 1) one of the following four types ($\mathbf{A} \neq \mathbf{O}$):

1. If $D(n_0, n_1) = \langle n_0, u_0\rangle = \langle n_1, u_1\rangle = 0$ then the curve is *empty set*;
2. If $D(n_0, n_1) = 0$, $\langle n_0, u_0\rangle \neq 0$ or $D(n_0, n_1) = 0$, $\langle n_1, u_1\rangle \neq 0$ then the curve is *line*;
3. If $D(n_0, n_1) \neq 0$ and $\det \mathbf{A} = 0$ then the curve is *union of vertical and horizontal lines*;
4. If $D(n_0, n_1) \neq 0$ and $\det \mathbf{A} \neq 0$ then the curve is *hyperbola*.

Let us consider the case when the maximum principle is valid. For example, consider equation $-u'' + u = 0$. If functionals $n_0 = \delta(x - \xi_0)$, $n_1 = \delta(x - \xi_1)$, $0 < \xi_0, \xi_1 < 1$ are used, then we have the Bitsadze-Samarskii type nonlocal boundary conditions: $u(0) = \gamma_0 u(\xi_0)$, $u(1) = \gamma_1 u(\xi_1)$. For the fundamental system (3.5) from the maximum principle it follows $0 < \langle n_j, u_i \rangle = u_i(\xi_j) < 1$, $i, j = 0, 1$. In this case we have

$$\det \mathbf{A} = -\langle n_0, u_1 \rangle \cdot \langle n_1, u_0 \rangle < 0$$

and from Lemma 2 and Remark 6 we derive that the Characteristic Curve can be hyperbola ($D(n_0, n_1) \neq 0$) or line ($D(n_0, n_1) = 0$). The Characteristic Curve crosses the coordinate axes at the points $\gamma_0 = 1/\langle n_0, u_0 \rangle > 0$ and $\gamma_1 = 1/\langle n_1, u_1 \rangle > 0$ (see Figure 1).

If we take in the Bitsadze-Samarskii type nonlocal boundary conditions $\xi_0 = 0$ or $\xi_1 = 1$ then we get the degenerative case $\det \mathbf{A} = 0$, i.e. the union of vertical and horizontal lines (see Figure 2(a)). The lines intersection point corresponds to the case when both nonlocal boundary conditions are trivial. If $n_0 = \delta(x - \xi_0)$, $n_1 = 0$ then $D(n_0, n_1) = 0$, $\langle n_1, u_1 \rangle \neq 0$, $\det \mathbf{A} = 0$, and the characteristic curve is a vertical line (see Figure 2(b)). Finally, for $n_0 = n_1 = 0$ we have classical boundary conditions $u(0) = \gamma_0 \cdot 0 = 0$, $u(1) = \gamma_1 \cdot 0 = 0$, and the Characteristic Curve is empty set.

*Example 3.* For the boundary value problem

$$u'' + \pi^2 u = 0, \tag{4.13}$$

$$u(0) = \gamma_0 u'(0), \quad u(1) = \gamma_1 u(0) \tag{4.14}$$

we cannot use the fundamental system (3.5). In this case we use fundamental system (3.6), i.e. $\{u_0, u_1\} = \{\cos(\pi x), \sin(\pi x)/\pi\}$:

$$D(k_0, k_1)[\boldsymbol{u}] = \begin{vmatrix} u_0(0) & u_1(0) \\ u_0(1) & u_1(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ u_0(1) & u_1(1) \end{vmatrix} = u_1(1) = 0;$$

$$D(n_0, k_1)[\boldsymbol{u}] = \begin{vmatrix} u_0'(0) & u_1'(0)\rangle \\ u_0(1) & u_1(1) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ u_0(1) & u_1(1) \end{vmatrix} = -u_0(1) = 1;$$

$$D(k_0, n_1)[\boldsymbol{u}] = \begin{vmatrix} u_0(0) & u_1(0) \\ u_0(0) & u_1(0) \end{vmatrix} = 0;$$

