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Non-Monotone Convergence Schemes*

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Abstract. We consider the second order BVP $x'' = f(t, x, x')$, $x'(a) = A$, $x'(b) = B$ provided that there exist α and β (lower and upper functions) such that: $\beta'(a) < A < \alpha'(a)$ and $\beta'(b) < B < \alpha'(b)$. We consider monotone and non-monotone approximations of solutions to the Neumann problem. The results and examples are provided.

Keywords: nonlinear boundary value problem, monotone iterations, Neumann boundary condition, non-monotone iterations.

AMS Subject Classification: 34B15.

1 Introduction

The classical results (see, for instance, [10]) say that the quasi-linear problem

$$x'' + p(t)x' + q(t)x = f(t, x, x'),$$
$$k_1x(a) + k_2x'(a) = A, \quad k_3x(a) + k_4x'(a) = B$$

is solvable (all the involved functions are continuous and f is bounded) if the homogeneous problem

$$x'' + p(t)x' + q(t)x = 0, \quad k_1x(a) + k_2x'(a) = 0, \qquad k_3x(a) + k_4x'(a) = 0$$

has only the trivial solution. If f is not bounded this is not true. There are results, however, which ensure the existence of a solution to boundary value problems with unbounded f.

The solvability results are well known [5, 9] for the second-order differential equation with unbounded function f

$$x'' = f(t, x, x'), \tag{1.1}$$

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given together with appropriate boundary conditions. We mention the existence results given in terms of the *lower* and *upper* functions α and β. Functions $\alpha(t)$ and $\beta(t)$, according to the definition, satisfy the following conditions:

$$\alpha \leq \beta, \quad \alpha'' \geq f(t, \alpha, \alpha'), \quad \beta'' \leq f(t, \beta, \beta'), \quad \forall t \in [a, b]. \tag{1.2}$$

First results using the lower and upper functions α and β simply state the existence of a solution to the boundary value problem under consideration. Later it was observed in the works by L.K. Jackson, K.W. Schrader [5] and H.W. Knobloch [7, 8] that solutions with a specific property (B) exist if α and β as above exist. These B-solutions, roughly speaking, can be obtained as lower (upper) limits of sequences of neighboring solutions on the whole interval $I = [a, b]$. In the work by L.K. Jackson, K.W. Schrader [5] this question was studied with respect to the Dirichlet boundary conditions

$$x(a) = A, \quad x(b) = B. \tag{1.3}$$

It was shown that in presence of α and β and a Nagumo type condition a specific solution $x(t)$ of the problem exists which can be obtained as a uniform limit of monotone sequences of solutions of auxiliary the Dirichlet type problems.

In our work [4] we considered different solutions of the problem $x'' = f(t, x)$, (1.3) and we have shown that there exist solutions which are limits of monotone sequences of solutions, and there may be also solutions, which can not be approximated by monotone sequences.

The objective of this paper is to consider the Neumann problem. We show that in presence of the lower and upper functions α and β there exists a solution with the specific property which can be obtained as a limit of a monotone sequence of solutions of auxiliary the Neumann type problems. We show that there may exist also solutions which cannot be approximated by monotone sequences. We introduce the scheme of non-monotone iterations for non-linear second-order boundary value problem and explain how it works on particular examples.

The structure of the paper is the following: in Section 2 lower and upper solutions are described, in Section 3 definitions are given, in Section 4 the main result is formulated: characteristics of solutions of the analyzed problems are given and the example of its application is shown.

2 Upper and Lower Solutions

We consider the problem

$$x'' = f(t, x, x'), \quad f(t, x, x') \in C^1\big([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big) \tag{2.1}$$

with boundary conditions

$$x'(a) = A, \quad x'(b) = B. \tag{2.2}$$

A solution of problem (2.1)–(2.2) exists according to the following result.

