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Green's Function and Existence of a Unique Solution for a Third-Order Three-Point Boundary Value Problem

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Abstract. The solutions of third-order three-point boundary value problem

$$x''' + f(t,x) = 0,\ t \in [a,b], x(a) = x'(a) = 0,\ x(b) = kx(\eta),$$

where $\eta \in (a,b)$, $k \in \mathbb{R}$, $f \in C([a,b] \times \mathbb{R}, \mathbb{R})$ and $f(t,0) \neq 0$, are the subject of this investigation. In order to establish existence and uniqueness results for the solutions, attention is focused on applications of the corresponding Green's function. As an application, also one example is given to illustrate the result.

Keywords: Green's function, nonlinear boundary value problems, three-point boundary conditions, existence and uniqueness of solutions.

AMS Subject Classification: 34B10; 34B15.

1 Introduction

We study third-order three-point boundary value problem

$$x''' + f(t,x) = 0,\ t \in [a,b], \tag{1.1}$$
$$x(a) = x'(a) = 0,\ x(b) = kx(\eta), \tag{1.2}$$

where $\eta \in (a,b)$, $k \in \mathbb{R}$, $f \in C([a,b] \times \mathbb{R}, \mathbb{R})$ and $f(t,0) \neq 0$.

Third-order three-point boundary value problems for ordinary differential equations are an important and actual field of research since they appear in physics, engineering and in various branches of applied mathematics, and as a consequence, have generated a lot of interest over the years. Nonlocal conditions are connected not only with the values of a solution on the boundary, but also with the values inside the domain. The purpose of using nonlocal boundary conditions is that metering at several locations can be combined to get more precise models.

The goal of the paper is to provide a result on the existence of a unique solution of (1.1)–(1.2) for a class of functions f. To arrive at this result we construct the corresponding Green's function and then use the contracting mapping theorem. If $k = 0$ we get two-point boundary value problem. Thus two-point boundary value problem is a particular case of (1.1)–(1.2).

The existence of solutions for nonlinear multipoint boundary value problems have been investigated by many authors. For instance Du, Lin, Ge [4], Feng, Ge [5] and references therein. In [3], to prove the existence of a solution for third-order three-point boundary value problem the authors employ the upper and the lower solution method and the Schauder fixed point theorem.

The Green's function plays an important role in the theory of boundary value problems. We refer the reader to [8], where the history of the subject is well-represented. Third-order three-point boundary value problems using the Green's function method were intensively studied by Anderson, Roman, Štikonas. We mention some papers [1,7], which motivated present investigation. Also the author wants to mention the paper [6] by Paukštaitė and Štikonas, where the relation between the Green's matrix for the system and the Green's function for the differential equation was presented. The existence of at least one solution for the fourth-order three-point boundary value problem using the Leray-Schauder nonlinear alternative was proved in [2].

The rest of the paper is organized as follows. In Section 2, we construct Green's function employing variation of parameters formula and some additional assumptions. Section 3 is devoted to estimation of the Green's function. In Section 4, we prove our main theorem on the existence and uniqueness for solution of the problem. Also one example is given to illustrate the result.

2 Construction of the Green's function

First of all let us construct Green's function for the two-point boundary value problem

$$u''' + h(t) = 0,\ t \in [a, b], \tag{2.1}$$

$$u(a) = u'(a) = 0,\ u(b) = 0, \tag{2.2}$$

and then, assuming that the solution of the three-point boundary value problem

$$x''' + h(t) = 0,\ t \in [a, b], \tag{2.3}$$

$$x(a) = x'(a) = 0,\ x(b) = kx(\eta), \tag{2.4}$$

can be expressed as

$$x(t) = u(t) + \left(\lambda_0 + \lambda_1 t + \lambda_2 t^2\right) u(\eta),$$

where λ_0, λ_1 and λ_2 are constants that will be determined, we will obtain Greens's function for the problem (2.3)–(2.4).

