



Convergence Order in Trajectory Estimation by Piecewise-Cubics and Exponential Parameterization

Ryszard Kozera^{a,b,c} and Magdalena Wilkołazka^b

^a*Faculty of Applied Informatics and Mathematics, Warsaw University of Life Sciences - SGGW*

Nowoursynowska str. 159, 02-776 Warsaw, Poland

^b*Faculty of Mathematics, Informatics and Landscape Architecture, The John Paul II Catholic University of Lublin*

Konstantynów 1H, 20-708 Lublin, Poland

^c*Department of Computer Science and Software Engineering, The University of Western Australia*

35 Stirling Highway, WA 6009 Crawley, Perth, Australia

E-mail(*corresp.*): ryszard.kozera@gmail.com

E-mail: magda.wilkolazka@gmail.com

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Abstract. This paper discusses the problem of estimating the trajectory of the unknown curve γ from the sequence of $m + 1$ interpolation points $Q_m = \{\gamma(t_i)\}_{i=0}^m$ in arbitrary Euclidean space \mathbb{E}^n . The respective knots $\mathcal{T}_m = \{t_i\}_{i=0}^m$ (in ascending order) are assumed to be unknown. Such Q_m is coined *reduced data*. In our setting, a piecewise-cubic Lagrange interpolation $\hat{\gamma}_3 : [0, \hat{T}] \rightarrow \mathbb{E}^n$ is applied to fit Q_m . Here, the missing knots \mathcal{T}_m are replaced by their estimates $\hat{\mathcal{T}}_m = \{\hat{t}_i\}_{i=0}^m$ in accordance with the exponential parameterization. The latter is controlled by a single parameter $\lambda \in [0, 1]$. This work analyzes the intrinsic asymptotics in approximating γ by $\hat{\gamma}_3$ based on the exponential parameterization and Q_m . The multiple goals are achieved. Firstly, the existing result established for $\lambda = 1$ (i.e. for the cumulative chord parameterization) is extended to the remaining cases of $\lambda \in [0, 1)$ and more-or-less uniformly sampled Q_m . As demonstrated herein, a quartic convergence order $\alpha(1) = 4$ in trajectory estimation drops discontinuously to the linear one $\alpha(\lambda) = 1$, for all $\lambda \in [0, 1)$. Secondly, the asymptotics derived in this paper is also analytically proved to be sharp with the aid of illustrative examples. Thirdly, the latter is verified in affirmative upon conducting numerical testing. Next, the necessity of more-or-less uniformity imposed on Q_m is shown to be indispensable. In addition, several sufficient conditions for $\hat{\gamma}_3$ to be

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reparameterizable to the domain of γ are formulated. Lastly, the motivation for using the exponential parameterization with $\lambda \in [0, 1)$ is also outlined.

Keywords: interpolation, reduced data, convergence order and sharpness.

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1 Introduction

Let $\gamma : [0, T] \rightarrow \mathbb{E}^n$ be a smooth regular curve (i.e. $\dot{\gamma}(t) \neq 0$) over $t \in [0, T]$, for $0 < T < \infty$. The term reduced data Q_m determines $m + 1$ interpolation points $\{q_i\}_{i=0}^m$ (with $q_i = \gamma(t_i)$) in arbitrary Euclidean space \mathbb{E}^n . The corresponding interpolation knots $\mathcal{T} = \{t_i\}_{i=0}^m$ satisfying $t_0 = 0 < t_1 < \dots < t_m = T$ are assumed here to be unknown. Any fitting scheme $\hat{\gamma}$ interpolating Q_m with some pre-selected knots' estimates $\hat{\mathcal{T}} = \{\hat{t}_i\}_{i=0}^m \approx \mathcal{T}$ subject to $q_i = \hat{\gamma}(\hat{t}_i)$ is called non-parametric interpolation (see e.g. [3], [14] or [15]).

Having chosen an interpolation scheme $\hat{\gamma} : [0, \hat{T}] \rightarrow \mathbb{E}^n$ and some substitutes $\{\hat{t}_i\}_{i=0}^m$ of the respective missing knots $\{t_i\}_{i=0}^m$, the question of the intrinsic asymptotic order α in γ approximation by $\hat{\gamma}$ arises naturally. Recall first the basic definition (see e.g. [1], [6] or [19]):

DEFINITION 1. We say that $\{t_i\}_{i=0}^m$ is admissible if:

$$\lim_{m \rightarrow \infty} \delta_m \rightarrow 0^+, \quad \text{where } \delta_m = \max_{1 \leq i \leq m} \{t_i - t_{i-1} : i = 1, 2, \dots, m\}. \quad (1.1)$$

The following subclass of (1.1) is here of particular relevance (see [9]):

DEFINITION 2. The sampling $\{t_i\}_{i=0}^m$ is more-or-less uniform if for some constants $0 < K_l \leq K_u$ and sufficiently large m : $\frac{K_l}{m} \leq t_i - t_{i-1} \leq \frac{K_u}{m}$ holds for all $i = 1, 2, \dots, m$. Equivalently, the last two inequalities can be replaced by

$$\beta \delta_m \leq t_{i+1} - t_i \leq \delta_m, \quad (1.2)$$

satisfied for some $\beta \in (0, 1]$ and sufficiently large m .

Recall now the following (see e.g. [1] or [6]):

DEFINITION 3. Consider a family $\{f_{\delta_m}, \delta_m > 0\}$ of functions $f_{\delta_m} : [0, T] \rightarrow \mathbb{E}$. We say that f_{δ_m} is of order $O(\delta_m^\alpha)$ (denoted as $f_{\delta_m} = O(\delta_m^\alpha)$), if there is a constant $K > 0$ such that, for some $\bar{\delta} > 0$ the inequality $|f_{\delta_m}(t)| \leq K \delta_m^\alpha$ holds for all $\delta_m \in (0, \bar{\delta})$, uniformly over $[0, T]$. In case of vector-valued functions $F_{\delta_m} : [0, T] \rightarrow \mathbb{E}^n$ by $F_{\delta_m} = O(\delta_m^\alpha)$ it is understood that $\|F_{\delta_m}\| = O(\delta_m^\alpha)$.

This paper deals with a special family of discrete exponential parametrizations designed to estimate the missing knots $\{t_i\}_{i=0}^m$ (see e.g. [14]) according to:

$$\hat{t}_{0,\lambda} = 0, \quad t_{i+1,\lambda} = \hat{t}_{i,\lambda} + \|q_{i+1} - q_i\|^\lambda, \quad (1.3)$$

for arbitrary $\lambda \in [0, 1]$ and $i = 0, 1, 2, \dots, m - 1$. In order to preserve the ascending order of $\hat{\mathcal{T}}_\lambda = \{\hat{t}_{i,\lambda}\}_{i=0}^m$ an extra constraint $q_{i+1} \neq q_i$ is imposed here on the admitted class of reduced data Q_m . Visibly, the case of $\lambda = 0$

in (1.3) yields the uniform distribution of $\{t_{i,0}\}_{i=0}^m$ with no account given to the geometrical layout of Q_m . On the other hand, the so-called cumulative chord parameterization (i.e. when $\lambda = 1$ in (1.3)) incorporates the dispersion of Q_m . Noticeably, the case of $\lambda = 0,5$ in (1.3) covers the so-called centripetal parameterization (see [14]). To abbreviate the notation, from now on (unless needed otherwise) the parameter λ in subscript of $\hat{t}_{i,\lambda}$ determined by (1.3) is omitted.

Upon selecting specific interpolation scheme (e.g. piecewise r -degree Lagrange polynomials $\hat{\gamma} = \hat{\gamma}_r$, with either $r = 2$ or $r = 3$) and then varying the parameter $\lambda \in [0, 1]$ in (1.3) different interpolants $\hat{\gamma}_r$ can be constructed. Recall that for $\{q_i, q_{i+1}, q_{i+2}\}$ with $r = 2$ the Lagrange quadratic $\hat{\gamma}_{2,i} : [\hat{t}_i, \hat{t}_{i+2}] \rightarrow \mathbb{E}^n$ satisfies $\hat{\gamma}_{2,i}(\hat{t}_{i+j}) = q_{i+j}$, for $j = 0, 1, 2$ (where e.g. \hat{t}_{i+j} are defined according to (1.3)). The track sum of $\hat{\gamma}_{2,i}$ defines a piecewise-quadratic interpolant $\hat{\gamma}_2 : [\hat{t}_0 = 0, \hat{t}_m] \rightarrow \mathbb{E}^n$ based on reduced data Q_m and $\{\hat{t}_i\}_{i=0}^m$. Upon extending the domain of each $\hat{\gamma}_{2,i}$ from $\hat{I}_i = [\hat{t}_i, \hat{t}_{i+2}]$ into \mathbb{R} one defines a modified collection of quadratics $\check{\gamma}_{2,i} : \mathbb{R} \rightarrow \mathbb{E}^n$ fulfilling the interpolation conditions $\check{\gamma}_{2,i}(\hat{t}_{i+j}) = q_{i+j}$ and satisfying $\check{\gamma}_{2,i}|_{[\hat{t}_i, \hat{t}_{i+2}]} = \hat{\gamma}_{2,i}$. The latter yields $\check{\gamma}_{2,i} \circ \psi_i = \hat{\gamma}_{2,i} \circ \psi_i$, for arbitrary mapping $\psi_i : I_i = [t_i, t_{i+2}] \rightarrow [\hat{t}_i, \hat{t}_{i+2}]$. In particular, this holds for ψ_i reparameterizing I_i into \hat{I}_i (i.e. for ψ_i strictly increasing). However, for $\psi_i([t_i, t_{i+2}]) \not\subseteq [\hat{t}_i, \hat{t}_{i+2}]$ the asymptotic analysis of the difference $\hat{\gamma}_{2,i} \circ \psi_i - \gamma$ over I_i needs to be adjusted to $\check{\gamma}_{2,i} \circ \psi_i - \gamma$ over I_i .

Assume now that $\psi_i = \psi_{2,i}$ is a unique Lagrange quadratic satisfying $\psi_{2,i}(t_{i+j}) = \hat{t}_{i+j}$ (for $j = 0, 1, 2$), where \hat{t}_{i+j} are defined as in (1.3). The following result holds (see [9] and [17]):

Theorem 1. *Let γ be a $C^3([0, T])$ regular curve in \mathbb{E}^n sampled more-or-less uniformly (see Definition 2). Assume that Q_m forms reduced data with the unknown knots estimated by (1.3). Then, uniformly over $[0, T]$:*

$$\check{\gamma}_{2,i} \circ \psi_{2,i} - \gamma = O(\delta_m^1), \quad \text{for } \lambda \in [0, 1) \quad (1.4)$$

and for either uniform $\{t_i\}_{i=0}^m = iT/m$ or $\lambda = 1$ we have:

$$\check{\gamma}_{2,i} \circ \psi_{2,i} - \gamma = O(\delta_m^3). \quad (1.5)$$

Moreover, if the mapping $\psi_{2,i}$ is a reparameterization of I_i into \hat{I}_i (e.g. when $\{t_i\}_{i=0}^m$ is uniform or for $\lambda = 1$) then, in both (1.4) and (1.5), a phrase $\check{\gamma}_{2,i}$ can be replaced by $\hat{\gamma}_{2,i}$.

Note that if $\lambda = 1$ the condition (1.2) can be replaced by the weaker assumption imposed on samplings \mathcal{T} i.e. by the condition (1.1). The asymptotics from Theorem 1 is also proved to be *sharp* (see [9]) understood as:

DEFINITION 4. For a given interpolation scheme $\hat{\gamma}$ based on reduced data Q_m and some estimates $\hat{\mathcal{T}}$ of the unknown knots \mathcal{T} (and subject to some selected mapping $\psi : [0, T] \rightarrow [0, \hat{T}]$) we say that asymptotics $\gamma - \hat{\gamma} \circ \psi = O(\delta_m^\alpha)$ over $[0, T]$ is sharp within the prescribed family of curves $\gamma \in \mathcal{J}$ and family of samplings $\mathcal{T} \in \mathcal{K}$, if for some $\gamma \in \mathcal{J}$ and some sampling $\mathcal{T} \in \mathcal{K}$, there exist $\bar{t} \in [0, T]$ and some positive constant K such that $\|(\hat{\gamma} \circ \psi)(\bar{t}) - \gamma(\bar{t})\| = K\delta_m^\alpha + O(\delta_m^\eta)$, where $\eta > \alpha$.

In particular, for sharpness of (1.4): $\mathcal{J} = C^3$, the more-or-less uniform samplings (1.2) forms \mathcal{K} , $\hat{\gamma} = \hat{\gamma}_2$ (built from $\hat{\gamma}_2$) with $\psi = \psi_2$. Here $\hat{\gamma}_2$ (and ψ_2) is the track-sum of $\hat{\gamma}_{2,i}$ (and of $\psi_{2,i}$).