$$D(n_0, n_1)[\boldsymbol{u}] = \begin{vmatrix} u_0'(0) & u_1'(0)\rangle \\ u_0(0) & u_1(0) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1,$$

and Characteristic Curve consists of two lines (Case 4b)

$$\gamma_0(\gamma_1 + 1) = 0. \tag{4.15}$$

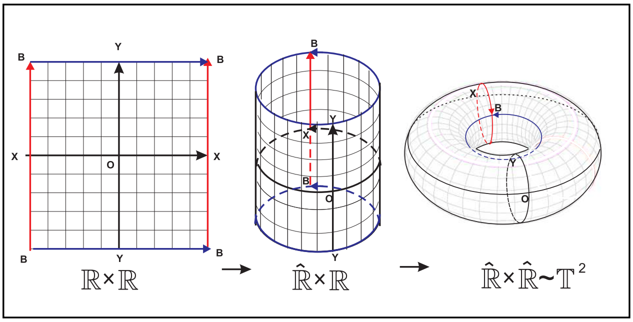



**Figure 3.** Plane, cylinder and torus, $O=(0,0)$, $X=(\infty,0)$, $Y=(0,\infty)$, $B=(\infty,\infty)$, in the plane $\hat{\mathbb{R}}^2$.

## 5 Characteristic Curve on Torus

The example of NBC (4.2)–(4.3) is

$$u(0)=\gamma_0\int_0^1\alpha(x)u(x)\,\mathrm{d}x,\quad u(1)=\gamma_1\int_0^1\beta(x)u(x)\,\mathrm{d}x. \tag{5.1}$$

Sometimes instead of NBCs (5.1) conditions (one or both) are used for Sturm–Liouville problem:

$$\int_0^1\alpha(x)u(x)\,\mathrm{d}x=0,\quad \int_0^1\beta(x)u(x)\,\mathrm{d}x=0. \tag{5.2}$$

Formally, we can say that such cases are realized for $\gamma_0=\infty$ or $\gamma_1=\infty$. More general problem will be if we consider NBCs:

$$\int_0^1\alpha(x)u(x)\,\mathrm{d}x=\tilde{\gamma}_0u(0),\quad \int_0^1\beta(x)u(x)\,\mathrm{d}x=\tilde{\gamma}_1u(1). \tag{5.3}$$

Now left hand side of these BC is "classical" and right hand side "nonlocal". In the case NBC (4.2)–(4.3) we must investigate problem (4.1) with NBC:

$$\langle n_0,u\rangle=\tilde{\gamma}_0\langle k_0,u\rangle,\quad \tilde{\gamma}_0\in\mathbb{R},\quad \langle n_1,u\rangle=\tilde{\gamma}_1\langle k_1,u\rangle,\quad \tilde{\gamma}_1\in\mathbb{R}. \tag{5.4}$$

The Characteristic Curve in the plane $\mathbb{R}^2_1:=\mathbb{R}^2_{\tilde{\gamma}_0,\tilde{\gamma}_1}$ is described by equation

$$D(k_0,k_1)\tilde{\gamma}_0\tilde{\gamma}_1-D(k_0,n_1)\tilde{\gamma}_0-D(n_0,k_1)\tilde{\gamma}_1+D(n_0,n_1)=0.$$

So, we have five cases again (see column $\mathbb{R}^2_1$ in Table 1). If NBCs have form

$$\langle k_0,u\rangle=\gamma_0\langle n_0,u\rangle,\quad \gamma_0\in\mathbb{R},\quad \langle n_1,u\rangle=\tilde{\gamma}_1\langle k_1,u\rangle,\quad \tilde{\gamma}_1\in\mathbb{R}; \tag{5.5}$$

$$\langle n_0,u\rangle=\tilde{\gamma}_0\langle k_0,u\rangle,\quad \tilde{\gamma}_0\in\mathbb{R},\quad \langle k_1,u\rangle=\gamma_1\langle n_1,u\rangle,\quad \gamma_1\in\mathbb{R}, \tag{5.6}$$