Proposition 1. *If $h : [a, b] \to \mathbb{R}$ is continuous function, then boundary value problem* (2.1)–(2.2) *has a unique solution*

$$u(t)=\int_a^t \left[\frac{(a-t)^2(s-b)^2}{2(a-b)^2}-\frac{(s-t)^2}{2}\right] h(s)ds+\int_t^b \left[\frac{(a-t)^2(s-b)^2}{2(a-b)^2}\right] h(s)ds,$$

that we can rewrite as

$$u(t) = \int_a^b R(t,s)h(s)ds,$$

where

$$R(t,s) = \begin{cases} \frac{(a-t)^2(s-b)^2}{2(a-b)^2} - \frac{(s-t)^2}{2}, & a \le s \le t \le b, \\ \frac{(a-t)^2(s-b)^2}{2(a-b)^2}, & a \le t \le s \le b. \end{cases} \tag{2.5}$$

Proof. To prove the proposition we use the variation of parameters formula

$$u(t) = c_1 + c_2 t + c_3 t^2 - \frac{1}{2}\int_a^t (s-t)^2 h(s)ds.$$

Using boundary conditions (2.2), we can obtain

$$c_1 = \frac{a^2}{2(a-b)^2}\int_a^b (s-b)^2 h(s)ds, \quad c_2 = -\frac{a}{(a-b)^2}\int_a^b (s-b)^2 h(s)ds,$$
$$c_3 = \frac{1}{2(a-b)^2}\int_a^b (s-b)^2 h(s)ds.$$

Thus, we get

$$\begin{aligned} u(t) &= \int_a^b \frac{(a-t)^2(s-b)^2}{2(a-b)^2} h(s)ds - \int_a^t \frac{(s-t)^2}{2} h(s)ds \\ &= \int_a^t \frac{(a-t)^2(s-b)^2}{2(a-b)^2} h(s)ds + \int_t^b \frac{(a-t)^2(s-b)^2}{2(a-b)^2} h(s)ds - \int_a^t \frac{(s-t)^2}{2} h(s)ds \\ &= \int_a^t \left[\frac{(a-t)^2(s-b)^2}{2(a-b)^2} - \frac{(s-t)^2}{2}\right] h(s)ds + \int_t^b \left[\frac{(a-t)^2(s-b)^2}{2(a-b)^2}\right] h(s)ds. \end{aligned}$$

The uniqueness follows from the fact, that the corresponding homogeneous problem has only the trivial solution. Hence the proof. □

Proposition 2. *Assume $h : [a, b] \to \mathbb{R}$ is continuous function. If $k(a-\eta)^2 \neq (a-b)^2$, $(a \neq \eta)$, then boundary value problem* (2.3)–(2.4) *has a unique solution*

$$x(t) = u(t) + \frac{k(a-t)^2}{(a-b)^2 - k(a-\eta)^2}\, u(\eta),$$

that we can rewrite as

$$x(t) = \int_a^b G(t,s)h(s)ds,$$

where

$$G(t,s) = R(t,s) + \frac{k(a-t)^2}{(a-b)^2 - k(a-\eta)^2} R(\eta,s). \tag{2.6}$$

Proof. Let $x(t) = u(t) + \left(\lambda_0 + \lambda_1 t + \lambda_2 t^2\right) u(\eta)$, where $\lambda_0, \lambda_1, \lambda_2$ are constants that will be determined and $u(t) = \int_a^b R(t,s)h(s)ds$. So,

$$\begin{aligned}
x(a) &= u(a) + \left(\lambda_0 + \lambda_1 a + \lambda_2 a^2\right) u(\eta) = \left(\lambda_0 + \lambda_1 a + \lambda_2 a^2\right) u(\eta),\\
x'(a) &= u'(a) + (\lambda_1 + 2\lambda_2 a)\, u(\eta) = (\lambda_1 + 2\lambda_2 a)\, u(\eta),\\
x(b) &= u(b) + \left(\lambda_0 + \lambda_1 b + \lambda_2 b^2\right) u(\eta) = \left(\lambda_0 + \lambda_1 b + \lambda_2 b^2\right) u(\eta),\\
x(\eta) &= u(\eta) + \left(\lambda_0 + \lambda_1 \eta + \lambda_2 \eta^2\right) u(\eta) = u(\eta)\left(\lambda_0 + \lambda_1 \eta + \lambda_2 \eta^2 + 1\right).
\end{aligned}$$