Theorem 1. [1] *Assume there exist* $\alpha(t)$, $\beta(t)$ *such that* $\forall t \in [a,b]$ $\alpha(t) \le \beta(t)$, $\alpha''(t) \ge f(t,\alpha(t),\alpha'(t))$, $\beta''(t) \le f(t,\beta(t),\beta'(t))$. *If* A *and* B *are such that* $\beta'(a) \le A \le \alpha'(a)$ *and* $\beta'(b) \ge B \ge \alpha'(b)$, *and the Nagumo condition*[2] *fulfils, then the BVP* (2.1)–(2.2) *has a solution* $x(t)$ *satisfying*

$$\alpha(t) \le x(t) \le \beta(t), \quad \forall t \in [a,b].$$

Many authors, for example [1, 2, 3] considered the monotone iterations in presence of upper and lower functions. The monotone sequences of problem (2.1)–(2.2) exist according to the following theorem.

Theorem 2. *Assume there exist* $\alpha(t)$, $\beta(t)$ *for problem* (2.1)–(2.2) *such that* $\alpha(t) < \beta(t)$, $\alpha''(t) \ge f(t,\alpha(t),\alpha'(t))$, $\beta''(t) \le f(t,\beta(t),\beta'(t))$ *and the Nagumo condition is satisfied. If* A *and* B *are such that* $\beta'(a) < A < \alpha'(a)$ *and* $\beta'(b) > B > \alpha'(b)$, *then there exist sequences* $\{\overline{x}_i\}$, $\{\underline{x}_i\}$ *such that*

$$\begin{aligned} &\beta(t) > \overline{x}_1(t) > \overline{x}_2(t) > \cdots > \overline{x}_n(t) > \cdots \quad x^*(t) \\ &\alpha(t) < \underline{x}_1(t) < \underline{x}_2(t) < \cdots < \underline{x}_n(t) < \cdots \quad x_*(t). \end{aligned}$$

Proof. We construct the sequence $\{\overline{x}_n(t)\}$. Choose $\overline{A}_i \in (\beta'(a), A)$ such that $\overline{A}_i$ monotonically converge to A. Choose $\overline{B}_i \in (B, \beta'(b))$ such that $\overline{B}_i$ monotonically converge to B. Consider the problem

$$x'' = f(t,x,x'), \;\; x'(a) = \overline{A}_1, \;\; x'(b) = \overline{B}_1, \;\; \beta'(a) < \overline{A}_1, \;\; \overline{B}_1 < \beta'(b).$$

Then a solution $\overline{x}_1(t)$ exists, such that $\alpha(t) < \overline{x}_1(t) < \beta(t)$. Set $\beta_1(t) = \overline{x}_1(t)$. Consider the problem

$$x'' = f(t,x,x'), \quad x'(a) = \overline{A}_2, \;\; x'(b) = \overline{B}_2.$$

This is solvable because $\alpha(t)$, $\beta_1(t) = \overline{x}_1(t)$ exist and $\overline{A}_1 = \beta_1'(a) < \overline{A}_2$, $\overline{B}_1 = \beta_1'(b) > \overline{B}_2$. Then $\overline{x}_2(t)$ exists such that $\alpha(t) < \overline{x}_2(t) < \overline{x}_1(t)$. Proceeding this way, we construct a sequence $\{\overline{x}_n(t)\}$ such that

$$\alpha(t) < \cdots < \overline{x}_n(t) < \cdots < \overline{x}_2(t) < \overline{x}_1(t) < \beta(t).$$

Applying the Arzela–Ascoli criterium we can show that some subsequence of $\{\overline{x}_n\}$ converges to x^*.[3] The same type arguments show that $\{\underline{x}_n\}$ also exists, $x_n \to x_*$.[4] Notice that $\underline{x}_n < x^*$ by construction and, therefore, $x_* \le x^*$. □

To illustrate the existence of monotone sequence let us consider the simple example.

Example 1. Consider the BVP

$$x'' = x^3, \quad x'(-1) = 0, \;\; x'(1) = 0. \tag{2.3}$$

The upper and lower functions are defined as $\beta(t) = t^2 + 2$, $\alpha(t) = -t^2 - 2$. Conditions (1.2) are satisfied. This problem has a unique trivial solution. The elements of monotone sequences $\{\underline{x}_n\}$ and $\{\overline{x}_n\}$ and the respective phase plane are shown in Fig. 1 and 2.

[1] Theorem 2, p. 35 in [10], adapted for the Neumann conditions.
[2] $|f(t,x,y)| \le \varphi(|y|)$, $(t,x) \in \{a \le t \le b, \alpha \le x \le \beta\}$, $\int_0^\infty \frac{s\,ds}{\varphi(s)} = +\infty$.
[3] x^* is called a maximal solution.
[4] x_* is called a minimal solution.