Visibly, (1.4) and (1.5) introduce the unexpected left-hand side discontinuity in convergence orders $\alpha(\lambda)$ at $\lambda = 1$ dropping sharply from $\alpha(1) = 3$ to $\alpha(\lambda) = 1$ (for all $\lambda \in [0, 1)$). The sufficient conditions guaranteeing $\psi_{2,i} : [t_i, t_{i+2}] \rightarrow [\hat{t}_i, \hat{t}_{i+2}]$ to be a genuine reparameterization for some $\lambda \in [0, 1)$ and $\beta \in (0, 1]$ are given in [9] or [10].

On the other hand, a full convergence analysis for piecewise cubic Lagrange interpolation $\hat{\gamma}_{3,i} : [\hat{t}_i, \hat{t}_{i+3}] \rightarrow \mathbb{E}^n$ based on exponential parameterization (1.3) and $\psi_{3,i} : J_i = [t_i, t_{i+3}] \rightarrow \hat{J}_i = [\hat{t}_i, \hat{t}_{i+3}]$ (satisfying $\psi_{3,i}(t_{i+j}) = \hat{t}_{i+j}$, for $j = 0, 1, 2, 3$) is so-far completed merely for $\lambda = 1$. Indeed, as shown in [13] and [17] a sharp quartic convergence order in estimating $\hat{\gamma}_{3,i} \circ \psi_{3,i} - \gamma$ over $[0, T]$ prevails. Here, similarly to the case of $r = 2$ and $\lambda = 1$ the cubic $\psi_{3,i}$ defines a genuine reparameterization.

2 Main results and motivation

We emphasize now the main contributions and the motivation of this paper:

1. This paper extends the analysis of approximation error in $\hat{\gamma}_{3,i} \circ \psi_{3,i} - \gamma$ for the remaining $\lambda \in [0, 1)$ - the case $\lambda = 1$ is covered in [17]. In fact, as proved in the next section an analogous sharp deceleration in convergence orders claimed by Theorem 1 appears also for $r = 3$ and $\lambda \in [0, 1)$. Before formulating the main result we adopt a similar notation for the adjusted cubics $\tilde{\gamma}_{3,i} : \mathbb{R} \rightarrow \mathbb{E}^n$ satisfying $\tilde{\gamma}_{3,i}(\hat{t}_{i+j}) = q_{i+j}$ and $\tilde{\gamma}_{3,i}|_{[\hat{t}_i, \hat{t}_{i+3}]} = \hat{\gamma}_{3,i}$. As previously if $\psi_{3,i}$ maps J_i into \hat{J}_i then $\tilde{\gamma}_{3,i} \circ \psi_{3,i} = \hat{\gamma}_{3,i} \circ \psi_{3,i}$. The latter holds e.g. if $\psi_{3,i}$ defines a genuine reparameterization of J_i into \hat{J}_i (i.e. $\psi_{3,i} > 0$).

The following result is proved in this paper (see Section 3):

Theorem 2. *Let γ be a $C^4([0, T])$ regular curve in \mathbb{E}^n sampled more-or-less uniformly (see Definition 2). Suppose that Q_m defines reduced data with the missing knots $\mathcal{T} = \{t_i\}_{i=0}^m$ compensated according to (1.3). Then the following holds (uniformly over $[0, T]$):*

$$\tilde{\gamma}_{3,i} \circ \psi_{3,i} - \gamma = O(\delta_m^1), \quad \text{for } \lambda \in [0, 1). \tag{2.1}$$

In addition, for either $\mathcal{T}_u = \{t_i\}_{i=0}^m$ or $\lambda = 1$ (for $\lambda = 1$ - see [17]) we obtain:

$$\tilde{\gamma}_{3,i} \circ \psi_{3,i} - \gamma = O(\delta_m^4). \tag{2.2}$$

Moreover, if additionally the mapping $\psi_{3,i}$ is asymptotically a reparameterization of J_i into \hat{J}_i (e.g. when $\{t_i\}_{i=0}^m$ is uniform or for $\lambda = 1$) then, in both (2.1) and (2.2), the curve $\tilde{\gamma}_{3,i}$ can be replaced by $\hat{\gamma}_{3,i}$.

Recently a similar linear convergence order as in (2.1) is established in [7] for the so-called modified Hermite interpolant $\hat{\gamma}_H \in C^1$ combined with the exponential parameterization (1.3) and $\lambda \in [0, 1)$ (the case of $\lambda = 1$ was covered in [6] or [8]). The proof used in [7] exploits the final claim of Theorem 2.

Indeed, in order to substantiate $\rho = \hat{\gamma}_H \circ \phi - \gamma = O(\delta_m^1)$ one decomposes ρ into $\rho = (\hat{\gamma}_H \circ \phi - \hat{\gamma}_3 \circ \psi_3) + (\hat{\gamma}_3 \circ \psi_3 - \gamma)$. The $O(\delta_m^1)$ asymptotics for the second component in ρ follows directly from (2.1). The linear asymptotics determining the first component $\hat{\gamma}_H \circ \phi - \hat{\gamma}_3 \circ \psi_3$ results upon conducting more advanced argument (as compared to Section 3) - see [7].

2. The convergence orders established in Theorem 2 are justified analytically as sharp in accordance with Definition 4 - see Section 5.

3. Section 7 supplements a) and b) with the numerical tests conducted in *Mathematica* confirming experimentally the sharpness of the asymptotics determined in (2.1) and (2.2).

4. In Section 6 (see Example 3) it is also demonstrated with the aid of symbolic computation performed in *Mathematica* that more-or-less uniformity (1.2) cannot be dropped in Theorem 2. Remarkably, by [6] or [17] the case of $\lambda = 1$ does not require more-or-less uniformity (1.2). Here, merely the general class of admissible samplings (1.1) is sufficient to ascertain (1.5) or (2.2). Noticeably, for the uniform sampling \mathcal{T}_u the coefficient $\beta = 1$ as introduced in (1.2).

5. Sufficient conditions enforcing $\psi_{3,i}$ to be a genuine reparameterization of J_i into \hat{J}_i are additionally formulated in the closing part of Section 4 (they are visualized in Figure 1). In general, the question of $\psi : [0, T] \rightarrow [0, \hat{T}]$ rendering a reparameterization is vital e.g. for estimating the length $d(\gamma) = \int_0^T \|\dot{\gamma}(t)\| dt$ of γ with $d(\hat{\gamma}) = \int_0^{\hat{T}} \|\hat{\gamma}'(\hat{t})\| d\hat{t}$ (see e.g. [2] and [6]). However, in certain circumstances, the non-parameterization cases (yielding loops in trajectory path $\hat{\gamma} \circ \psi$ over some segments J_i) may also be desirable for some specific applications like robot or airborne flying device trajectory planning (e.g. for inspecting electrical poles by drones).

6. Fitting reduced data Q_m (dense or sparse) constitutes an important task in computer graphics and vision, engineering, physics and in other applications like medical or biological image processing - see e.g. [4], [5], [11], [14] or [18]. The resulting abrupt deceleration in estimating $\hat{\gamma}_3 \circ \psi_3 - \gamma = O(\delta_m^{\alpha(\lambda)})$ dropping from $\alpha(1) = 4$ to $\alpha(\lambda) = 1$ (with $\lambda \in [0, 1)$) yields an interesting theoretical result complementing [17], Theorem 1 and [7]. However, the interpolation conditions $\hat{\gamma}(\hat{t}_i) = q_i$ are frequently accompanied with some extra collateral constraints (e.g. minimizing the curvature of the $\hat{\gamma}$ in curve modelling - see [12]). The exponential parameterization (1.3) may serve here as a possible tool in selecting the respective "optimal knots" $\hat{\mathcal{T}}_\lambda$. Indeed, upon relaxing a single parameter $\lambda \in [0, 1)$ at the cost of maintaining a slow order $\alpha(\lambda) = 1$ in γ estimation, an optimal $\lambda_{opt} \in [0, 1)$ can be found to reinforce the extra constraint in question. Evidently, such degree of freedom is lost once λ in (1.3) is fixed. In particular, the latter eventuates for $\lambda = 1$ frozen. Here despite resulting a fast convergence order in trajectory estimation (as in (1.5) or in (2.2)) no flexibility of adjusting optimal parameter λ enforcing additional constraint is left anymore.

3 The proof of Theorem 2

In this section the proof of Theorem 2 is given. In doing so recall first the Hadamard’s Lemma (see e.g. [16]):

Lemma 1. *Let $f : [a, b] \rightarrow \mathbb{E}^n$ be of class C^l , where $l \geq 1$ and assume that $f(t_0) = \mathbf{0}$, for some $t_0 \in (a, b)$. There exists a C^{l-1} function $g : [a, b] \rightarrow \mathbb{E}^n$ for which $f(t) = (t - t_0)g(t)$. Additionally, $g(t) = O(\frac{df}{dt})$.*

If function $f(t)$ has multiple zeros $t_0 \leq t_1 \leq \dots \leq t_k$ with $k + 1 \leq l$, then Hadamard’s Lemma applied $k + 1$ times yields:

$$f(t) = (t - t_0)(t - t_1)\dots(t - t_k)h(t), \tag{3.1}$$

where h is $C^{l-(k+1)}$ and $h = O(d^{k+1}f/dt^{k+1})$. From now on, to justify Theorem 2, the abbreviated notation for both cubics $\psi_{3,i} = \psi_i$ and $\check{\gamma}_{3,i} = \check{\gamma}_i$ is used.

Proof. The proof of Theorem 2 admits an arbitrary $\lambda \in [0, 1)$ and any more-or-less uniform sampling. In contrast, the special case $\lambda = 1$ (omitted here as already justified in [17]) extends to all admissible samplings (1.1) but is still restricted to $\gamma \in C^4$. Noticeably, for $\lambda = 1$ each cubic ψ_i defines a genuine reparameterization of J_i into \hat{J}_i - see [6] or [17].

Define now the following “error function” $f_i(t) = (\check{\gamma}_i \circ \psi_i)(t) - \gamma(t)$. Combining $f_i(t_{i+j}) = 0$ (for $j = 0, 1, 2, 3$) with Hadamard’s Lemma yields (see here (3.1)):

$$\begin{aligned} f_i(t) &= (t - t_i)(t - t_{i+1})(t - t_{i+2})(t - t_{i+3})[(\check{\gamma}_i \circ \psi_i)^{(4)}(t) - \gamma^{(4)}(t)] \\ &= O(\delta_m^4) \cdot O((\check{\gamma}_i \circ \psi_i)^{(4)}(t) - \gamma^{(4)}(t)). \end{aligned} \tag{3.2}$$

Since $\check{\gamma}_i$ and ψ_i are cubics the chain rule applied to (3.2) results in:

$$(\check{\gamma}_i \circ \psi_i)^{(4)}(t) = 6\check{\gamma}_i'''(\hat{t})\dot{\psi}_i^2(t)\ddot{\psi}_i(t) + 3\check{\gamma}_i''(\hat{t})\dot{\psi}_i^2(t) + 4\check{\gamma}_i''(\hat{t})\dot{\psi}_i(t)\ddot{\psi}_i(t) \tag{3.3}$$

with the respective derivatives over t or $\hat{t} = \psi_i(t)$ expressed by either dotted or apostrophed notations, respectively. Coupling the latter with (3.2) and $\gamma \in C^4$ defined over a compact set $[0, T]$ leads to:

$$\begin{aligned} f_i(t) &= O(\delta_m^4)[O(\check{\gamma}_i'''(\hat{t})\dot{\psi}_i^2(t)\ddot{\psi}_i(t) \\ &\quad + O(\check{\gamma}_i''(\hat{t})\dot{\psi}_i^2(t) + O(\check{\gamma}_i''(\hat{t})\dot{\psi}_i(t)\ddot{\psi}_i(t) + O(1))]. \end{aligned} \tag{3.4}$$

We pass now to the determination of the respective asymptotics for all contributing terms appearing in (3.4).