then Characteristic Curves in the planes $\mathbb{R}^2_2:=\mathbb{R}^2_{\gamma_0,\tilde{\gamma}_1}$, $\mathbb{R}^2_3:=\mathbb{R}^2_{\tilde{\gamma}_0,\gamma_1}$ are

$$D(n_0,k_1)\gamma_0\tilde{\gamma}_1-D(n_0,n_1)\gamma_0-D(k_0,k_1)\tilde{\gamma}_1+D(k_0,n_1)=0,$$
$$D(k_0,n_1)\tilde{\gamma}_0\gamma_1-D(k_0,k_1)\tilde{\gamma}_0-D(n_0,n_1)\gamma_1+D(n_0,k_1)=0.$$

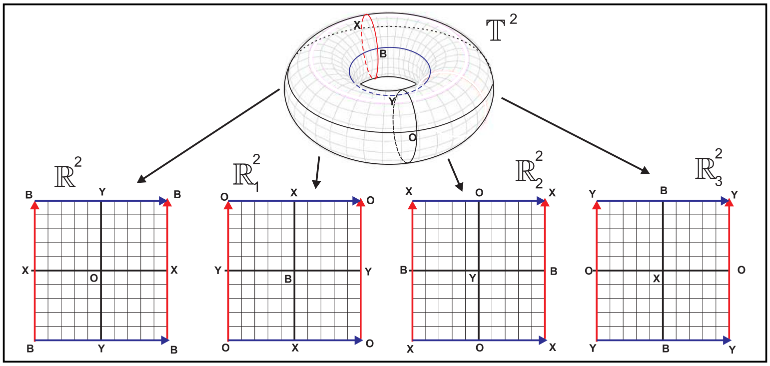



**Figure 4.** Torus $\mathbb{T}^2$ and planes $\mathbb{R}^2$, $\mathbb{R}_1^2$, $\mathbb{R}_2^2$, $\mathbb{R}_3^2$.

If Characteristic Curve $\mathcal{C}$ in the plane $\mathbb{R}^2$ is described by matrix $\mathbf{A}$ (see (4.11)) then Characteristic Curve in the planes $\mathbb{R}_1^2$, $\mathbb{R}_2^2$, $\mathbb{R}_3^2$ is described by matrices

$$\mathbf{A}_1 = \begin{pmatrix} a_{11} & a_{10} \\ a_{01} & a_{00} \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} a_{01} & a_{11} \\ a_{00} & a_{10} \end{pmatrix}, \quad \mathbf{A}_3 = \begin{pmatrix} a_{10} & a_{11} \\ a_{00} & a_{01} \end{pmatrix},$$

accordingly. Columns $\mathbb{R}_1^2$, $\mathbb{R}_2^2$, $\mathbb{R}_3^2$ in Table 1 show the type of Characteristic Curve $\mathcal{C}$ in these planes. For example, line in the case 3f) corresponds to hyperbola in the planes $\mathbb{R}_1^2$, $\mathbb{R}_2^2$, $\mathbb{R}_3^2$ (the cases 5e,5d,5c).

If $\gamma_i \neq 0$ and $\tilde{\gamma}_i \neq 0$, $i = 0, 1$ , then relation between these parameters are $\tilde{\gamma}_i = 1/\gamma_i$. Now we consider that $\gamma_i \in \mathbb{R} \cup \{\infty\} = \hat{\mathbb{R}} = \mathbb{R}P^1$, i.e. real projective line, which is a homogeneous space, in fact homeomorphic to a circle $S^1$.

Topologically, a torus (see Figure 3) is a closed surface defined as the product of two circles: $\mathbb{T}^2 = S^1 \times S^1$. We can consider the parameters planes $\mathbb{R}^2$, $\mathbb{R}_1^2$, $\mathbb{R}_2^2$, $\mathbb{R}_3^2$ as four charts which form an atlas for the torus (see Figure 4).