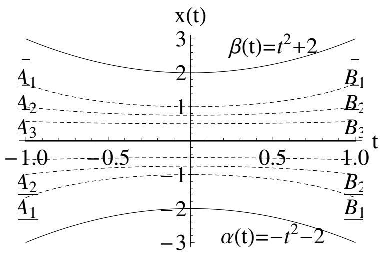

Figure 1. Monotone sequence converging to the unique (trivial) solution of the Neumann problem (2.3).

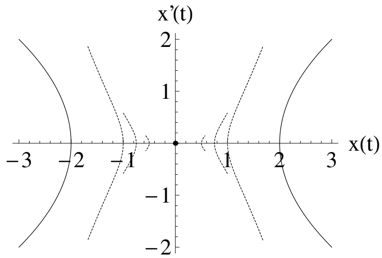

Figure 2. The phase plane of monotone subsequences for problem (2.3).

3 Definitions

If a function $f(t,x,x')$ has partial derivatives $f_x(t,x,x')$ and $f_{x'}(t,x,x')$ then the equation of variations for a solution $\xi(t)$ of problem (2.1)–(2.2) can be considered.

Definition 1. We say that the type of a solution $\xi(t)$ of the Neumann BVP is i $(i \geq 0)$, if a solution of initial value problem of variations with respect to $\xi(t)$

$$y'' = f_x\big(t,\xi(t),\xi'(t)\big)y + f_{x'}\big(t,\xi(t),\xi'(t)\big)y', \quad y(a)=1, \;\; y'(a)=0 \tag{3.1}$$

has exactly i zeros in the interval (a,b) and $y(b) \neq 0$. We denote $type(\xi) = i$. If moreover $y(b)=0$, then we denote $type(\xi) = (i, i+1)$.

In order to construct non-monotone sequences let us introduce "diagonal" sequences. By diagonal sequence we mean a sequence of solutions of equation (1.1) which satisfy the following boundary conditions

$$\begin{aligned} &x'(a) = A_i < A, \quad x'(b) = B_i < B \;\text{ or} \\ &x'(a) = A_i > A, \quad x'(b) = B_i > B. \end{aligned} \tag{3.2}$$

Remark. We use the term "diagonal" because in our previous work [4] on the Dirichlet problem $x(a)=A$, $x(b)=B$ we constructed non-monotone sequences as solutions of the auxiliary problems $x(a)=A_i>A$, $x(b)=B_i<B$ (or $x(a)=A_i<A$, $x(b)=B_i>B$), which look "diagonal".

4 Main Result

Theorem 3. [5] *If there exists a sequence $\{x_n\}$, consisting of solutions of the same type i $(i \neq 0)$ of the above auxiliary problems and $\alpha(t) < x_n < \beta(t)$, then there exists a subsequence converging to a similar type solution $x(t)$ of the problem* (2.1)–(2.2) *and $type(x) = (i-1,i)$, or $type(x) = i$, or $type(x) = (i,i+1)$.*

[5] Iterations in Theorem 3 need not to be monotone.

Proof. Let us construct the initial value problem of variations for the solution $x_n(t)$

$$y_n'' = f_x\big(t, x_n(t), x_n'(t)\big)y_n + f_{x'}\big(t, x_n(t), x_n'(t)\big)y_n' \tag{4.1}$$
$$y_n(a) = 1, \quad y_n'(a) = 0. \tag{4.2}$$

If the Nagumo conditions hold, then a constant $N > 0$ exists such that any solution of the problem which satisfies $\alpha(t) \le x(t) \le \beta(t)$, $t \in [a,b]$ satisfies also $|x'(t)| < N$, $\forall t \in [a,b]$. Therefore we consider $f(t,x,x')$ only for $\alpha \le x \le \beta$, $|x'| < N$. Thus we may assume that f is bounded: $|f(t,x,x')| < M$.