3.1 Estimation of derivatives of ψ_i

To examine the asymptotics of the derivatives of ψ_i , recall now Newton’s interpolation formula (see [1] or [19]) reading as:

$$\begin{aligned} \psi_i(t) &= \psi_i[t_i] + \psi_i[t_i, t_{i+1}](t - t_i) + \psi_i[t_i, t_{i+1}, t_{i+2}](t - t_i)(t - t_{i+1}) \\ &\quad + \psi_i[t_i, t_{i+1}, t_{i+2}, t_{i+3}](t - t_i)(t - t_{i+1})(t - t_{i+2}), \end{aligned}$$

over J_i . Hence

$$\begin{aligned} \dot{\psi}_i(t) &= \psi_i[t_i, t_{i+1}] + (2t - t_i - t_{i+1})\psi_i[t_i, t_{i+1}, t_{i+2}] + [(t - t_{i+1})(t - t_{i+2}) \\ &\quad + (t - t_i)(t - t_{i+1}) + (t - t_i)(t - t_{i+2})]\psi_i[t_i, t_{i+1}, t_{i+2}, t_{i+3}]. \end{aligned} \quad (3.5)$$

Consequently

$$\begin{aligned} \ddot{\psi}_i(t) &= 2\psi_i[t_i, t_{i+1}, t_{i+2}] + 2(3t - t_{i+2} - t_{i+1} - t_i)\psi_i[t_i, t_{i+1}, t_{i+2}, t_{i+3}], \\ \dddot{\psi}_i(t) &= 6\psi_i[t_i, t_{i+1}, t_{i+2}, t_{i+3}]. \end{aligned} \quad (3.6)$$

Thus by (3.5) and (3.6), the examination of the asymptotics of $\dot{\psi}_i$, $\ddot{\psi}_i$, $\dddot{\psi}_i$ over each J_i relies on estimating the asymptotic orders of divided differences $\psi_i[t_i, t_{i+1}]$, $\psi_i[t_i, t_{i+1}, t_{i+2}]$ and $\psi_i[t_i, t_{i+1}, t_{i+2}, t_{i+3}]$. Taylor's expansion combined with (1.3) (see also [9], where quadratics $\psi_{2,i}$ instead of cubics $\psi_{3,i}$ are analyzed) yields:

$$\begin{aligned} \psi_i[t_i, t_{i+1}] &= \frac{\psi_i(t_{i+1}) - \psi_i(t_i)}{t_{i+1} - t_i} = \frac{\|\gamma(t_{i+1}) - \gamma(t_i)\|^\lambda}{t_{i+1} - t_i} \\ &= (t_{i+1} - t_i)^{\lambda-1} + O((t_{i+1} - t_i)^{\lambda+1}) = O(\delta_m^{\lambda-1}). \end{aligned} \quad (3.7)$$

Note that since $\lambda-1 < 0$ the last step in (3.7) resorts to more-or-less uniformity since then $0 < (t_{i+1} - t_i)^{\lambda-1} \leq \beta^{\lambda-1} \delta_m^{1-\lambda}$. In addition, the justification of (3.7) relies also on the regularity of the curve γ . Indeed, such curves are reparameterizable to arc-length rendering $\|\dot{\gamma}(t)\| = 1$ (for each $t \in [0, T]$) - see [15] and [2]. Thus, the dot product $\langle \dot{\gamma}(t) | \ddot{\gamma}(t) \rangle \equiv 0$ nullifies slow terms in Taylor expansion of $\|\gamma(t_{i+1}) - \gamma(t_i)\|$. Similarly, for $j = 0, 1$ we have

$$\begin{aligned} \psi_i[t_{i+j+1}, t_{i+j+2}] &= (t_{i+j+2} - t_{i+j+1})^{\lambda-1} \\ &\quad + O((t_{i+j+2} - t_{i+j+1})^{\lambda+1}) = O(\delta_m^{\lambda-1}). \end{aligned} \quad (3.8)$$

The formula for $\psi_i[t_i, t_{i+1}, t_{i+2}]$ and $0 < (t_{i+j+1} - t_{i+j})(t_{i+2} - t_i)^{-1} < 1$ (for $j = 0, 1$) coupled together with (3.7) and (3.8) yields:

$$\begin{aligned} \psi_i[t_i, t_{i+1}, t_{i+2}] &= \frac{(t_{i+2} - t_{i+1})^{\lambda-1} - (t_{i+1} - t_i)^{\lambda-1}}{t_{i+2} - t_i} \\ &\quad + O((t_{i+2} - t_{i+1})^\lambda) + O((t_{i+1} - t_i)^\lambda) = O(\delta_m^{\lambda-2}). \end{aligned} \quad (3.9)$$

Again (1.2) is exploited while justifying the last step of (3.9). Analogously:

$$\begin{aligned} \psi_i[t_{i+1}, t_{i+2}, t_{i+3}] &= \frac{(t_{i+3} - t_{i+2})^{\lambda-1} - (t_{i+2} - t_{i+1})^{\lambda-1}}{t_{i+3} - t_{i+1}} \\ &\quad + O((t_{i+3} - t_{i+2})^\lambda) + O((t_{i+2} - t_{i+1})^\lambda) = O(\delta_m^{\lambda-2}). \end{aligned} \quad (3.10)$$

Thus by (3.9) and (3.10) the fourth divided difference

$$\begin{aligned} \psi_i[t_i, t_{i+1}, t_{i+2}, t_{i+3}] &= \frac{(t_{i+3} - t_{i+2})^{\lambda-1} - (t_{i+2} - t_{i+1})^{\lambda-1}}{(t_{i+3} - t_{i+1})(t_{i+3} - t_i)} \\ &\quad - \frac{(t_{i+2} - t_{i+1})^{\lambda-1} - (t_{i+1} - t_i)^{\lambda-1}}{(t_{i+2} - t_i)(t_{i+3} - t_i)} + \frac{O((t_{i+3} - t_{i+2})^{\lambda+1}) + O((t_{i+2} - t_{i+1})^{\lambda+1})}{(t_{i+3} - t_{i+1})(t_{i+3} - t_i)} \\ &\quad + \frac{O((t_{i+2} - t_{i+1})^{\lambda+1}) + O((t_{i+1} - t_i)^{\lambda+1})}{(t_{i+2} - t_i)(t_{i+3} - t_i)}. \end{aligned} \quad (3.11)$$

Upon observing $0 < (t_{i+j+2} - t_{i+j+1})^2((t_{i+3} - t_{i+1})(t_{i+3} - t_i))^{-1} < 1$ and $0 < (t_{i+j+1} - t_{i+j})^2((t_{i+2} - t_i)(t_{i+3} - t_i))^{-1} < 1$ (for $j = 0, 1$) formula (3.11) used together with more-or-less uniformity leads to:

$$\psi_i[t_i, t_{i+1}, t_{i+2}, t_{i+3}] = O(\delta_m^{\lambda-3}) + O(\delta_m^{\lambda-1}) = O(\delta_m^{\lambda-3}). \tag{3.12}$$

Coupling together (3.5), (3.7), (3.9) and (3.12) leads to:

$$\dot{\psi}_i(t) = O(\delta_m^{\lambda-1}) + O(\delta_m) \cdot O(\delta_m^{\lambda-2}) + O(\delta_m^2) \cdot O(\delta_m^{\lambda-3}) = O(\delta_m^{\lambda-1}). \tag{3.13}$$

Similarly by (3.6), (3.7), (3.9) and (3.12) the following holds:

$$\ddot{\psi}_i(t) = O(\delta_m^{\lambda-2}) + O(\delta_m) \cdot O(\delta_m^{\lambda-3}) = O(\delta_m^{\lambda-2}), \quad \dddot{\psi}_i(t) = O(\delta_3^{\lambda-3}). \tag{3.14}$$

The latter exploits the inequalities

$$|t - t_k| \leq t_{i+3} - t_i = (t_{i+1} - t_i) + (t_{i+2} - t_{i+1}) + (t_{i+3} - t_{i+2}) \leq 3\delta_m,$$

holding for each $t, t_k \in [t_i, t_{i+3}]$ (with $k = 0, 1, 2$).

For the special case of uniform samplings $\mathcal{T}_u = \{\frac{iT}{m}\}_{i=0}^m$ (where $\beta = 1$ - see (1.2)) by (3.7), (3.9) and (3.11) we have (for $\lambda \in [0, 1)$):

$$\begin{aligned} \psi_i[t_i, t_{i+1}] &= \delta_m^{\lambda-1} + O(\delta_m^{\lambda+1}), & \psi_i[t_i, t_{i+1}, t_{i+2}] &= O(\delta_m^\lambda), \\ \psi_i[t_i, t_{i+1}, t_{i+2}, t_{i+3}] &= O(\delta_m^{\lambda-1}). \end{aligned} \tag{3.15}$$

Consequently for $\beta = 1$, by (3.5), (3.6) and (3.15) we have (over each J_i):

$$\dot{\psi}_i(t) = \delta_m^{\lambda-1} + O(\delta_m^{\lambda+1}), \quad \ddot{\psi}_i(t) = O(\delta_m^\lambda) \quad \text{and} \quad \dddot{\psi}_i(t) = O(\delta_m^{\lambda-1}). \tag{3.16}$$

Hence by (3.16), $\dot{\psi}_i > 0$ asymptotically. Thus for uniform samplings $\{iT/m\}_{i=0}^m$ ψ_i defines a genuine reparameterization of J_i into \hat{J}_i , for sufficiently large m .

3.2 Estimation of derivatives of $\check{\gamma}_i$

We estimate now the derivatives of $\check{\gamma}_i : [t_i, t_{i+3}] \rightarrow \mathbb{E}^n$ present in (3.2) and (3.3). Again Newton's interpolation formula (see e.g. [1]) applied to $\check{\gamma}_i$ yields (for any $\hat{t} \in \mathbb{R}$):

$$\begin{aligned} \check{\gamma}_i(\hat{t}) &= \check{\gamma}_i[\hat{t}_i] + \check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}](\hat{t} - \hat{t}_i) + \check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}](\hat{t} - \hat{t}_i)(\hat{t} - \hat{t}_{i+1}) \\ &\quad + \check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}, \hat{t}_{i+3}](\hat{t} - \hat{t}_i)(\hat{t} - \hat{t}_{i+1})(\hat{t} - \hat{t}_{i+2}). \end{aligned}$$

Hence

$$\begin{aligned} \check{\gamma}_i''(\hat{t}) &= 2\check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}] + 2(3\hat{t} - \hat{t}_{i+2} - \hat{t}_{i+1} - t_i)\check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}, \hat{t}_{i+3}], \\ \check{\gamma}_i'''(\hat{t}) &= 6\check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}, \hat{t}_{i+3}]. \end{aligned} \tag{3.17}$$

As previously, the orders of $\check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}]$, $\check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}]$ and $\check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}, \hat{t}_{i+3}]$ need to be examined. In doing so, observe first that:

$$\check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}] = \frac{\check{\gamma}_i(\hat{t}_{i+1}) - \check{\gamma}_i(\hat{t}_i)}{\hat{t}_{i+1} - \hat{t}_i} = \frac{\gamma(t_{i+1}) - \gamma(t_i)}{\|\gamma(t_{i+1}) - \gamma(t_i)\|^\lambda}. \tag{3.18}$$

Taylor's expansion applied to γ yields:

$$\gamma(t_{i+1}) - \gamma(t_i) = (t_{i+1} - t_i) \left[\dot{\gamma}(t_i) + \frac{(t_{i+1} - t_i)}{2} \ddot{\gamma}(t_i) + O((t_{i+1} - t_i)^2) \right]. \quad (3.19)$$

Since $\|\dot{\gamma}(t)\| = 1$ (and thus $\langle \dot{\gamma}(t) | \dot{\gamma}(t) \rangle = 0$) we arrive at:

$$\begin{aligned} \hat{t}_{i+1} - \hat{t}_i &= \|\gamma(t_{i+1}) - \gamma(t_i)\|^\lambda = (t_{i+1} - t_i)^\lambda [1 + O((t_{i+1} - t_i)^2)]^{\frac{\lambda}{2}} \\ &= (t_{i+1} - t_i)^\lambda [1 + O((t_{i+1} - t_i)^2)]. \end{aligned} \quad (3.20)$$

The last step exploits $(1+x)^{\frac{\lambda}{2}} = 1 + \frac{\lambda}{2}x + O(x^2)$ (for x separated from -1). Hence by (3.18), (3.19) and (3.20) we obtain

$$\begin{aligned} \tilde{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}] &= \frac{(t_{i+1} - t_i)^{1-\lambda} \left[\dot{\gamma}(t_i) + \frac{(t_{i+1} - t_i)}{2} \ddot{\gamma}(t_i) + O((t_{i+1} - t_i)^2) \right]}{1 + O((t_{i+1} - t_i)^2)} \\ &= (t_{i+1} - t_i)^{1-\lambda} \left[\dot{\gamma}(t_i) + \frac{(t_{i+1} - t_i)}{2} \ddot{\gamma}(t_i) + O((t_{i+1} - t_i)^2) \right] \end{aligned} \quad (3.21)$$

as $(1+O(x))^{-1} = 1 + O(x)$ (for x separated from -1). Analogously

$$\begin{aligned} \tilde{\gamma}_i[\hat{t}_{i+1}, \hat{t}_{i+2}] &= (t_{i+2} - t_{i+1})^{1-\lambda} \left[\dot{\gamma}(t_{i+1}) \right. \\ &\quad \left. + \frac{(t_{i+2} - t_{i+1})}{2} \ddot{\gamma}(t_{i+1}) + O((t_{i+2} - t_{i+1})^2) \right]. \end{aligned} \quad (3.22)$$