We get

$$\begin{aligned}
&\left(\lambda_0 + \lambda_1 a + \lambda_2 a^2\right) u(\eta) = 0,\\
&(\lambda_1 + 2\lambda_2 a)\, u(\eta) = 0,\\
&\left(\lambda_0 + \lambda_1 b + \lambda_2 b^2\right) u(\eta) = k u(\eta)\left(\lambda_0 + \lambda_1 \eta + \lambda_2 \eta^2 + 1\right),
\end{aligned}$$

or

$$\begin{cases}
\lambda_0 + \lambda_1 a + \lambda_2 a^2 = 0,\\
\lambda_1 + 2\lambda_2 a = 0,\\
(1-k)\lambda_0 + (b - k\eta)\lambda_1 + (b^2 - k\eta^2)\lambda_2 = k.
\end{cases}$$

Solving the system, we have

$$\begin{aligned}
\lambda_0 &= \frac{a^2 k}{(a-b)^2 - k(a-\eta)^2}, \quad \lambda_1 = \frac{-2ak}{(a-b)^2 - k(a-\eta)^2},\\
\lambda_2 &= \frac{k}{(a-b)^2 - k(a-\eta)^2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
x(t) = u(t) &+ \left(\frac{a^2 k}{(a-b)^2 - k(a-\eta)^2} - \frac{2akt}{(a-b)^2 - k(a-\eta)^2} \right.\\
&\left. + \frac{kt^2}{(a-b)^2 - k(a-\eta)^2} \right) u(\eta) = u(t) + \frac{k(a-t)^2}{(a-b)^2 - k(a-\eta)^2}\, u(\eta).
\end{aligned}$$

Let us prove the uniqueness. Assume that $y(t)$ is also a solution of (2.3)–(2.4), that is

$$\begin{aligned}
&y'''(t) + h(t) = 0,\ t \in [a,b],\\
&y(a) = y'(a) = 0,\ y(b) = ky(\eta).
\end{aligned}$$

Let $z(t) = y(t) - x(t),\ t \in [a,b]$. Thus we have

$$z'''(t) = y'''(t) - x'''(t) = h(t) - h(t) = 0,\ t \in [a,b].$$

Therefore $z(t) = c_1t^2 + c_2t + c_3$, where c_1, c_2 and c_3 are constants that we will determine. We have

$$\begin{aligned} z(a) &= y(a) - x(a) = 0, \\ z'(a) &= y'(a) - x'(a) = 0, \\ z(b) &= y(b) - x(b) = ky(\eta) - kx(\eta) = k(y(\eta) - x(\eta)) = kz(\eta), \end{aligned}$$

or

$$\begin{aligned} z(a) &= c_1a^2 + c_2a + c_3 = 0, \\ z'(a) &= 2c_1a + c_2 = 0, \\ z(b) &= c_1b^2 + c_2b + c_3 = k(c_1\eta^2 + c_2\eta + c_3) = kz(\eta). \end{aligned}$$

We get homogeneous system

$$\begin{cases} a^2c_1 + ac_2 + c_3 = 0, \\ 2ac_1 + c_2 = 0, \\ (b^2 - k\eta^2)c_1 + (b - k\eta)c_2 + (1 - k)c_3 = 0 \end{cases}$$

with determinant

$$\begin{vmatrix} a^2 & a & 1 \\ 2a & 1 & 0 \\ b^2 - k\eta^2 & b - k\eta & 1 - k \end{vmatrix} = k(a - \eta)^2 - (a - b)^2 \neq 0.$$

So the homogeneous system has only the trivial solution and hence $z(t) \equiv 0,\ t \in [a, b]$ or $x(t) \equiv y(t),\ t \in [a, b]$. The proof is complete. □

3 Estimation of the Green's function

Proposition 3. *The Green's function $R(t, s)$ from* (2.5) *satisfies*

$$\int_a^b |R(t, s)|\, ds \leq \frac{(b - a)^3}{3}$$

for $t \in [a, b]$.