According to Arzela–Ascoli criterium [6], any infinite compact sequence contains a convergent subsequence. In order to do it, compactness of the infinite number of functions $\{x_n\}$ and $\{y_n\}$ within the space C^1 is to be shown, however it means that $\{x_n\}$, $\{x_n'\}$, $\{y_n\}$ and $\{y_n'\}$, are equicontinuous and equibounded. Let us show compactness of the sequence $\{x_n(t)\}$ of the solution of the problem (2.1)–(2.2). First of all, show that the infinite sequence $\{x_n(t)\}$ is bounded. That is because

$$\big|x_n(t)\big| < \max\big\{\big|\beta(t)\big|, \big|\alpha(t)\big|\big\}$$

Introduce a constant $K = \max\{|\beta(t)|, |\alpha(t)|\}$. We get that $\forall t \in [a,b]$, $n \in \mathbb{N}$, $|x_n(t)| < K$ is bounded.

Let us prove the equicontinuity. First of all, show this feature for the infinite number of functions $\{x_n(t)\}$. According to the definition of that equicontinuity $\forall \varepsilon > 0$, $\exists \delta > 0$, such that as soon as $|t_2 - t_1| < \delta \Rightarrow |x_n(t_1) - x_n(t_2)| < \varepsilon$. One has, according to Lagrange's Mean Value Theorem, that

$$x_n(t_1) - x_n(t_2) = x_n'(\xi)(t_1 - t_2),$$

where $t_1 < \xi < t_2$, $\forall t_1, t_2 \in [a,b]$. We can evaluate the modulus of the difference

$$\big|x_n(t_1) - x_n(t_2)\big| = \big|x_n'(\xi)\big||t_1 - t_2| < N|t_1 - t_2|.$$

As a result, get the value $\delta > 0$ as $\delta = \varepsilon/N$.

Now let us evaluate the modulus of the difference $|x_n'(t_1) - x_n'(t_2)|$ and find the corresponding δ. Using Lagrange's Mean Value Theorem, it is possible to state: $|x_n'(t_1) - x_n'(t_2)| = |x_n''(\eta)||t_1 - t_2|$, $t_1 < \eta < t_2$, $\forall t_1, t_2 \in [a,b]$.

Using the condition of the problem (2.1), in the last expression change $x_n''(\eta)$ to $f(\eta, x_n(\eta), x_n'(\eta))$, and then apply the fact that the function f is bounded within the interval $[a,b]$:

$$\big|x_n'(t_1) - x_n'(t_2)\big| = \big|f\big(\eta, x_n(\eta), x_n'(\eta)\big)\big||t_1 - t_2| < M|t_1 - t_2|.$$

As the result of this analysis we get $\delta = \varepsilon/M$.

We wish to show now that $\{y_n(t)\}$ contains a converging to $\{y(t)\}$ subsequence. Let us prove that the sequences $\{y_n(t)\}$ and $\{y_n'(t)\}$ are uniformly bounded. Denote $f_x(t, x_n(t), x_n'(t)) = \varphi(t)$ and $f_{x'}(t, x_n(t), x_n'(t)) = \psi(t)$. It is clear that $|\varphi(t)| < M_1 = \mathit{const}$ and $|\psi(t)| < K_1 = \mathit{const}$ because $a \le t \le b$ and $\alpha < x_n < \beta$. We want to show that there exists $N_1 = \mathit{const}$ and $|y_n(t)| < N_1$ for all $t \in [a,b]$. In order to show that the limiting function $x(t)$ possesses the property described in the statement of theorem let us write equation (4.1) as a system

$$\begin{cases} y' = u, \\ u' = \varphi(t)y + \psi(t)u. \end{cases} \tag{4.3}$$

and introduce polar coordinates

$$\begin{cases} u' = \rho' \cos\theta - \rho \sin\theta, \\ y' = \rho' \sin\theta + \rho\cos\theta, \end{cases} \tag{4.4}$$

where $u = \rho\cos\theta, \quad y = \rho\sin\theta$. Using formulas (4.4) we get

$$\rho' = \frac{\begin{vmatrix} u' & -\rho\sin\theta \\ y' & \rho\cos\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix}} = \frac{u'\rho\cos\theta + y'\rho\sin\theta}{\rho\cos^2\theta + \rho\sin^2\theta} = u'\cos\theta + y'\sin\theta.$$

Using the system (4.3) we get

$$\begin{aligned} \rho' &= \big(\varphi(t)y + \psi(t)u\big)\cos\theta + u\sin\theta \\ &= \varphi(t)\rho\sin\theta\cos\theta + \psi(t)\rho\cos^2\theta + \rho\cos\theta\sin\theta \\ &= \frac{1}{2}\rho\big[\varphi(t)\sin 2\theta + \psi(t)(1+\cos 2\theta) + \sin 2\theta\big]. \end{aligned}$$

According to the initial conditions (4.2) is $\rho(0) = 1$.