By (3.21), (3.22) and $\hat{t}_{i+k+2} - \hat{t}_{i+k} = (\hat{t}_{i+k+2} - \hat{t}_{i+k+1}) + (\hat{t}_{i+k+1} - \hat{t}_{i+k})$ (for $k = 0, 1$) the asymptotic bound for $\tilde{\gamma}_i[\hat{t}_{i+k}, \hat{t}_{i+k+1}, \hat{t}_{i+k+2}] = a$

$$\begin{aligned} a &= \frac{(t_{i+k+2} - t_{i+k+1})^{1-\lambda} \left[\dot{\gamma}(t_{i+k+1}) + \frac{t_{i+k+2} - t_{i+k+1}}{2} \ddot{\gamma}(t_{i+k+1}) \right]}{(\hat{t}_{i+k+2} - \hat{t}_{i+k+1}) + (\hat{t}_{i+k+1} - \hat{t}_{i+k})} \\ &\quad - \frac{(t_{i+k+1} - t_{i+k})^{1-\lambda} \left[\dot{\gamma}(t_{i+k}) + \frac{t_{i+k+1} - t_{i+k}}{2} \ddot{\gamma}(t_{i+k}) \right]}{(\hat{t}_{i+k+2} - \hat{t}_{i+k+1}) + (\hat{t}_{i+k+1} - \hat{t}_{i+k})} \\ &\quad + \frac{O((t_{i+k+2} - t_{i+k+1})^2) + O((t_{i+k+1} - t_{i+k})^2)}{(\hat{t}_{i+k+2} - \hat{t}_{i+k+1}) + (\hat{t}_{i+k+1} - \hat{t}_{i+k})}. \end{aligned} \quad (3.23)$$

Using hybrids of (3.20) with (3.23) renders $\|\tilde{\gamma}_i[\hat{t}_{i+k}, \hat{t}_{i+k+1}, \hat{t}_{i+k+2}]\| = b$

$$\begin{aligned} b &\leq \frac{(t_{i+k+2} - t_{i+k+1})^{1-2\lambda} \left\| \dot{\gamma}(t_{i+k+1}) + \frac{t_{i+k+2} - t_{i+k+1}}{2} \ddot{\gamma}(t_{i+k+1}) \right\|}{1 + O((t_{i+k+2} - t_{i+k+1})^2)} \\ &\quad + \frac{(t_{i+k+1} - t_{i+k})^{1-2\lambda} \left\| \dot{\gamma}(t_{i+k}) + \frac{t_{i+k+1} - t_{i+k}}{2} \ddot{\gamma}(t_{i+k}) \right\|}{1 + O((t_{i+k+1} - t_{i+k})^2)} \\ &\quad + \frac{\|O((t_{i+k+2} - t_{i+k+1})^{2-\lambda})\|}{1 + O((t_{i+k+2} - t_{i+k+1})^2)} + \frac{\|O((t_{i+k+1} - t_{i+k})^{2-\lambda})\|}{1 + O((t_{i+k+1} - t_{i+k})^2)}. \end{aligned} \quad (3.24)$$

In the latter the asymptotic positivity of both factors from (3.20) is used as for the exponential parameterization (1.3) with $q_{i+1} \neq q_i$ we have $\hat{t}_l - \hat{t}_s > 0$ for $l > s$. Since $1 - 2\lambda < 2 - \lambda$ (for $\lambda \in [0, 1)$) and $f(x) = (1+x)^{-1} = 1 + O(x)$

(holding for x separated from -1), more-or-less uniformity of $\{t_i\}_{i=0}^m$ (needed when $1 - 2\lambda < 0$) combined with $\|\dot{\gamma}(t)\|^2 = \langle \dot{\gamma}(t) | \dot{\gamma}(t) \rangle \equiv 1$ leads to:

$$\|\check{\gamma}_i[\hat{t}_{i+k}, \hat{t}_{i+k+1}, \hat{t}_{i+k+2}]\| = O(\delta_m^{1-2\lambda}) + O(\delta_m^{2-\lambda}) = O(\delta_m^{1-2\lambda}),$$

which by Definition 3 reformulates into:

$$\check{\gamma}_i[\hat{t}_{i+k}, \hat{t}_{i+k+1}, \hat{t}_{i+k+2}] = O(\delta_m^{1-2\lambda}). \tag{3.25}$$

In order to examine the asymptotics of $\check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}, \hat{t}_{i+3}]$, formula (3.23) and $0 < \hat{t}_{i+3} - \hat{t}_i = (\hat{t}_{i+3} - \hat{t}_{i+2}) + (\hat{t}_{i+2} - \hat{t}_{i+1}) + (\hat{t}_{i+1} - \hat{t}_i)$ are combined together. Thus as previously for exponential parameterization (1.3) we have $(\hat{t}_{i+3} - \hat{t}_i)^{-1} \leq (\hat{t}_{i+l+1} - \hat{t}_{i+l})^{-1}$ (for $l = 0, 1, 2$). Hence by (3.23) the fourth divided difference

$$\begin{aligned} \check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}, \hat{t}_{i+3}] &= \frac{\check{\gamma}_i[\hat{t}_{i+1}, \hat{t}_{i+2}, \hat{t}_{i+3}] - \check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}]}{\hat{t}_{i+3} - \hat{t}_i} \\ &= \frac{1}{\hat{t}_{i+3} - \hat{t}_i} \left(\frac{(t_{i+3} - t_{i+2})^{1-\lambda} [\dot{\gamma}(t_{i+2}) + \frac{t_{i+3}-t_{i+2}}{2} \ddot{\gamma}(t_{i+2})]}{(\hat{t}_{i+3} - \hat{t}_{i+2}) + (\hat{t}_{i+2} - \hat{t}_{i+1})} \right. \\ &\quad - \frac{(t_{i+2} - t_{i+1})^{1-\lambda} [\dot{\gamma}(t_{i+1}) + \frac{t_{i+2}-t_{i+1}}{2} \ddot{\gamma}(t_{i+1})]}{(\hat{t}_{i+3} - \hat{t}_{i+2}) + (\hat{t}_{i+2} - \hat{t}_{i+1})} \\ &\quad + \frac{O((t_{i+3} - t_{i+2})^2)}{(\hat{t}_{i+3} - \hat{t}_{i+2}) + (\hat{t}_{i+2} - \hat{t}_{i+1})} + \frac{O((t_{i+2} - t_{i+1})^2)}{(\hat{t}_{i+3} - \hat{t}_{i+2}) + (\hat{t}_{i+2} - \hat{t}_{i+1})} \\ &\quad \left. - \frac{(t_{i+2} - t_{i+1})^{1-\lambda} ([\dot{\gamma}(t_{i+1}) + \frac{t_{i+2}-t_{i+1}}{2} \ddot{\gamma}(t_{i+1})] + [\dot{\gamma}(t_i) + \frac{t_{i+1}-t_i}{2} \ddot{\gamma}(t_i)])}{(\hat{t}_{i+2} - \hat{t}_{i+1}) + (\hat{t}_{i+1} - \hat{t}_i)} \right. \\ &\quad \left. + \frac{O((t_{i+2} - t_{i+1})^2)}{(\hat{t}_{i+2} - \hat{t}_{i+1}) + (\hat{t}_{i+1} - \hat{t}_i)} + \frac{O((t_{i+1} - t_i)^2)}{(\hat{t}_{i+2} - \hat{t}_{i+1}) + (\hat{t}_{i+1} - \hat{t}_i)} \right). \tag{3.26} \end{aligned}$$

Thus as for (3.24), formula (3.26) yields an upper bound for

$$\begin{aligned} \|\check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}, \hat{t}_{i+3}]\| &\leq \frac{(t_{i+3} - t_{i+2})^{1-3\lambda} \|\dot{\gamma}(t_{i+2}) + \frac{t_{i+3}-t_{i+2}}{2} \ddot{\gamma}(t_{i+2})\|}{1 + O((t_{i+3} - t_{i+2})^2)} \\ &\quad + 2 \frac{(t_{i+2} - t_{i+1})^{1-3\lambda} \|\dot{\gamma}(t_{i+1}) + \frac{t_{i+2}-t_{i+1}}{2} \ddot{\gamma}(t_{i+1})\|}{1 + O((t_{i+2} - t_{i+1})^2)} \\ &\quad + \frac{\|O((t_{i+3} - t_{i+2})^{2-2\lambda})\|}{1 + O((t_{i+3} - t_{i+2})^2)} + \frac{\|O((t_{i+2} - t_{i+1})^{2-2\lambda})\|}{1 + O((t_{i+2} - t_{i+1})^2)}. \end{aligned}$$

Hence as for proving (3.25), given $1 - 3\lambda < 2 - 2\lambda$ (for $\lambda \in [0, 1]$) we obtain:

$$\check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}, \hat{t}_{i+3}] = O(\delta_m^{1-3\lambda}).$$

Consequently by (3.17) the following holds:

$$\check{\gamma}_i'''(\hat{t}) = O(\delta_m^{1-3\lambda}). \tag{3.27}$$

Furthermore, again by (3.17) and (3.25) (for $\hat{t} \in \psi_i([t_i, t_{i+3}])$) we have:

$$\check{\gamma}_i''(\hat{t}) = O(\delta_m^{1-2\lambda}) + 2[(\hat{t} - \hat{t}_{i+2}) + (\hat{t} - \hat{t}_{i+1}) + (\hat{t} - \hat{t}_i)] \cdot O(\delta_m^{1-3\lambda}). \tag{3.28}$$

Thus by (3.13) and Lagrange Theorem for arbitrary $\hat{t} = \psi_i(t) \in \psi_i([t_i, t_{i+3}])$ (and for $j = 0, 1, 2$ and $t \in [t_i, t_{i+3}]$) one arrives at:

$$\hat{t} - \hat{t}_{i+j} = \frac{\psi_i(t) - \psi_i(t_{i+j})}{t - t_{i+j}}(t - t_{i+j}) = \dot{\psi}_i(\xi)(t - t_{i+j}) = O(\delta_m^\lambda). \quad (3.29)$$

Hence, coupling (3.28) with (3.29) results in:

$$\check{\gamma}_i''(\hat{t}) = O(\delta_m^{1-2\lambda}). \quad (3.30)$$

A special case of uniform samplings $\mathcal{T}_u = \{\frac{iT}{m}\}_{i=0}^m$ (with $\beta = 1$ in (1.2)) leads to some simplifications. Indeed upon incorporating into (3.23) both $t_{i+k+2} - t_{i+k+1} = t_{i+k+1} - t_{i+k} = \delta_m$ and $\dot{\gamma}(t_{i+k+1}) = \dot{\gamma}(t_{i+k}) + O((t_{i+k+1} - t_{i+k}))$ the term δ_m factorizes in the first two expressions accelerating the third divided differences (here $0 \leq 2 - 2\lambda \leq 2 - \lambda$ for $\lambda \geq 0$ and $k = 0, 1$) by order one to:

$$\check{\gamma}_i[\hat{t}_{i+k}, \hat{t}_{i+k+1}, \hat{t}_{i+k+2}] = O(\delta_m^{2-2\lambda}) + O(\delta_m^{2-\lambda}) = O(\delta_m^{2-2\lambda}). \quad (3.31)$$

A similar reduction follows in (3.26) upon using $\hat{t}_{i+3} - \hat{t}_{i+2} = \hat{t}_{i+2} - \hat{t}_{i+1} = \hat{t}_{i+1} - \hat{t}_i$ with $\dot{\gamma}(t_{i+k+1}) = \dot{\gamma}(t_{i+k}) + O(t_{i+k+1} - t_{i+k})$ (where $k = 0, 1$) rendering:

$$\check{\gamma}_i'''(\hat{t}) = 6\check{\gamma}_i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}, \hat{t}_{i+3}] = O(\delta_m^{2-3\lambda}) + O(\delta_m^{2-2\lambda}) = O(\delta_m^{2-3\lambda}). \quad (3.32)$$

Consequently the latter combined with (3.17), (3.29) and (3.31) yields:

$$\check{\gamma}_i''(\hat{t}) = O(\delta_m^{2-2\lambda}) + O(\delta_m^{2-3\lambda}) \cdot O(\delta_m^\lambda) = O(\delta_m^{2-2\lambda}). \quad (3.33)$$

3.3 Final estimation of differences $f_i = \check{\gamma}_i \circ \psi_i - \gamma$

We examine now the underlying asymptotics in $f_i = \check{\gamma}_i \circ \psi_i - \gamma$ uniformly over each J_i and for arbitrary $\lambda \in [0, 1)$ (see (3.4)). Indeed, upon feeding (3.13), (3.14), (3.27) and (3.30) into (3.4) one obtains:

$$\begin{aligned} f_i(t) &= O(\delta_m^4) \cdot [O(\delta_m^{1-3\lambda}) \cdot O(\delta_m^{2\lambda-2}) \cdot O(\delta_m^{\lambda-2}) + O(\delta_m^{1-2\lambda}) \cdot O(\delta_m^{2\lambda-4}) \\ &\quad + O(\delta_m^{1-2\lambda}) \cdot O(\delta_m^{\lambda-1}) \cdot O(\delta_m^{\lambda-3}) + O(1)], \end{aligned}$$

which ultimately leads to:

$$f_i(t) = O(\delta_m^4) \cdot [O(\delta_m^{-3}) + O(\delta_m^{-3}) + O(\delta_m^{-3}) + O(1)] = O(\delta_m^1).$$

The latter justifies the first claim (2.1) of Theorem 2.