Proof.

$$\begin{aligned} \int_a^b |R(t, s)|\, ds &= \int_a^t |R(t, s)|\, ds + \int_t^b |R(t, s)|\, ds \\ &= \int_a^t \left| \frac{(a - t)^2(s - b)^2}{2(a - b)^2} - \frac{(s - t)^2}{2} \right| ds + \int_t^b \frac{(a - t)^2(s - b)^2}{2(a - b)^2} ds \\ &\leq \int_a^t \left(\frac{(a - t)^2(s - b)^2}{2(a - b)^2} + \frac{(s - t)^2}{2} \right) ds + \int_t^b \frac{(a - t)^2(s - b)^2}{2(a - b)^2} ds \\ &= \left(\frac{(a - t)^2(s - b)^3}{6(a - b)^2} + \frac{(s - t)^3}{6} \right) \Bigg|_a^t + \frac{(a - t)^2(s - b)^3}{6(a - b)^2} \Bigg|_t^b = \frac{(a - t)^2(t - b)^3}{6(a - b)^2} \end{aligned}$$

$$- \frac{(a-t)^2(a-b)^3}{6(a-b)^2} - \frac{(a-t)^3}{6} - \frac{(a-t)^2(t-b)^3}{6(a-b)^2} = \frac{(t-a)^3}{6} + \frac{(t-a)^2(b-a)}{6}$$
$$\leq \frac{(b-a)^3}{6} + \frac{(b-a)^2(b-a)}{6} = \frac{(b-a)^3}{3}.$$

□

Proposition 4. *The Green's function $G(t,s)$ from (2.6) satisfies*

$$\int_a^b |G(t,s)|\, ds \leq \frac{(b-a)^3}{3} + \frac{|k|}{3} \frac{(b-a)^5}{|(b-a)^2 - k(\eta-a)^2|}$$

for $t \in [a,b]$.

Proof.

$$\int_a^b |G(t,s)|\, ds = \int_a^b \left| R(t,s) + \frac{k(a-t)^2}{(a-b)^2 - k(a-\eta)^2} R(\eta,s) \right| ds$$
$$\leq \int_a^b |R(t,s)|\, ds + \left| \frac{k(a-t)^2}{(a-b)^2 - k(a-\eta)^2} \right| \int_a^b |R(\eta,s)|\, ds$$
$$\leq \frac{(b-a)^3}{3} + \frac{\left|k(a-t)^2\right|}{|(a-b)^2 - k(a-\eta)^2|} \cdot \frac{(b-a)^3}{3}$$
$$\leq \frac{(b-a)^3}{3} + \frac{|k|}{3} \frac{(b-a)^5}{|(b-a)^2 - k(\eta-a)^2|}.$$

□

4 Existence of a unique solution

Theorem 1. *Suppose that $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a uniform Lipschitz condition with respect to x on $[a,b] \times \mathbb{R}$, namely there is a constant L such that, for every $(t,x), (t,y) \in [a,b] \times \mathbb{R}$,*

$$|f(t,x) - f(t,y)| \leq L\,|x-y|\,.$$

If $(b-a)^2 \neq k(a-\eta)^2$, $(a \neq \eta)$ and $(b-a)$ is so small that

$$\frac{(b-a)^3}{3} + \frac{|k|}{3} \frac{(b-a)^5}{|(b-a)^2 - k(\eta-a)^2|} < \frac{1}{L}, \tag{4.1}$$

then there exists a unique solution of (1.1)–(1.2).