Further, $\rho(t) = \rho(0)\exp\left(\int R(t)\,dt\right)$, therefore

$$R(t) = \frac{1}{2}\big[\varphi(t)\sin 2\theta + \psi(t)(1+\cos 2\theta) + \sin 2\theta\big] \le \frac{1 + M_1 + 2K_1}{2}.$$

Hence, $|y_n(t)| < N_1$, where $N_1 = e^{0.5(1+M_1+2K_1)(b-a)}$. One has that $\{y_n'(t)\}$ is bounded because $y' = \rho'\sin\theta + \rho\cos\theta$.

Now let us evaluate the modulus of the difference $|y_n(t_1) - y_n(t_2)|$ and find the corresponding δ. For this, we write

$$y_n(t) = y_n(0) + \int_0^t y_n'(s)\,ds, \quad \big|y_n(t_1) - y_n(t_2)\big| = \left|\int_{t_1}^{t_2} y_n'(s)\,ds\right| < P|t_2 - t_1|,$$

where $|y_n'(t)| < P = const$, $\forall t \in [a,b]$. As result, get the value $\delta = \varepsilon/P$.

Consider the $\{y_n'(t)\}$ and evaluate the modulus of the difference

$$\begin{aligned} &\big|y_n'(t_1) - y_n'(t_2)\big| = \left|\int_{t_1}^{t_2} y_n''(s)\,ds\right| \\ &\quad = \left|\int_{t_1}^{t_2} f_x\big(s, x_n(s), x_n'(s)\big)y_n(s)\,ds + \int_{t_1}^{t_2} f_{x'}\big(s, x_n(s), x_n'(s)\big)y_n'(s)\,ds\right| \\ &\quad \le \left|\int_{t_1}^{t_2} f_x\big(s, x_n(s), x_n'(s)\big)y_n(s)\,ds\right| + \left|\int_{t_1}^{t_2} f_{x'}\big(s, x_n(s), x_n'(s)\big)y_n'(s)\,ds\right| \\ &\quad \le M_1\int_{t_1}^{t_2} \big|y_n(s)\big|\,ds + M_2\int_{t_1}^{t_2}\big|y_n'(s)\big|\,ds \le (M_1N_1 + M_2P)|t_2 - t_1|. \end{aligned}$$

As a result we get that $\delta = \varepsilon/(M_1N_1 + M_2P)$.

There are sequences $\{x_n(t)\}$, $\{y_n(t)\}$ and limiting functions $x(t)$, $y(t)$. It follows from the hypotheses of the theorem that a polar function $\theta_n(t)$ corresponding to $\{y_n(t)\}$ fulfils the condition $\pi(i+1) > \theta_n(b) > \pi i$. This is the same as $y_n(t)$ have exactly i zeros on the interval, $y_n(b) \neq 0$. Then the limiting function $y(t)$ is such that the respective θ satisfies the inequalities $\pi(i+1) \geq \theta(b) \geq \pi i$, and this is the same as either $type(x) = i$, $type(x) = (i-1, i)$, or $type(x) = (i, i+1)$. The proof is complete. □

Example 2. We consider the BVP

$$x'' = x^3 - k^2 x, \quad x'(0) = x'(2) = 0, \quad k = 2. \tag{4.5}$$

The upper and lower functions are $\beta(t) = 3.5 + (t-1)^2$, $\alpha(t) = -3.5 - (t-1)^2$. The solutions of problem (4.5) are $\xi_1 \equiv 2$, $\xi_2 \equiv -2$, $\xi_3 \equiv 0$. Some elements of monotone and diagonal sequences are shown in Fig. 3.