**Lemma 3.** *On torus Characteristic Curve for problem* (4.1)–(4.3) *can be one of the following three types:*

(1) *If* $\mathbf{A} \in GL_2(\mathbb{R})$ *then the curve is homeomorphic to a circle, and this curve winds around the torus one time (one time in one direction and one in the other direction);*

(2) *If* $\mathbf{O} \neq \mathbf{A} \notin GL_2(\mathbb{R})$ *then the curve is the union of two circles (strictly 'latitudinal' and strictly 'longitudinal') with one common point;*

(3) *Otherwise (i.e.,* $\mathbf{A} = \mathbf{O}$*) the curve is whole torus.*

*Proof.* Let begin with Case 4d in the plane $\mathbb{R}^2$. In this case Characteristic Curve $\mathcal{C}$ is the union of two (vertical and horizontal) lines $\gamma_0 = a \neq 0$, $\gamma_1 = b \neq 0$ with intersection point $(a, b)$ (see Corollary 3). In the plane $\mathbb{R}_1^2$ we again have the union of two lines $\tilde{\gamma}_0 = 1/a \neq 0$, $\tilde{\gamma}_1 = 1/b \neq 0$. Points $(1/a, 0)$ and $(0, 1/b)$ in this plane corresponds to points $(a, \infty)$ and $(\infty, b)$ in $\hat{\mathbb{R}}^2$. So, line

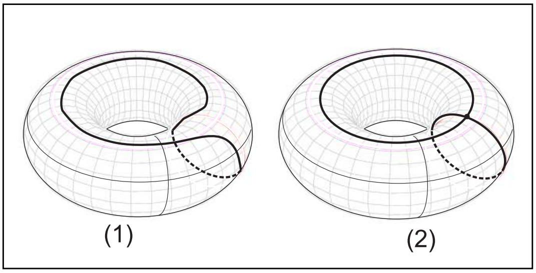



**Figure 5.** Characteristic Curve on torus in Case 1 and Case 2.

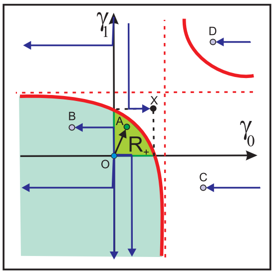



**Figure 6.** Domain of regularity $\mathcal{R}$ and connected components.

$l_a := \{(a,\gamma_1)\colon \gamma_1 \in \mathbb{R}\}$ in the $\mathbb{R}^2$ is part of a curve $l_a \cup \{(a,\infty)\}$ in the $\hat{\mathbb{R}}^2$ which is homeomorphic to a circle, and the line $l_b := \{(\gamma_0,b)\colon \gamma_0 \in \mathbb{R}\}$ in $\mathbb{R}^2$ is part of a curve $l_b \cup \{(\infty,b)\}$ in the $\hat{\mathbb{R}}^2$ which is homeomorphic to a circle, too. The point of intersection of these two curves is $(a,b)$. In Cases 4a, 4b, 4c we have the same situation, but $a$ or $b$ are zero. So, the line $l_a$ or the line $l_b$ can be axes. If only the vertical line $l_a$ is axis in the plane $\mathbb{R}^2$ (Case 4b) then we have one horizontal line in the plane $\mathbb{R}^2_1$ (Case 3b) and it is better to use the plane $\mathbb{R}^2_2$ for investigation of vertical line $l_a$ in the point $Y=(0,\infty)$. We note, that if $l_a$ and $l_b$ are axes then in $\mathbb{R}^2_1$ Characteristic Curve is empty set (Case 2), and in planes $\mathbb{R}^2_2$, $\mathbb{R}^2_3$ we have only one line. Finally, Case 4 (two lines in $\mathbb{R}^2$) corresponds to the circles on torus. Moreover, we have two circles in Case 2, 3a, 3b, 3c, 3d, too, because the case of two lines is in one of the planes $\mathbb{R}^2_1$, $\mathbb{R}^2_2$, $\mathbb{R}^2_3$ (see Table 1).