The special case of uniform samplings addressed by (2.2) follows once (3.16), (3.32) and (3.33) are incorporated into (3.4) resulting in (uniformly over J_i):

$$\begin{aligned} f_i(t) &= O(\delta^4) \cdot [O(\delta_m^{2-3\lambda}) \cdot O(\delta_m^{2\lambda-2}) \cdot O(\delta_m^\lambda) + O(\delta_m^{2-2\lambda}) \cdot O(\delta_m^{2\lambda}) \\ &\quad + O(\delta_m^{2-2\lambda}) \cdot O(\delta_m^{\lambda-1}) \cdot O(\delta_m^{\lambda-1}) + O(1)], \end{aligned}$$

which in turn yields:

$$f_i(t) = O(\delta_m^4) \cdot [O(1) + O(\delta_m^2) + O(1) + O(1)] = O(\delta_m^4).$$

The proof is thus completed. \square

4 Sufficient conditions on reparameterization

In this section *several sufficient conditions* for the cubic ψ_i to yield a *genuine reparameterization* are formulated. The case of $\lambda = 1$ renders ψ_i as a parameterization for arbitrary admissible sampling (1.1) as proved in [17]. The importance of $\dot{\psi}_i > 0$ is outlined in Section 2 - see item e). Noticeably to enforce ψ_i as a reparameterization it suffices to let the quadratic $\dot{\psi}_i(t) = a_i t^2 + b_i t + c_i > 0$ over each J_i . The latter follows if e.g. (see Figure 1):

$$\begin{aligned}
 (i) \quad & a_i < 0 \quad \wedge \quad \dot{\psi}_i(t_i) > 0 && \wedge \quad \dot{\psi}_i(t_{i+3}) > 0, \\
 (ii) \quad & a_i > 0 \quad \wedge \quad \dot{\psi}_i(t_i) > 0 && \wedge \quad -b_i/2a_i < t_i, \\
 (iii) \quad & a_i > 0 \quad \wedge \quad -b_i/2a_i > t_{i+3} && \wedge \quad \dot{\psi}_i(t_{i+3}) > 0, \\
 (iv) \quad & a_i > 0 \quad \wedge \quad \dot{\psi}_i(-b_i/2a_i) > 0. && \tag{4.1}
 \end{aligned}$$

Clearly for a given collection of Q_m and \mathcal{T} , the testing of $\dot{\psi}_i > 0$ over different J_i can vary between constraints (i) – (iv) (or between any other ones). Assuming that we admit the subfamily of more-or-less uniform samplings (1.2) satisfying $0 < \beta_0 \leq \beta \leq 1$ (with some β_0 fixed) the conditions (4.1) can be expressed in terms of β_0 . A full treatment of solving (4.1) for arbitrary $\lambda \in [0, 1)$ exceeds the scope and page limit for this paper. Some hints can be found in [10], where the parameterization issue for piecewise-quadratic Lagrange interpolation $\hat{\gamma}_2$ based on Q_m and exponential parameterization (1.3) is thoroughly studied.

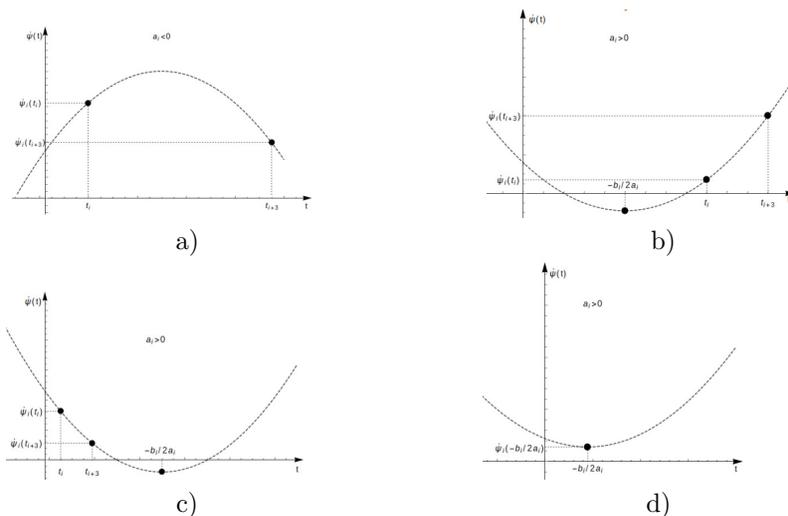


Figure 1. The plot of (4.1) with $\dot{\psi}_i > 0$ over $[t_i, t_{i+3}]$ for: a) (i), b) (ii), c) (iii) & d) (iv).

5 Sharpness of the asymptotics in trajectory estimation

The sharpness of the asymptotics from Theorem 2 (see also Definition 4) is proved below. In doing so, assume the straight line $\gamma_l(t) = (t, 0) \in \mathbb{E}^2$ to be

sampled more-or-less uniformly (over consecutive J_i) according to:

$$t_i = \frac{i}{m}, \quad t_i = \frac{i+1}{m}, \quad t_{i+2} = \frac{i+2}{m} \quad \text{and} \quad t_{i+3} = \frac{i+4}{m}. \quad (5.1)$$

Here $\delta_m = \frac{2}{m}$ and $\beta = \frac{1}{2}$ - see Definition 2. The reduced data $q_i = \gamma_l(t_i)$ are supplemented with $\hat{\mathcal{T}} \approx \mathcal{T}$ determined by the exponential parameterization (1.3). The analysis to follow (for $\lambda \in [0, 1)$) results in

$$f_i^l = \tilde{\gamma} \circ \psi_i - \gamma_l = K\delta_m + O(\delta_m^\eta),$$

holding asymptotically for some $\eta > 1$ (uniformly over each J_i). To prove the sharpness of Theorem 2 it suffices to show the latter for some particular $\bar{t} \in J_i$ and a special i . The constant K and those from $O(\delta_m^\eta)$ are independent of m . The sharpness of (2.1) is proved in the next example.

Example 1. Given $\bar{\delta} = \delta_m/2$ the sampling (5.1) over J_0 reads as:

$$t_0 = 0, \quad t_1 = \bar{\delta}, \quad t_2 = 2\bar{\delta} \quad \text{and} \quad t_3 = 4\bar{\delta}. \quad (5.2)$$

The exponential parameterization (1.3) for $\{\gamma_l(t_i)\}_{i=0}^3$ yields:

$$\hat{t}_0 = 0, \quad \hat{t}_1 = \bar{\delta}^\lambda, \quad \hat{t}_2 = 2\bar{\delta}^\lambda \quad \text{and} \quad \hat{t}_3 = (2 + 2^\lambda)\bar{\delta}^\lambda. \quad (5.3)$$

The corresponding divided differences for a cubic $\psi_0 : J_0 \rightarrow \mathbb{R}$ are equal to:

$$\psi_0[0] = 0, \quad \psi_0[0, \bar{\delta}] = \psi_0[\bar{\delta}, 2\bar{\delta}] = \bar{\delta}^{\lambda-1} \quad \text{and} \quad \psi_0[0, \bar{\delta}, 2\bar{\delta}] = 0. \quad (5.4)$$

Coupling now $\psi_0[\bar{\delta}, 2\bar{\delta}, 4\bar{\delta}] = (1/3)(2^{\lambda-1} - 1)\bar{\delta}^{\lambda-2}$ with (5.4) leads to:

$$\psi_0[0, \bar{\delta}, 2\bar{\delta}, 4\bar{\delta}] = \frac{1}{12}(2^{\lambda-1} - 1)\bar{\delta}^{\lambda-3}. \quad (5.5)$$

Thus by (3.5), (5.2), (5.4) and (5.5) we arrive at:

$$\psi_0(t) = \bar{\delta}^{\lambda-1}t + \frac{1}{12}(2^{\lambda-1} - 1)\bar{\delta}^{\lambda-3}t(t - \bar{\delta})(t - 2\bar{\delta}), \quad (5.6)$$

for any $t \in [0, 4\bar{\delta}] = J_0$. Substituting $\bar{t} = 3\bar{\delta} \in J_0$ into (5.6) yields:

$$\psi_0(3\bar{\delta}) = \left(\frac{5}{2} + 2^{\lambda-2}\right)\bar{\delta}^\lambda. \quad (5.7)$$

Similarly, by (5.3) for γ_l we obtain the following:

$$\check{\gamma}_0(0) = \vec{0}, \quad \check{\gamma}_0(\bar{\delta}^\lambda) = (\bar{\delta}, 0), \quad \check{\gamma}_0(2\bar{\delta}^\lambda) = (2\bar{\delta}, 0), \quad \check{\gamma}_0((2^\lambda + 2)\bar{\delta}^\lambda) = (4\bar{\delta}, 0).$$

Consequently:

$$\check{\gamma}_0[0] = \vec{0}, \quad \check{\gamma}_0[0, \bar{\delta}^\lambda] = \check{\gamma}_0[\bar{\delta}^\lambda, 2\bar{\delta}^\lambda] = (\bar{\delta}^{1-\lambda}, 0), \quad \check{\gamma}_0[0, \bar{\delta}^\lambda, 2\bar{\delta}^\lambda] = \vec{0}. \quad (5.8)$$

In addition, as $\check{\gamma}_0[2\bar{\delta}^\lambda, (2 + 2^\lambda)\bar{\delta}^\lambda] = (2^{1-\lambda}\bar{\delta}^{1-\lambda}, 0)$ we have:

$$\check{\gamma}_0[\bar{\delta}^\lambda, 2\bar{\delta}^\lambda, (2 + 2^\lambda)\bar{\delta}^\lambda] = \left(\frac{2^{1-\lambda} - 1}{2^\lambda + 1}\bar{\delta}^{1-2\lambda}, 0\right). \quad (5.9)$$

Hence upon combining now (5.8) with (5.9) we arrive at:

$$\check{\gamma}_0[0, \bar{\delta}^\lambda, 2\bar{\delta}^\lambda, (2 + 2^\lambda)\bar{\delta}^\lambda] = \left(\frac{2^{1-\lambda} - 1}{(2^\lambda + 1)(2 + 2^\lambda)} \bar{\delta}^{1-3\lambda}, 0 \right). \quad (5.10)$$

Again coupling together (5.3), (5.8) and (5.10) yields, for $\hat{t} = \psi_0(t)$ with $t \in [0, 4\bar{\delta}]$, the following formula:

$$\check{\gamma}_0(\hat{t}) = \left(\bar{\delta}^{1-\lambda}\hat{t} + \frac{2^{1-\lambda} - 1}{(2^\lambda + 1)(2 + 2^\lambda)} \bar{\delta}^{1-3\lambda}\hat{t}(\hat{t} - \bar{\delta}^\lambda)(\hat{t} - 2\bar{\delta}^\lambda), 0 \right). \quad (5.11)$$

By (5.7) and (5.11) the “error difference”:

$$f_0^l(3\bar{\delta}) = \check{\gamma}_0(\psi_0(3\bar{\delta})) - \gamma(3\bar{\delta}) = ((1/2)\rho(\lambda)\delta_m, 0),$$

where for $\lambda \in [0, 1)$ we have

$$\rho(\lambda) = \left(\frac{5}{2} + 2^{\lambda-2} \right) \left(1 + \left(\frac{1}{2} + 2^{\lambda-2} \right) \left(\frac{3}{2} + 2^{\lambda-2} \right) \frac{2^{1-\lambda} - 1}{(2^\lambda + 1)(2 + 2^\lambda)} \right) - 3. \quad (5.12)$$

The plots of the function ρ over $[0, 1)$ together with its magnification over $[0.899, 1)$ are presented in Figure 2. Evidently the coefficient $\rho(\lambda)$ standing along δ_m is non-vanishing. Noticeably $\rho(1) = 0$ and thus $f_0^l(3\bar{\delta}) = 0$.

A moment of reflection reveals that here both γ_l and $\check{\gamma}$ coincide (since $t_i = \hat{t}_i$) resulting in $f^l \equiv 0$ over $[0, T]$. By (5.12) a linear convergence order for f_0^l holds at $\hat{t} = 3\bar{\delta}$ which implies that f^l defined as a track-sum of f_i^l cannot be of faster order than a linear one (uniformly over $[0, T]$). On the other by (2.1) the difference f^l satisfies $O(\delta_m^1)$. Consequently the sharpness of (2.1) is proved (see also Definition 4). Note that the sharpness of (2.2) for $\lambda = 1$ and general admissible samplings (1.1) follows as a spin-off of calculations from Example 3.

□

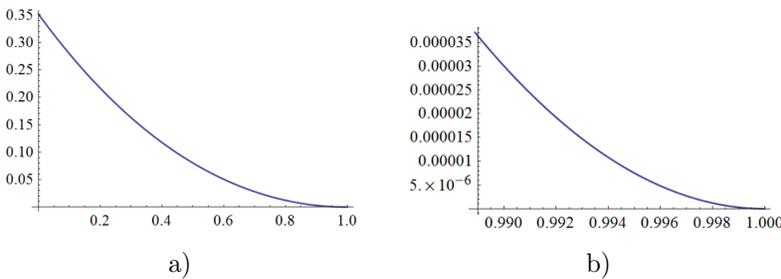


Figure 2. The plot of $\rho(\lambda) > 0$ from (5.12) over a) $[0, 1)$ and b) $[0.899, 1)$.