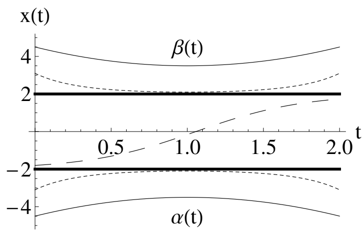

Figure 3. Sequences converging to the solutions of Neumann problem (4.5).

We consider the equation of variations. For ξ_1 and ξ_2 this problem is given by

$$y'' = f_x\big(t, \xi(t), \xi'(t)\big)y = 8y, \quad y(0) = 1, \quad y'(0) = 0$$

Solution for this problem is not equal to zero in the interval $[0, 2]$. Therefore $type(\xi_1) = 0$ and $type(\xi_2) = 0$. Hence, these solutions can be approximated with monotone sequences.

For ξ_3 equation of variations is $y'' = -4y$. This problem with initial conditions $y(0) = 1$, $y'(0) = 0$ has a solution $y(t)$, which has one zero in the interval $[0, 2]$. Hence, $type(\xi_3) = 1$ and ξ_3 can be approximated with non-monotone sequence.

Theorem 4. *Suppose a solution ξ of the Neumann problem of type i $(i > 0)$ exists. Then ξ can be approximated by solutions of similar type (as formulated in Theorem 3*[6]*) of auxiliary problems (as formulated in* (3.2)*).*

Proof. Suppose i is an even number. A solution $\xi(t)$ solves the problem

$$x'' = f(t, x, x'), \quad x'(a) = A, \;\; x'(b) = B$$

[6] $type(x) = (i-1, i)$, or $type(x) = i$, or $type(x) = (i, i+1)$.

and is such that a solution $y(t)$ of the Cauchy problem (3.1) has exactly i zeros in (a,b) and $y(b) \neq 0$. This means that the difference $w(t) := x(t;\varepsilon) - \xi(t)$ is not zero at $t = a$, has exactly i zeros in the interval (a,b) and is not equal to zero at $t = b$. Also $w'(a) = 0$. Let $x(t;\varepsilon)$ be a solution to the Cauchy problem

$$x'' = f(t,x,x'), \quad x(a) = \xi(a) + \varepsilon, \quad x'(a) = \xi'(a) = 0.$$

Due to the assumption that i is an even integer one has that $x(b;\varepsilon) - \xi(b) > 0$ for small enough $\varepsilon > 0$. Then, by continuous dependence of solutions with respect to the initial data, a solution $x(t;\varepsilon,\delta)$ of the Cauchy problem

$$x'' = f(t,x,x'), \quad x(a) = \xi(a) + \varepsilon, \quad x'(a) = \xi'(a) + \delta = 0$$

is such that the difference $x(t;\varepsilon,\delta) - \xi(t)$ has exactly i zeros in the interval (a,b) and $x(b;\varepsilon,\delta) - \xi(b) > 0$ for sufficiently small $\delta > 0$. Moreover, a solution $y(t;\varepsilon,\delta)$ of the respective equation of variations behaves like a solution $y(t)$ of the equation of variations for $\xi(t)$. Therefore $\xi(t)$ and $x(t;\varepsilon,\delta)$ have the same index. Then a sequence $\{\varepsilon_n, \delta_n\} \to (0,0)$ (as $n \to \infty$) exists such that solutions of the Cauchy problems

$$x'' = f(t,x,x'), \quad x(a) = \xi(a) + \varepsilon_n, \quad x'(a) = \xi'(a) + \delta_n = 0$$

all have the same index i.

It turns out that $x(t;\varepsilon_n,\delta_n)$ are solutions of the equation $x'' = f(t,x,x')$, which locate between α and β and satisfy the following boundary conditions

$$x'(a) = A + \delta_n, \quad x'(b) = x'(b;\varepsilon_n,\delta_n) := C_n > B.$$

The proof for odd i is similar with the only difference that δ_n are negative and $C_n < B$. $\square$

5 Conclusion

If α and β exist and a Nagumo type condition holds, then monotone sequences of solutions of auxiliary problems can be constructed converging to maximal x^* and minimal x_* solutions of the Neumann problem.

If there exist multiple solutions of the Neumann problem then we have to distinguish solutions by types. Solutions of nonzero type cannot be approximated by monotone sequences. They can be approximated, however, by solutions of auxiliary problems, which do not converge monotonically.

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