In Case 3e we have line $l$ in $\mathbb{R}^2$ and we must investigate the point $B=(\infty,\infty)$. In the plain $\mathbb{R}^2_1$ this point is in origin and we have a line again. So, $l \cup B$ is homeomorphic to a circle. Moreover, we have circle in Case 5b, too (see Table 1). This case corresponds to hyperbola and axis are it's asymptotes. If we add points $X=(\infty,0)$ and $Y=(0,\infty)$, then we get curve which is homeomorphic to a circle. All other hyperbolae (Cases 5a, 5c, 5d, 5e) have vertical and horizontal asymptotes and give such curve, analogously. A line in Case 3f is hyperbola (Case 5e) in $\mathbb{R}^2_1$.

If $\mathbf{A}=\mathbf{O}$ then we have whole planes $\mathbb{R}^2$, $\mathbb{R}^2_1$, $\mathbb{R}^2_2$, $\mathbb{R}^2_3$. So, in this case Characteristic Curve is a whole torus. Note, that in the case $\mathbf{A}\neq\mathbf{O}$ we have two circles on torus if and only if $\det\mathbf{A}=0$. Lemma is proved. $\square$

Case 1 and Case 2 are presented in Figure 5.

Let us consider two Characteristic Curves $\mathcal{C}_1$ and $\mathcal{C}_2$ which correspond to matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. We can define distance between curves as distance between matrices. If we introduce matrix elementwise *max* norm

$$\|\mathbf{A}\| := \max\{|a_{00}|,|a_{01}|,|a_{10}|,|a_{11}|\} \text{ for } \mathbf{A}=\begin{pmatrix} a_{00} & a_{01}\\ a_{10} & a_{11}\end{pmatrix},$$

then a set of Characteristic Curves becomes metric space with distance

$$d(\mathcal{C}_1,\mathcal{C}_2) := \|\mathbf{A}_1-\mathbf{A}_2\|.$$

For max norm an inequality

$$|\det \mathbf{A}_1 - \det \mathbf{A}_2| \leqslant (\|\mathbf{A}_1\| + \|\mathbf{A}_2\|)\, \|\mathbf{A}_1 - \mathbf{A}_2\| \tag{5.7}$$

is valid. So, a difference between matrix determinants can be estimated via $\|\mathbf{A}_1 - \mathbf{A}_2\|$.

*Remark 7.* $GL_2(\mathbb{R})$ is open set in $\mathbb{R}^4$. So, small changes of coefficients of matrix $\mathbf{A}$ do not change Characteristic Curve type (i.e., we have circle on torus again). Meanwhile, if $\det \mathbf{A} = 0$ or $\mathbf{A} = \mathbf{O}$ then there exist matrices from $GL_2(\mathbb{R})$ in the neighborhood of the matrix $\mathbf{A}$. In the plane $\mathbb{R}^2$ the first case on torus corresponds to the cases: 3e, 3f (line) and 5 (hyperbola). Matrices of hyperbolae make open set in $GL_2(\mathbb{R})$, too. If we consider special cases of nonlocal boundary conditions the class of matrices can satisfy condition $\det A = 0$ or can describe lines only.

Every strictly 'latitudinal' circle $\gamma_0 = const \in \hat{\mathbb{R}}$ or strictly 'longitudinal' circle $\gamma_1 = const \in \hat{\mathbb{R}}$ on torus crosses Characteristic Curve. The intersection is a circle in Case 3, a point in Case 1 and a point or circle in Case 2.

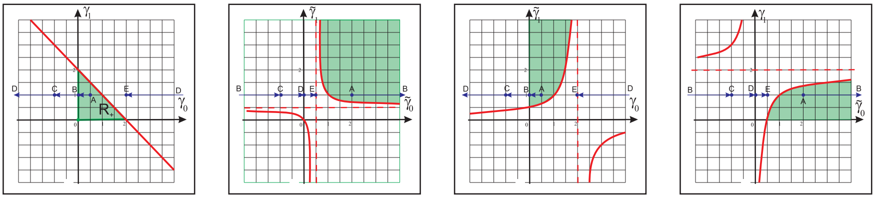
We name the point $(\gamma_0, \gamma_1)$ which does not belong to Characteristic Curve *regular point*. All regular points make a domain of regularity $\mathcal{R}$. For Case 3 $\mathcal{R}$ is empty set.