The next example justifies the sharpness of (2.2) (see also Definition 4) established for uniform sampling (here $\delta_m = T/m$). Without loss one may assume that $[0, T]$ is shifted left by fixed $\varepsilon > 0$ to $[-\varepsilon, T - \varepsilon]$ so that, asymptotically it contains all interpolation knots $t_i = -2\delta_m, t_{i+1} = -\delta_m, t_{i+2} = 0$ and $t_{i+3} = \delta_m$. In order to simplify the notation used for t_{i+j} (with $j = 0, 1, 2, 3$)

we set $i = 0$. The regular curve tested now coincides with $\gamma_q(t) = (t^4 + t, 0)$. The asymptotics of the respective difference $h^q = \tilde{\gamma} \circ \psi - \gamma_q$ is examined over $J_0 = [-2\delta_m, \delta_m]$. The detailed symbolic calculations used in Example 2 is stored in *Mathematica Notebook* under the URL link [20].

Example 2. The uniform knots (with $\delta_m = T/m$):

$$t_0 = -2\delta_m, \quad t_1 = -\delta_m, \quad t_2 = 0, \quad t_3 = \delta_m \quad (5.13)$$

together with the exponential parameterization (1.3) and $\{\gamma_q(t_i)\}_{i=0}^3$ give (asymptotically):

$$\begin{aligned} \hat{t}_0 &= 0, & \hat{t}_1 &= (\delta_m - 15\delta_m^4)^\lambda, & \hat{t}_2 &= [(\delta_m - 15\delta_m^4)^\lambda + (\delta_m - \delta_m^4)^\lambda], \\ \hat{t}_3 &= [(\delta_m - 15\delta_m^4)^\lambda + (\delta_m - \delta_m^4)^\lambda + (\delta_m + \delta_m^4)^\lambda]. \end{aligned} \quad (5.14)$$

The divided differences of order one or two for a cubic $\psi : [-2\delta_m, \delta_m] \rightarrow \mathbb{R}$ amount to:

$$\begin{aligned} \psi[-2\delta_m] &= 0, & \psi[-2\delta_m, -\delta_m] &= (\delta_m - 15\delta_m^4)^\lambda / \delta_m, \\ \psi[-\delta_m, 0] &= (\delta_m - \delta_m^4)^\lambda / \delta_m, & \psi[0, \delta_m] &= (\delta_m + \delta_m^4)^\lambda / \delta_m. \end{aligned} \quad (5.15)$$

Note that we omit here the subscript 0 in ψ_0 . The latter yields:

$$\begin{aligned} \psi[-2\delta_m, -\delta_m, 0] &= ((\delta_m - \delta_m^4)^\lambda - (\delta_m - 15\delta_m^4)^\lambda) / (2\delta_m^2), \\ \psi[-\delta_m, 0, \delta_m] &= ((\delta_m + \delta_m^4)^\lambda - (\delta_m - \delta_m^4)^\lambda) / (2\delta_m^2), \end{aligned} \quad (5.16)$$

which in turn renders:

$$\psi[-2\delta_m, -\delta_m, 0, \delta_m] = \frac{(\delta_m - 15\delta_m^4)^\lambda - 2(\delta_m - \delta_m^4)^\lambda + (\delta_m + \delta_m^4)^\lambda}{6\delta_m^3}. \quad (5.17)$$

Combining (3.5) with (5.13), (5.15), (5.16), (5.17) and upon using *Collect Mathematica function* (*Collect[Collect[$\psi[t, \lambda], \lambda], t]$) leads to:*

$$\begin{aligned} \psi(t) &= (\delta_m - 15\delta_m^4)^\lambda + (\delta_m - \delta_m^4)^\lambda + [(-(\delta_m - 15\delta_m^4)^\lambda + 5(\delta_m - \delta_m^4)^\lambda \\ &\quad + 2(\delta_m + \delta_m^4)^\lambda) / (6\delta_m)]t + [((\delta_m + \delta_m^4)^\lambda - (\delta_m - \delta_m^4)^\lambda) / (2\delta_m^2)]t^2 \\ &\quad + [((\delta_m - 15\delta_m^4)^\lambda - 2(\delta_m - \delta_m^4)^\lambda + (\delta_m + \delta_m^4)^\lambda) / (6\delta_m^3)]t^3, \end{aligned} \quad (5.18)$$

for an arbitrary $t \in [-2\delta_m, \delta_m]$. The evaluation of (5.18) at $\bar{t} = (\delta_m/2)$ with *Mathematica* symbolic computation *Collect[$\psi[\delta_m/2, \lambda], \lambda]$* results in:

$$\psi(\bar{t}) = (15/16)(\delta_m - 15\delta_m^4)^\lambda + (5/4)(\delta_m - \delta_m^4)^\lambda + (5/16)(\delta_m + \delta_m^4)^\lambda. \quad (5.19)$$

As γ_q and $\tilde{\gamma}$ coincide at interpolation points $\{q_{i+j}\}_{j=0}^3$ we have $\tilde{\gamma}(\hat{t}_2) = \vec{0}$ and:

$$\tilde{\gamma}(\hat{t}_0) = (-2\delta_m + 16\delta_m^4, 0), \quad \tilde{\gamma}(\hat{t}_1) = (-\delta_m + \delta_m^4, 0), \quad \tilde{\gamma}(\hat{t}_3) = (\delta_m + \delta_m^4, 0).$$

Hence and by (5.14) the respective divided differences for $\tilde{\gamma}$ are equal:

$$\begin{aligned} \tilde{\gamma}[\hat{t}_0] &= (-2\delta_m + 16\delta_m^4, 0), & \tilde{\gamma}[\hat{t}_0, \hat{t}_1] &= ((\delta_m - 15\delta_m^4)^{1-\lambda}, 0), \\ \tilde{\gamma}[\hat{t}_1, \hat{t}_2] &= ((\delta_m - \delta_m^4)^{1-\lambda}, 0), & \tilde{\gamma}[\hat{t}_2, \hat{t}_3] &= ((\delta_m + \delta_m^4)^{1-\lambda}, 0). \end{aligned} \quad (5.20)$$

Thus the third divided differences satisfy:

$$\begin{aligned} \tilde{\gamma}[\hat{t}_0, \hat{t}_1, \hat{t}_2] &= \left(\frac{-(\delta_m - 15\delta_m^4)^{1-\lambda} + (\delta_m - \delta_m^4)^{1-\lambda}}{(\delta_m - 15\delta_m^4)^\lambda + (\delta_m - \delta_m^4)^\lambda}, 0 \right), \\ \tilde{\gamma}[\hat{t}_1, \hat{t}_2, \hat{t}_3] &= \left(\frac{-(\delta_m - \delta_m^4)^{1-\lambda} + (\delta_m + \delta_m^4)^{1-\lambda}}{(\delta_m - \delta_m^4)^\lambda + (\delta_m + \delta_m^4)^\lambda}, 0 \right), \end{aligned} \tag{5.21}$$

which in turn with the aid of *Mathematica Simplify* function leads to:

$$\tilde{\gamma}[\hat{t}_0, \hat{t}_1, \hat{t}_2, \hat{t}_3] = \left(\frac{\frac{(\delta_m - 15\delta_m^4)^{1-\lambda} - (\delta_m - \delta_m^4)^{1-\lambda}}{(\delta_m - 15\delta_m^4)^\lambda + (\delta_m - \delta_m^4)^\lambda} + \frac{-(\delta_m - \delta_m^4)^{1-\lambda} + (\delta_m + \delta_m^4)^{1-\lambda}}{(\delta_m - \delta_m^4)^\lambda + (\delta_m + \delta_m^4)^\lambda}}{(\delta_m - 15\delta_m^4)^\lambda + (\delta_m - \delta_m^4)^\lambda + (\delta_m + \delta_m^4)^\lambda}, 0 \right). \tag{5.22}$$

Applying (5.14), (5.19), (5.20), (5.21) and (5.22) with Newton's interpolation formula yields for $h^q(\bar{t}) = (\tilde{\gamma} \circ \psi - \gamma_q)(\bar{t})$ (with $\hat{t} = \psi(\bar{t})$) the following:

$$\begin{aligned} h^q(\bar{t}) &= \left(-2\delta_m + 16\delta_m^4 + \bar{t}(\delta_m - 15\delta_m^4)^{1-\lambda} \right. \\ &\quad + \frac{\bar{t}(\bar{t} - (\delta_m - 15\delta_m^4)^\lambda)((\delta_m - \delta_m^4)^{1-\lambda} - (\delta_m - 15\delta_m^4)^{1-\lambda})}{(\delta_m - 15\delta_m^4)^\lambda + (\delta_m - \delta_m^4)^\lambda} \\ &\quad + \frac{\bar{t}(\bar{t} - (\delta_m - 15\delta_m^4)^\lambda)(\bar{t} - (\delta_m - 15\delta_m^4)^\lambda - (\delta_m - \delta_m^4)^\lambda)}{(\delta_m - 15\delta_m^4)^\lambda + (\delta_m - \delta_m^4)^\lambda + (\delta_m + \delta_m^4)^\lambda} \\ &\quad \left. \times \left(\frac{(\delta_m - 15\delta_m^4)^{1-\lambda} - (\delta_m - \delta_m^4)^{1-\lambda}}{(\delta_m - 15\delta_m^4)^\lambda + (\delta_m - \delta_m^4)^\lambda} + \frac{(\delta_m + \delta_m^4)^{1-\lambda} - (\delta_m - \delta_m^4)^{1-\lambda}}{(\delta_m - \delta_m^4)^\lambda + (\delta_m + \delta_m^4)^\lambda} \right), 0 \right). \end{aligned} \tag{5.23}$$

Setting $\lambda = 1$ or $\lambda = 0$ in (5.23) and applying *Mathematica Simplify* function leads to $h^q(\bar{t}) = \frac{15}{16}\delta_m^4$. The latter proves the sharpness of (2.2) for uniformly sampled regular curves and two special cases of either $\lambda = 1$ or $\lambda = 0$. In order to recalculate (5.23) for other $\lambda \in (0, 1)$, *Series*[$f_i[\delta_m, \{\delta_m, 0, 3\}]$] in *Mathematica* is used to expand the expressions f_i (for $i = 1, 2, 3$) into Taylor series (up to the 3rd order at 0) yielding $f_1(\delta_m) = (1 - 15\delta_m^3)^\lambda = 1 - 15\lambda\delta_m^3 + O(\delta_m^4)$, $f_2(\delta_m) = (1 - \delta_m^3)^\lambda = 1 - \lambda\delta_m^3 + O(\delta_m^4)$, and $f_3(\delta_m) = (1 + \delta_m^3)^\lambda = 1 + \lambda\delta_m^3 + O(\delta_m^4)$. The latter passed into (5.14) yields:

$$\begin{aligned} \hat{t}_1 &= \delta_m^\lambda (1 - 15\lambda\delta_m^3 + O(\delta_m^4)), \\ \hat{t}_2 &= \delta_m^\lambda (1 - 15\lambda\delta_m^3 + O(\delta_m^4)) + \delta_m^\lambda (1 - \lambda\delta_m^3 + O(\delta_m^4)), \\ \hat{t}_3 &= \delta_m^\lambda (1 - 15\lambda\delta_m^3 + O(\delta_m^4)) + \delta_m^\lambda (1 + O(\delta_m^4)). \end{aligned} \tag{5.24}$$

Define now in *Mathematica* for the first coordinate of $h^q(t) = \tilde{\gamma} \circ \psi - \gamma_q$ the function:

$$\text{fun}[\lambda] := \tilde{\gamma}[\text{Factor}[\text{Collect}[\psi[\delta_m/2, \lambda], \lambda], \lambda] - ((\delta_m/2)^4 + \delta_m/2)]. \tag{5.25}$$

Upon passing (5.24) into (5.25) (without *Simplify*) we obtain that

$$\begin{aligned} \text{fun}(\lambda) &= \\ &= \left(\frac{255}{16} + \frac{15}{8}(14 - 14\lambda) - 15\lambda + \frac{5}{8}(-6 + 6\lambda) + \frac{5}{2}(-15 + 15\lambda) \right) \delta_m^4 + O(\delta_m^5), \end{aligned}$$

or with *Simplify* we have $\text{fun}(\lambda) = \frac{15}{16}\delta_m^4 + O(\delta_m^5)$ - see also the *Mathematica Notebook* under the *URL link* [20]. The latter justifies the sharpness of (2.2) for the remaining $\lambda \in (0, 1)$ and any uniformly sampled $\gamma \in C^4$. \square

6 Necessity of more-or-less uniformity

We demonstrate now that more-or-less-uniformity for (2.1) (with $\lambda \in [0, 1)$) is essential and cannot be omitted in Theorem 2. A simple inspection shows that the admissible sampling (6.1) does not satisfy more-or-less uniformity (1.2) (with $\delta_m \rightarrow 0$). As previously we simplify the notation used for t_{i+j} (with $j = 0, 1, 2, 3$) by setting $i = 0$. The asymptotics of $h^q(t) = (\tilde{\gamma} \circ \psi)(t) - \gamma_q(t)$ is examined over the sub-interval $J_0 = [-\delta_m, \delta_m]$. The detailed symbolic computations used in Example 3 are conducted in *Mathematica Notebook* stored under the *URL link* [21].