Proof. Let X be the Banach space of continuous functions on $[a,b]$ with max norm

$$\|x\| = \max\{|x(t)| : a \leq t \leq b\}.$$

Note that x is a solution of (1.1)–(1.2) if and only if x is a solution of (2.3)–(2.4) with $h(t) = f(t,x(t))$. But (2.3)–(2.4) has a unique solution

$$x(t) = \int_a^b G(t,s) f(s,x(s)) ds,$$

where $G(t,s)$ is defined by (2.6). Define a mapping $T : X \to X$ by

$$Tx(t) = \int_a^b G(t,s)f(s,x(s))ds, \; t \in [a,b].$$

Hence (1.1)–(1.2) has a unique solution if and only if T has a unique fixed point.

We apply the contraction mapping theorem (Banach fixed-point theorem) to show that T has a unique fixed point. Let $x, y \in X$ and consider

$$\begin{aligned}
|Tx(t) - Ty(t)| &= \left| \int_a^b G(t,s)\,(f(s,x(s)) - f(s,y(s)))\,ds \right| \\
&\le \int_a^b |G(t,s)| \cdot |f(s,x(s)) - f(s,y(s))|\,ds \\
&\le \int_a^b |G(t,s)| \cdot L\,|x(s) - y(s)|\,ds \le L \int_a^b |G(t,s)|\,ds\,\|x - y\| \\
&\le L \left[\frac{(b-a)^3}{3} + \frac{|k|}{3}\,\frac{(b-a)^5}{|(b-a)^2 - k(\eta - a)^2|} \right] \|x - y\|, \; \text{for } t \in [a,b].
\end{aligned}$$

It follows that

$$\|Tx - Ty\| \le \alpha \|x - y\|,$$

where

$$\alpha = L \left[\frac{(b-a)^3}{3} + \frac{|k|}{3} \cdot \frac{(b-a)^5}{|(b-a)^2 - k(\eta - a)^2|} \right].$$

In view of (4.1), $\alpha < 1$ and T is a contraction mapping on X and has a unique fixed point. Hence the proof. □

Example 1. Consider the problem

$$\begin{aligned}
&x''' + 1 + t + \sin x = 0, \\
&x(0) = x'(0) = 0, \;\; x(1) = \frac{2}{3}\,x\left(\frac{1}{\sqrt{2}}\right).
\end{aligned} \tag{4.2}$$

We have $f(t,x) = 1 + t + \sin x$ ($f(t,0) = 1 + t \neq 0$) and

$$\left| \frac{\partial f}{\partial x}(t,x) \right| = |\cos x| \le L = 1.$$

Since $(a-b)^3 = 1 \neq 1/3 = k(a-\eta)^2$ and

$$\frac{(b-a)^3}{3} + \frac{|k|}{3}\,\frac{(b-a)^5}{|(b-a)^2 - k(\eta - a)^2|} = \frac{1}{3} + \frac{2}{9}\,\frac{1}{1 - \frac{1}{3}} = \frac{2}{3} < \frac{1}{L} = 1,$$

the problem (4.2) has a unique solution $x(t)$. The graph of solution $x(t)$ is depicted in Figure 1.

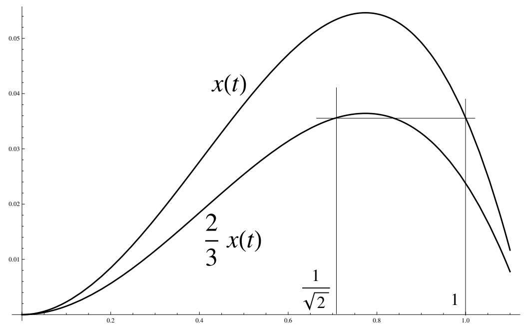

Figure 1. Solution of the problem (4.2).

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