*Corollary 9.* The domain $\mathcal{R} \subset \mathbb{T}^2$ is a path-connected open set.

*Remark 8.* The domain $\mathcal{R} \subset \mathbb{R}^2$ may consist of a few connected components (one, two or three).

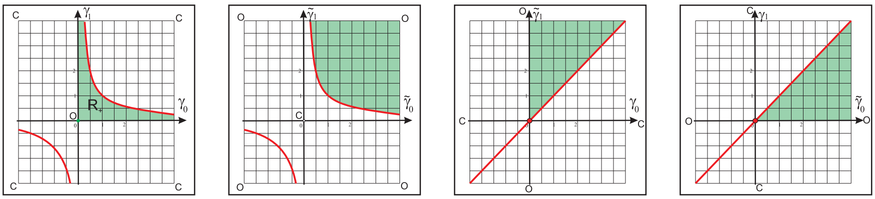
*Remark 9.* One of main results of paper [6] was that we can change continuously parameters $\gamma_0$ and $\gamma_1$ in the plane $\mathbb{R}^2$ from point $(0,0)$ (classical boundary condition) to some area $R_+$ limited by Characteristic Curve in the first quadrant (in the paper [6] only the case $\gamma_0, \gamma_1 \geqslant 0$ was considered). The second order problem with nonlocal boundary conditions as in Example 2 has unique solution in all points of this area. It is easy to expand this area into bigger domain if we use negative $\gamma_0$ and $\gamma_1$, but we cannot cross the Characteristic Curve if we want to connect two points in different connected components.

On torus the domain of regularity $\mathcal{R}$ is path-connected open set and $R_+ \subset \mathcal{R}$. We can connect two points of this domain by continuous path. Moreover, the path can be union of two parts and each of them belongs to circle $\gamma_0 = const \in \hat{\mathbb{R}}$ or circle $\gamma_1 = const \in \hat{\mathbb{R}}$ on torus. For each point in the domain $\mathcal{R}$ there exists only one other point in this domain such that we need to use three such parts in the case $\det \mathbf{A} \neq 0$. For some points we need only one such path. In Figure 6, we have three connected components in $\mathbb{R}^2$ and one on torus. The area $R_+$ corresponds to results of paper [6]. The domain with points $A$ and $B$ under the left part of hyperbola is one of the three connected domains in the plain. On torus we can continuously move to points $C$, $D$, $X$, too. If we connect origin $O$ and point $X$ then we must use path with three linear (the vertical or horizontal) parts.



$\mathbb{R}^2$: $\gamma_1 = 2 - \gamma_0$ (a) | $\mathbb{R}^2_1$: $\tilde\gamma_1 = \frac{\tilde\gamma_0}{2\tilde\gamma_0 - 1}$ (b) | $\mathbb{R}^2_2$: $\tilde\gamma_1 = \frac{1}{2-\gamma_0}$ (c) | $\mathbb{R}^2_3$: $\gamma_1 = 2 - \frac{1}{\tilde\gamma_0}$ (d)

**Figure 7.** Characteristic Curve for integral nonlocal boundary conditions.



$\mathbb{R}^2$: $\gamma_1 = 1/\gamma_0$ (a) | $\mathbb{R}^2_1$: $\tilde\gamma_1 = 1/\tilde\gamma_0$ (b) | $\mathbb{R}^2_2$: $\tilde\gamma_1 = \gamma_0$ (c) | $\mathbb{R}^2_3$: $\gamma_1 = \tilde\gamma_0$ (d)

**Figure 8.** Characteristic Curve and area paradox.

*Example 4* [integral NBCs]. For functionals $n_0 = 1$, $n_1 = 1$ we have integral type nonlocal boundary conditions [6, 7]:

$$u(0) = \gamma_0 \int_0^1 u(x)\,dx, \quad u(1) = \gamma_1 \int_0^1 u(x)\,dx. \tag{5.8}$$