Example 3. Consider now the following non-more-or-less uniform sampling:

$$t_0 = -\delta_m, \quad t_1 = 0, \quad t_2 = \delta_m^2 \quad \text{and} \quad t_3 = \delta_m. \quad (6.1)$$

The exponential parameterization (1.3) applied to $\{\gamma_q(t_i)\}_{i=0}^3$ and (6.1) yields (asymptotically):

$$\begin{aligned} \hat{t}_0 &= 0, & \hat{t}_1 &= (\delta_m - \delta_m^4)^\lambda, & \hat{t}_2 &= (\delta_m - \delta_m^4)^\lambda + (\delta_m^2 + \delta_m^8)^\lambda, \\ \hat{t}_3 &= (\delta_m - \delta_m^4)^\lambda + (\delta_m - \delta_m^2 + \delta_m^4 - \delta_m^8)^\lambda + (\delta_m^2 + \delta_m^8)^\lambda. \end{aligned} \quad (6.2)$$

The corresponding divided differences for a cubic $\psi : J_0 \rightarrow \mathbb{R}$ are equal to:

$$\begin{aligned} \psi[-\delta_m] &= 0, & \psi[-\delta_m, 0] &= (\delta_m - \delta_m^4)^\lambda / \delta_m, & \psi[0, \delta_m^2] &= (\delta_m^2 + \delta_m^8)^\lambda / \delta_m^2, \\ \psi[\delta_m^2, \delta_m] &= (\delta_m - \delta_m^2 + \delta_m^4 - \delta_m^8)^\lambda / (\delta_m - \delta_m^2). \end{aligned} \quad (6.3)$$

Hence latter yields:

$$\begin{aligned} \psi[-\delta_m, 0, \delta_m^2] &= -(\delta_m (\delta_m - \delta_m^4)^\lambda + (\delta_m^2 + \delta_m^8)^\lambda) / (1 + \delta_m) \delta_m^3, \\ \psi[0, \delta_m^2, \delta_m] &= -\frac{\delta_m (\delta_m - \delta_m^2 + \delta_m^4 - \delta_m^8)^\lambda - (\delta_m^2 + \delta_m^8)^\lambda + \delta_m (\delta_m^2 + \delta_m^8)^\lambda}{(-1 + \delta_m) \delta_m^3}, \end{aligned} \quad (6.4)$$

which with *Mathematica Factor* [$\psi[-\delta_m, 0, \delta_m^2, \delta_m]$] results in

$$\begin{aligned} \psi[-\delta_m, 0, \delta_m^2, \delta_m] &= (\delta_m (\delta_m - \delta_m^4)^\lambda + \delta_m^2 (\delta_m - \delta_m^4)^\lambda \\ &\quad - \delta_m (\delta_m - \delta_m^2 + \delta_m^4 - \delta_m^8)^\lambda + 2(\delta_m^4 + \delta_m^8)^\lambda - \delta_m^2 (\delta_m - \delta_m^2 + \delta_m^4 - \delta_m^8)^\lambda \\ &\quad - \delta_m (\delta_m^2 + \delta_m^8)^\lambda - \delta_m^2 (\delta_m^2 + \delta_m^8)^\lambda) / (2(-1 + \delta_m)(1 + \delta_m) \delta_m^4). \end{aligned} \quad (6.5)$$

Combining (3.5) with (6.1), (6.3), (6.4), (6.5) with *Collect* [*Factor* [$\psi[t, \lambda]$], t] yields (where $\kappa_1(\delta_m) = (\delta_m - \delta_m^2 + \delta_m^4 - \delta_m^8)^\lambda$ and $\kappa_2(\delta_m) = (\delta_m^2 + \delta_m^8)^\lambda$ over J_0 the following

$$\begin{aligned} \psi(t) &= \left[(\delta_m - \delta_m^4)^\lambda (-2\delta_m^4 + 2\delta^6) + t^3 \left((\delta_m - \delta_m^4)^\lambda (-\delta_m + \delta_m^2) \right. \right. \\ &\quad \left. \left. - \kappa_1(\delta_m)(\delta_m + \delta_m^2) + \kappa_2(\delta_m)(2 - \delta_m - \delta_m^2) \right) + t^2 \left((\delta_m^2 - \delta_m^4) \right. \right. \\ &\quad \left. \left. \times \left((\delta_m - \delta_m^4)^\lambda - \kappa_1(\delta_m) - \kappa_2(\delta_m) \right) \right) + t \left((\delta_m - \delta_m^4)^\lambda (-\delta_m^4 + \delta_m^5) \right) \right] \end{aligned}$$

$$+ \kappa_1(\delta_m)(\delta_m^4 + \delta_m^5) + \kappa_2(\delta_m)(-2\delta_m^2 + \delta^4 + \delta_m^5)] / (2(-1 + \delta_m)(1 + \delta_m)\delta_m^4). \quad (6.6)$$

The evaluation of (6.6) at $\bar{t} = (\delta_m/2)$ with $Factor[\psi[\delta/2, \lambda]]$ results in:

$$\begin{aligned} \psi(\bar{t}) = & \frac{3 \left((\delta_m - \delta_m^4)^\lambda (-5\delta_m - \delta_m^2 + 6\delta_m^3) - \kappa_1(\delta_m)(\delta_m - \delta_m^2 - 2\delta_m^3) \right)}{16(-1 + \delta_m)\delta_m(1 + \delta_m)} \\ & - 3\kappa_2(\delta_m)(2 + \delta_m - \delta_m^2 - 2\delta_m^3) / (16(-1 + \delta_m)\delta_m(1 + \delta_m)). \end{aligned} \quad (6.7)$$

Since γ_q and $\tilde{\gamma}$ coincide at interpolation points $\{q_{i+j}\}_{j=0}^3$ we have $\gamma(\hat{t}_1) = \vec{0}$,

$$\tilde{\gamma}(\hat{t}_0) = (-\delta_m + \delta_m^4, 0), \quad \tilde{\gamma}(\hat{t}_2) = (\delta_m^2 + \delta_m^8, 0), \quad \tilde{\gamma}(\hat{t}_3) = (\delta_m + \delta_m^4, 0).$$

Hence by (6.2) the respective divided differences for $\tilde{\gamma}$ are equal:

$$\begin{aligned} \tilde{\gamma}[\hat{t}_0] &= (-\delta_m + \delta_m^4, 0), \quad \tilde{\gamma}[\hat{t}_0, \hat{t}_1] = ((\delta_m - \delta_m^4)^{1-\lambda}, 0), \quad (6.8) \\ \tilde{\gamma}[\hat{t}_1, \hat{t}_2] &= ((\delta_m^2 + \delta_m^8)^{1-\lambda}, 0), \quad \tilde{\gamma}[\hat{t}_2, \hat{t}_3] = ((\delta_m - \delta_m^2 + \delta_m^4 - \delta_m^8)^{1-\lambda}, 0). \end{aligned}$$

Consequently we obtain:

$$\begin{aligned} \tilde{\gamma}[\hat{t}_0, \hat{t}_1, \hat{t}_2] &= \left(\frac{-(\delta_m - \delta_m^4)^{1-\lambda} + (\delta_m^2 + \delta_m^8)^{1-\lambda}}{(\delta_m - \delta_m^4)^\lambda + (\delta_m^2 + \delta_m^8)^\lambda}, 0 \right), \\ \tilde{\gamma}[\hat{t}_1, \hat{t}_2, \hat{t}_3] &= \left(\frac{(\delta_m - \delta_m^2 + \delta_m^4 - \delta_m^8)^{1-\lambda} - (\delta_m^2 + \delta_m^8)^{1-\lambda}}{(\delta_m - \delta_m^2 + \delta_m^4 - \delta_m^8)^\lambda + (\delta_m^2 + \delta_m^8)^\lambda}, 0 \right), \end{aligned} \quad (6.9)$$

which in turn leads to ($\lambda_1 = 1 - \lambda$):

$$\tilde{\gamma}[\hat{t}_0, \hat{t}_1, \hat{t}_2, \hat{t}_3] = \left(\frac{(\delta_m - \delta_m^4)^{\lambda_1} - (\delta_m^2 + \delta_m^8)^{\lambda_1} + (\delta_m - \delta_m^2 + \delta_m^4 - \delta_m^8)^{\lambda_1} - (\delta_m^2 + \delta_m^8)^{\lambda_1}}{(\delta_m - \delta_m^4)^\lambda + (\delta_m^2 + \delta_m^8)^\lambda} + \frac{(\delta_m - \delta_m^2 + \delta_m^4 - \delta_m^8)^{\lambda_1} - (\delta_m^2 + \delta_m^8)^{\lambda_1}}{(\delta_m - \delta_m^2 + \delta_m^4 - \delta_m^8)^\lambda + (\delta_m^2 + \delta_m^8)^\lambda}, 0 \right).$$

Incorporating (6.2), (6.7), (6.8), (6.9) and the latter into Newton’s interpolation formula yields for

$$\begin{aligned} h^q(\bar{t}) &= (\tilde{\gamma} \circ \psi - \gamma_q)(\bar{t}) = \left(-\frac{3\delta_m}{2} + \frac{15\delta_m^4}{16} + (\delta_m - \delta_m^4)^{1-\lambda} \bar{t} \right. \\ &+ \frac{\left((\delta_m^2 + \delta_m^8)^{1-\lambda} - (\delta_m - \delta_m^4)^{1-\lambda} \right) \bar{t} \left(\bar{t} - (\delta_m - \delta_m^4)^\lambda \right)}{(\delta_m - \delta_m^4)^\lambda + (\delta_m^2 + \delta_m^8)^\lambda} \\ &+ \bar{t} \left(\bar{t} - (\delta_m - \delta_m^4)^\lambda \right) \left(\bar{t} - (\delta_m - \delta_m^4)^\lambda - (\delta_m^2 + \delta_m^8)^\lambda \right) \\ &\times \left. \frac{(\delta_m - \delta_m^4)^{1-\lambda} - (\delta_m^2 + \delta_m^8)^{1-\lambda} + (\delta_m - \delta_m^2 + \delta_m^4 - \delta_m^8)^{1-\lambda} - (\delta_m^2 + \delta_m^8)^{1-\lambda}}{(\delta_m - \delta_m^4)^\lambda + (\delta_m^2 + \delta_m^8)^\lambda} + \frac{(\delta_m - \delta_m^2 + \delta_m^4 - \delta_m^8)^{1-\lambda} - (\delta_m^2 + \delta_m^8)^{1-\lambda}}{(\delta_m - \delta_m^2 + \delta_m^4 - \delta_m^8)^\lambda + (\delta_m^2 + \delta_m^8)^\lambda} \right), \end{aligned} \quad (6.10)$$

where $\bar{t} = \psi(\bar{t})$. Setting now e.g. $\lambda = 0$ in (6.10) and using *Mathematica Simplify* to $h^q(\bar{t})$ leads to (see *Mathematica Notebook* under the URL link [21]):

$$h^q(\bar{t}) = \frac{-144 + 432\delta_m + 328\delta_m^2 + 296\delta_m^3 - 5437\delta_m^4 - 2757\delta_m^5 + 17663\delta_m^6}{8192\delta_m^2 (-1 + \delta_m^2)^3} + \frac{7605\delta_m^7 - 25274\delta_m^8 - 8438\delta_m^9 + 8577\delta_m^{10} + 9801\delta_m^{11}}{8192\delta_m^2 (-1 + \delta_m^2)^3} + \frac{13845\delta_m^{12} - 5895\delta_m^{13} - 19566\delta_m^{14} + 1116\delta_m^{15} + 7560\delta_m^{16}}{8192\delta_m^2 (-1 + \delta_m^2)^3}.$$

Visibly the latter is of the sharp order:

$$h^q(\bar{t}) = (-144/8192)\delta_m^{-2} + O(\delta_m^{-1}),$$

not coinciding with the linear asymptotics from Theorem 2 established for any $\lambda \in [0, 1)$ and arbitrary more-or-less uniform sampling. Such deceleration effect is due to the fact that sampling (6.1) does not satisfy (1.2). A similar sharp asymptotics with the slowest term $K_\lambda \delta_m^{-2}$ (with $K_\lambda \neq 0$) follows upon substituting any $\lambda \in [0, 1)$ into (6.10) and applying *Mathematica Simplify* function. Due to the page limitation the *Mathematica Notebook* is attached (see ([21])) so that the latter can be verified upon performing symbolic computation for any fixed $\lambda \in [0, 1)$. Note also that feeding $\lambda = 1$ into (6.10) results in:

$$h^q(\bar{t}) = \frac{3}{16}(1 - 2\delta_m)\delta_m^4,$$

which yields an exact quartic convergence order in trajectory estimation at $\bar{t} = (\delta_m/2)$. The latter justifies sharpness of Theorem 2 claimed also for $\lambda = 1$ and the general class of admissible samplings. Evidently, though (6.1) does not fulfill more-or-less uniformity it still complies however with (1.1). \square