In this case $0 < \langle n_j, u_i\rangle = \int_0^1 u_i(x)dx < 1$, $i = 0, 1$. Since $D(n_0, n_1) = D(1, 1) = 0$, the Characteristic Curve in $\mathbb{R}^2$ is a line (see Figure 7(a)). If we investigate problem with integral boundary conditions on torus then in the planes $\mathbb{R}^2_1$, $\mathbb{R}^2_2$, $\mathbb{R}^2_3$ we have hyperbolae (see Figure 7) and they correspond to nonlocal boundary conditions:

$$\int_0^1 u(x)dx = \tilde\gamma_0 u(0), \quad \int_0^1 u(x)\,dx = \tilde\gamma_1 u(1); \tag{5.9}$$

$$u(0) = \gamma_0 \int_0^1 u(x)\,dx, \quad \int_0^1 u(x)dx = \tilde\gamma_1 u(1); \tag{5.10}$$

$$\int_0^1 u(x)\,dx = \tilde\gamma_0 u(0), \quad u(1) = \gamma_1 \int_0^1 u(x)\,dx; \tag{5.11}$$

respectively. Note, that the first boundary condition (5.10) coincides with the second condition for $\gamma_0 = \tilde\gamma_1 = 0$ and the origin of the coordinate plane belongs to hyperbola. On torus problems with boundary conditions (5.8)–(5.11) is the same problem and Characteristic Curve is the circle on the torus. In Figure 7 we see four various maps of this curve. The area $R_+$ (see the first quadrant) shows the domain investigated in [6]. In all maps we see a path between points

$A(\gamma_0 = 0.5, \gamma_1 = 1)$ and $E(\gamma_0 = 2, \gamma_1 = 1)$:

$$A(0.5, 1) \to B(0, 1) \to C(-1, 1) \to D(\infty, 1) \to E(2, 1),$$

and each closed interval $[A, B]$, $[B, C]$, $[C, D]$, $[D, E]$ is finite in at least one of the planes $\mathbb{R}^2$, $\mathbb{R}^2_1$, $\mathbb{R}^2_2$, $\mathbb{R}^2_3$.

*Example 5* [an area paradox]. Let us consider nonlocal boundary conditions (see Example 2, too)

$$u(0) = \gamma_0 u(1), \quad u(1) = \gamma_1 u(0). \tag{5.12}$$

In this case the Characteristic Curve in plane $\mathbb{R}^2$ is $\gamma_0\gamma_1 = 1$.

If we consider problem on torus then this curve is hyperbola or line in the planes $R^2_1$, $R^2_2$, $R^2_3$ (see Figure 8). The domain $R_+$ and domains corresponding to $R_+$ in these planes are shown in Figure 8(a)-(d). In the plane $R^2_1$ the formula for the Characteristic Curve is $\tilde{\gamma}_0\tilde{\gamma}_1 = 1$ and boundary conditions are

$$u(1) = \tilde{\gamma}_0 u(0), \quad u(0) = \tilde{\gamma}_1 u(1). \tag{5.13}$$

These conditions are the same as (5.12), thus domain $R_+$ for parameters $(\tilde{\gamma}_1, \tilde{\gamma}_0)$ must be the same as in Figure 8(a).

If we consider the domain of regularity $\mathcal{R}$ on torus then we do not have this paradox. Note, that in this example classical boundary conditions $u(0) = 0$ and $u(1) = 0$ correspond to two points $O(0, 0)$ and $(\infty, \infty)$ on torus (or (0,0) in planes $\mathbb{R}^2$ and $\mathbb{R}^2_1$).

Let us consider the two cases of the boundary conditions

$$\langle k_0, u\rangle = \gamma_0\langle n_0, u\rangle, \quad \langle k_1, u\rangle = \gamma_1\langle n_1, u\rangle, \tag{5.14}$$

$$\langle K_0, U\rangle = \gamma_0\langle N_0, U\rangle, \quad \langle K_1, U\rangle = \gamma_1\langle N_1, U\rangle, \tag{5.15}$$

and denote $\varPhi_{k_ij} := \langle k_i, u_j\rangle - \langle K_i, U_j\rangle$, $\varPhi_{n_ij} := \langle n_i, u_j\rangle - \langle N_i, U_j\rangle$. We have two matrices

$$\mathbf{A}_1 = \begin{pmatrix} D(n_0, n_1) & D(n_0, k_1) \\ D(k_0, n_1) & D(k_0, k_1) \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} D(N_0, N_1) & D(N_0, K_1) \\ D(K_0, N_1) & D(K_0, K_1) \end{pmatrix}.$$

From inequality (5.7) it follows

$$\begin{aligned} |D(n_0, n_1) - D(N_0, N_1)| &\leqslant C_1\|M(n_0, n_1)[u_0, u_1] - M(N_0, N_1)[U_0, U_1]\| \\ &\leqslant C_1 \max\{|\varPhi_{n_00}|, |\varPhi_{n_10}|, |\varPhi_{n_01}|, |\varPhi_{n_11}|\}. \end{aligned}$$

Similar inequalities are valid for

$$|D(n_0, k_1) - D(N_0, K_1)|, |D(k_0, n_1) - D(K_0, N_1)|, |D(k_0, k_1) - D(K_0, K_1)|.$$

Finally, we get estimate

$$\|\mathbf{A}_1 - \mathbf{A}_2\| \leqslant C \max_{j=0,1}\{|\varPhi_{k_ij}|, |\varPhi_{n_ij}|\}. \tag{5.16}$$

*Corollary 10.* The Characteristic Curve continuously depends on functionals in boundary conditions.

*Remark 10.* Lemma 3 is valid for the discrete Sturm–Liouville problem

$$\mathcal{L}^h U^h := -\delta(P^h \delta U^h) + Q^h U^h = 0 \quad \text{in } \omega^h, \tag{5.17}$$

$$\langle K_0^h, U^h\rangle = \gamma_0 \langle N_0^h, U^h\rangle, \quad \langle K_1^h, U^h\rangle = \gamma_1 \langle N_1^h, U^h\rangle, \tag{5.18}$$

$(\gamma_0, \gamma_1) \in \mathbb{T}^2$.

A proof is the same as for the differential case.

*Remark 11.* If we consider differential problem (4.1)–(4.3) (the case $\lambda = 0$) and discrete problem (5.17)–(5.18), then

$$\Phi_{k_i}^h := \langle k_i, u\rangle - \langle K_i^h, U^h\rangle, \quad \Phi_{n_i}^h := \langle n_i, u\rangle - \langle N_i^h, U^h\rangle, \quad i = 0, 1,$$

denote approximation errors in boundary conditions. Let $\mathcal{C}$ and $\mathcal{C}^h$ be Characteristic Curves for differential and discrete problems, respectively. If $\Phi_{k_i}^h = \mathcal{O}(h^\alpha)$, $\Phi_{n_i}^h = \mathcal{O}(h^\alpha)$, then from (5.16) we get $\|\mathbf{A}_1 - \mathbf{A}_2\| = \mathcal{O}(h^\alpha)$, i.e., Characteristic Curve $\mathcal{C}$ is approximated by Characteristic Curve $\mathcal{C}^h$ with the same order as functionals in the boundary conditions.

*Remark 12.* For complex $\gamma_i \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $i = 0, 1$ we must use $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ instead of torus $\hat{\mathbb{R}} \times \hat{\mathbb{R}}$. This is an open problem for future investigations.

Another open problem is classification of points in the regularity domain:

- Subsets where all eigenvalues are positive, or exists one or a few negative eigenvalues;
- Subsets where exists constant eigenvalue points or multiple eigenvalue points (see, [29]);
- Qualitative analysis of spectrum dependence on $\gamma_0$ and $\gamma_1$ and other parameters in NBCs.

## Conclusions

In this article the Sturm–Liouville problem with NBCs depending on two parameters is investigated. Zero eigenvalue existence conditions are investigated and geometric interpretation of the results are given. We propose to investigate these results not in a plane but on torus. Classification of Characteristic Curves in the plane and on the tore is done. Obtained results can be generalized to $n$-th order equation with $n$ NBCs.

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