7 Experiments

We verify now numerically the sharpness of the asymptotics established in Theorem 2 for some regular 2D and 3D curves. All tests are carried out in *Mathematica 10.0* (see [22]) and resort to either uniform or more-or-less uniform samplings (with $t_i \in [0, 1]$ and for $i = 0, 1, \dots, m$) defined according to:

$$t_i = \begin{cases} i/m + 1/(2m), & \text{for } i = 4k + 1, \\ i/m - 1/(2m), & \text{for } i = 4k + 3, \\ i/m, & \text{for } i \text{ even,} \end{cases} \quad (7.1)$$

and

$$(i) \ t_i = i/m + (-1)^{i+1}/(3m), \quad (ii) \ t_i = i/m. \quad (7.2)$$

Here, $\beta = 1$ for (7.2)(ii), $\beta = (1/3)$ for (7.1) and $\beta = (1/5)$ for (7.2)(i) - see Definition 2. For a given m , the error E_m , between two continuous functions γ and $\hat{\gamma}_3$ reads as $E_m = \max_{[0,1]} \|(\hat{\gamma}_3 \circ \psi_3)(t) - \gamma(t)\|$. To estimate numerically

the convergence order $\alpha(\lambda)$ in trajectory approximation, first $\{E_m\}_{m=m_{min}}^{m_{max}}$ is calculated, where m_{min} and m_{max} are sufficiently large fixed constants. Next a linear regression yielding a function $y(x) = \bar{\alpha}(\lambda)x + b$ is applied to $\{(\log(m), -\log(E_m))\}_{m=m_{min}}^{m_{max}}$. Mathematica built-in function *LinearModelFit* extracts a coefficient $\bar{\alpha}(\lambda) \approx \alpha(\lambda)$. The tests conducted here use the following three different C^∞ regular curves: a spiral γ_{sp} and a cubic γ_o in \mathbb{E}^2 and a Steinmetz curve γ_{st} in \mathbb{E}^3 .

Example 4. Consider a planar spiral $\gamma_{sp} : [0, 1] \rightarrow \mathbb{E}^2$ defined by $\gamma_{sp}(t) = ((0.2 + t) \cos(\pi(1 - t)), (0.2 + t) \sin(\pi(1 - t)))$ and sampled according to either (7.1) or (7.2). The numerical results for $\bar{\alpha}(\lambda) \approx \alpha(\lambda)$ listed in Table 1 for different $\lambda \in [0, 1]$, render the asymptotics consistent with this from Theorem 2. \square

Table 1. $\bar{\alpha}(\lambda) \approx \alpha(\lambda)$ for γ_{sp} sampled as in (7.1) and (7.2) with $m \in \{96, \dots, 120\}$.

λ	0.0	0.1	0.3	0.5	0.7	0.9	1.0
$\bar{\alpha}(\lambda)$ for (7.1)	1.005	1.009	1.021	1.038	1.076	1.269	4,006
$\bar{\alpha}(\lambda)$ for (7.2)(i)	1.028	1.028	1.026	1.023	1.015	1.007	4.071
$\alpha(\lambda)$ in Theorem 2	1.0	1.0	1.0	1.0	1.0	1.0	4.0
$\bar{\alpha}(\lambda)$ for (7.2)(ii)	4.027	4.020	4.013	4.037	4.047	4.060	4.067
$\alpha(\lambda)$ in Theorem 2	4.0	4.0	4.0	4.0	4.0	4.0	4.0

Example 5. Assume now a regular Steinmetz 3D curve $\gamma_{st} : [0, 1] \rightarrow \mathbb{E}^3$ defined as $\gamma_{st}(t) = (\cos(2\pi t), \sin(2\pi t), \sqrt{1.2^2 - (\sin(2\pi t))^2})$ is sampled along either (7.1) or (7.2). The computed $\bar{\alpha}(\lambda) \approx \alpha(\lambda)$ for various $\lambda \in [0, 1]$ are presented in Table 2. Thus the sharpness of (2.1) and (2.2) is numerically confirmed. \square

Table 2. $\bar{\alpha}(\lambda) \approx \alpha(\lambda)$ for γ_{st} sampled as in (7.1) and (7.2) with $m \in \{72, \dots, 162\}$.

λ	0.0	0.1	0.3	0.5	0.7	0.9	1.0
$\bar{\alpha}(\lambda)$ for (7.1)	1.000	1.000	1.003	1.001	1.008	1.200	4.981**
$\bar{\alpha}(\lambda)$ for (7.2)(i)	0.993	0.994	0.996	0.998	1.009	1.186	3.961**
$\alpha(\lambda)$ in Theorem 2	1.0	1.0	1.0	1.0	1.0	1.0	4.0
$\bar{\alpha}(\lambda)$ for (7.2)(ii)*	3.977	3.972	3.965	3.950	3.954	3.950	3.948
$\alpha(\lambda)$ in Theorem 2	4.0	4.0	4.0	4.0	4.0	4.0	4.0

*for $159 \leq m \leq 201$ **for $201 \leq m \leq 240$

Example 6. Finally, consider a planar regular curve $\gamma_o : [0, 1] \rightarrow \mathbb{E}^2$ given by $\gamma_o(t) = (\cos^2(t), 1.5^t)$ and sampled as previously according to either (7.1) or

Table 3. $\bar{\alpha}(\lambda) \approx \alpha(\lambda)$ for γ_o sampled as in (7.1) and (7.2) with $m \in \{72, \dots, 162\}$.

λ	0.0	0.1	0.3	0.5	0.7	0.9	1.0
$\bar{\alpha}(\lambda)$ for (7.1)	0.998	0.998	0.998	0.999	0.999	1.005	3.91**
$\bar{\alpha}(\lambda)$ for (7.2)(i)	0.999	0.999	0.999	0.999	1.000	1.010	3.94*
$\alpha(\lambda)$ in Theorem 2	1.0	1.0	1.0	1.0	1.0	1.0	4.0
$\bar{\alpha}(\lambda)$ for (7.2)(ii)	3.999	3.943	3.915	3.952*	3.947*	3.946*	3.95*
$\alpha(\lambda)$ in Theorem 2	4.0	4.0	4.0	4.0	4.0	4.0	4.0

*for $159 \leq m \leq 201$ **for $330 \leq m \leq 387$

(7.2). The respective estimates $\bar{\alpha}(\lambda)$ of $\alpha(\lambda)$ (for varying $\lambda \in [0, 1]$) are presented in Table 3. Visibly, the asymptotics established in Theorem 2 (together with its sharpness) is as above, numerically verified in affirmative. \square

8 Conclusions

The main results and the motivation of this work are fully listed in Section 2 (see items a)–f)). The principal findings can be summarized as follows:

Section 3 proves Theorem 2 which extends Theorem 1 holding merely for $\lambda = 1$. The combination of the latter yields a surprising abrupt discontinuity in convergence orders while estimating γ with $\hat{\gamma}_3 \circ \psi_3$. Here a fast quartic order $\alpha(1) = 4$ drops to the linear one $\alpha(\lambda) = 1$ holding for any $\lambda \in [0, 1]$ incorporated into the exponential parameterization (1.3).

Section 4 formulates several sufficient conditions for $\psi_3 : [0, T] \rightarrow [0, \hat{T}]$ to be a reparameterization (see (4.1)). Geometrical meaning of (4.1) is also given.

Section 5 justifies the sharpness of (2.1) and (2.2) with the aid of non-trivial analytic and symbolic computations in accordance with Definition 4.

Section 6 justifies the necessity of more-or-less uniformity (2) in proving Theorem 2. The analytic argument combined with symbolic computation is employed. Additionally, the case of $\lambda = 1$ relies merely on a general class of admissible samplings (1.1) as also confirmed herein.

Section 7 verifies independently with the aid of numerical tests performed in *Mathematica* the sharpness of the asymptotics established in Theorem 2.

Section 2 specifies the main motivation standing behind this paper including desirable parameterization or non-parametrization cases of ψ_3 . The related literature in the context of specific applications to fit Q_m in conjunction with exponential parametrization is also listed.

Future work may involve C^2 cubics splines (see e.g. [1]) and (1.3) as an extension of C^0 piecewise-cubic non-parametric interpolation discussed in this paper. The case of C^1 modified Hermite interpolation is covered in [7] or [8].

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References

- [1] C. de Boor. *A practical guide to splines*. Springer-Verlag, Berlin Heidelberg New York, 2001.
- [2] M.P. do Carmo. *Differential geometry of curves and surfaces*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1976.
- [3] M.P. Epstein. On the influence of parameterization in parametric interpolation. *SIAM J. Numer. Anal.*, **13**(2):261–268, 1976. <https://doi.org/10.1137/0713025>.
- [4] G. Farin. *Curves and surfaces for computer aided geometric design*. Academic Press, San Diego, 1993.
- [5] M. Janik, R. Kozera and P. Koziol. Reduced data for curve modeling - applications in graphics, computer vision and physics. *Adv. Sci. Tech.*, **7**(18):38–35, 2013. <https://doi.org/10.5604/20804075.1049599>.
- [6] R. Kozera. Curve modeling via interpolation based on multidimensional reduced data. *Studia Informatica*, **25**(4B-61):1–140, 2004. Available from Internet: https://dx.doi.org/10.21936/si2004_v25_n4B.
- [7] R. Kozera and M. Wilkołazka. A modified Hermite interpolation with exponential parameterization. *Math. Comput. Sci.*, 2018. <https://doi.org/10.1007/s11786-018-0362-4>.
- [8] R. Kozera and L. Noakes. C^1 interpolation with cumulative chord cubics for length and trajectory estimation. *Fund. Inform.*, **61**(3-4):281–301, 2004.
- [9] R. Kozera and L. Noakes. Piecewise-quadratics and exponential parameterization for reduced data. *Appl. Math. Comput.*, **221**:620–638, 2013. <https://doi.org/10.1016/j.amc.2013.06.060>.
- [10] R. Kozera and L. Noakes. Piecewise-quadratics and reparameterizations for interpolating reduced data. In V.P. Gerdt, W. Koepf, W.M. Seiler and E.V. Vorozhtsov(Eds.), *Computer Algebra in Scientific Computing, Proc. Int. Conf. CASC 2015*, volume 9301 of *LNCS*, pp. 260–274, Aachen, Germany, 2015. Springer International Publishing. https://doi.org/10.1007/978-3-319-24021-3_20.
- [11] R. Kozera and L. Noakes. Exponential parameterization and ε -uniformly sampled reduced data. *Appl. Math. Inf. Sci.*, **10**(1):33–48, 2016. <https://doi.org/10.18576/amis/100104>.
- [12] R. Kozera and L. Noakes. Optimal knots selection for sparse reduced data. In F. Huang and A. Sugimoto(Eds.), *Image and Video Technology - Pacific Symposium on Image and Video Technology Workshops, Proc. Workshops PSIVT 2015*, volume 9555 of *LNCS*, pp. 3–14, Auckland, New Zealand, 2016. Springer International Publishing, AG Switzerland. https://doi.org/10.1007/978-3-319-30285-0_1.
- [13] R. Kozera, L. Noakes and P. Szmielew. Quartic orders and sharpness in trajectory estimation for smooth cumulative chord cubics. In L.J. Chmielewski, R. Kozera, Bok-Suk Shin and K. Wojciechowski(Eds.), *Computer Vision and Graphics, Proc. Int. Conf. ICCVG 2014*, volume 8671 of *LNCS*, pp. 9–16, Warsaw,

- Poland, 2014. Springer International Publishing. https://doi.org/10.1007/978-3-319-11331-9_2.
- [14] B.I. Kvasov. *Methods of shape-preserving spline approximation*. World Scientific, Singapore, 2000. <https://doi.org/10.1142/4172>.
- [15] E.T.Y. Lee. Choosing nodes in parametric curve interpolation. *Comput. Aided Design*, **21**(6):363–370, 1989. [https://doi.org/10.1016/0010-4485\(89\)90003-1](https://doi.org/10.1016/0010-4485(89)90003-1).
- [16] J. Milnor. *Morse theory*. Ann. of Math. Stud. 51, Princeton Univ. Press, 1963.
- [17] L. Noakes and R. Kozera. Cumulative chords piecewise-quadratics and piecewise-cubics. *Comput. Imaging Vision, Geometric Properties from Incomplete Data*, **32**:59–75, 2006. https://doi.org/10.1007/1-4020-3858-8_4.
- [18] A. Prokopenya, M. Minglibayev and G. Mayemerova. Symbolic calculations in studying the problem of three bodies with variable masses. *Program. Comput. Softw.*, **40**(2):79–85, 2014. <https://doi.org/10.1134/S036176881402008X>.
- [19] A. Ralston. *A first course in numerical analysis*. McGraw-Hill, 1965.
- [20] Mathematica Notebook 1. Available at <https://goo.gl/6R6ouv>.
- [21] Mathematica Notebook 2. Available at <https://goo.gl/vz8y8f>.
- [22] S. Wolfram. *The Mathematica book*. Wolfram Media, 5th ed., 2003.