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$W_0^{1,p(x)}$ Versus C^1 Local Minimizers for Nonsmooth Functionals*

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Abstract. In this paper, we study that the local minimizers of a class of functionals (not necessarily differentiable) in the C^1-topology are still their local minimizers in $W_0^{1,p(x)}$.

Keywords: $p(x)$-Laplacian, Clarke's generalized gradient, local minimizers, nonsmooth functionals.

AMS Subject Classification: 34A60; 35R70; 49J52; 49J53.

1 Introduction

We consider the functional $\varphi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\varphi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx - \int_\Omega F(x,u)\, dx,$$

where Ω is bounded smooth domain in $\mathbb{R}^N$,

$$1 < p_- = \min_{x\in\overline{\Omega}} p(x) \le p(x) \le \max_{x\in\overline{\Omega}} p(x) = p^+ < +\infty$$

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and $F(t,x)$ is locally Lipschitz function in the t-variable integrand (in general it can be nonsmooth), and $\partial F(x,t)$ is defined in the sense of Clarke [3].

Such a result was first proven for functionals corresponding to elliptic equations with Dirichlet boundary values by Fan in [5]. They considered potentials of the form

$$\varphi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx - \int_\Omega F(x,u)\, dx,$$

where $F(x,u) = \int_0^u f(x,t)\, dt$ with some Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$. For variable exponent Neumann Sobolev spaces (i.e. $W_n^{1,p(x)}(\Omega)$), the result can be found in Gasinski–Papageorgiou [7]. For the case $p(x) = p = 2$, Brezis and Nirenberg in [2] proved that the local minimizers of a class of functionals in the C^1-topology are still their local minimizers in $W_0^{1,p}(\Omega)$. For the case when $p(x) \equiv p$, Motreanu and Papageorgiou [9] first considered the problem by using nonsmooth functionals defined on $W_0^{1,p}(\Omega)$ for the case $2 \le p < +\infty$. In Iannizzotto–Papageorgiou [8] an analogous result was proved for $p > 1$, Neumann boundary conditions and a nonsmooth potential.

The idea of the present paper was the generalization to the more general case of nonsmooth functionals defined on $W_0^{1,p(x)}(\Omega)$ with $1 < p^- \le p^+ < +\infty$. The main goal of this paper is to answer the following question:

(Q) If $u_0 \in W_0^{1,p(x)}(\Omega)$ is a local minimizer of φ in the C^1-topology, is it still a local minimizer of φ in $W_0^{1,p(x)}(\Omega)$?

2 Hypotheses

We suppose the following conditions on the nonsmooth potentials $F : \Omega \times \mathbb{R} \to \mathbb{R}$:

H(F):

(1) For all $t \in \mathbb{R}$, $x \mapsto F(x,t)$ is measurable;

(2) For almost all $x \in \mathbb{R}^N$, $t \mapsto F(x,t)$ is locally Lipschitz;

(3) For almost all $x \in \Omega$, all $t \in \mathbb{R}$ and $w \in \partial F(x,t)$, we have $|w| \le a_1 + a_2|t|^{\alpha(x)-1}$, where a_1, a_2 are positive constants, $\alpha \in C(\overline{\Omega})$ with $1 < \alpha^- \le \alpha(x) < p^*(x)$, $p^*(x) = \frac{Np(x)}{N-p(x)} (+\infty)$, if $p(x) < N (p(x) \ge N)$.

H(p): $p \in C^{0,\beta}(\overline{\Omega})$ with $\beta \in (0,1)$, $1 < p(x) < p^*(x)$ for every $x \in \overline{\Omega}$.

Remark 1. Note that the conditions above imply that the functional $\varphi : W_0^{1.p(x)}(\Omega) \to \mathbb{R}$ is locally Lipschitz (similar to the proof of Lemma 3.1 in [6]). This guarantees, in particular, that Clarke's generalized gradient $t \mapsto \partial\varphi(t)$ exists.

3 Sobolev Versus C^1 Local Minimizers

In this section we shall give a positive answer to our question **(Q)**. Our main result is stated in the following theorem.

Theorem 1. *Suppose that* $\mathbf{H}(\mathbf{F})$ *and* $\mathbf{H}(p)$ *hold, and that* $u_0 \in W_0^{1,p(x)}(\Omega)$ *is a local minimizer of* φ *in the* $C_0^1(\overline{\Omega})$ *topology, that is., there exists some* $r > 0$ *such that*

$$\varphi(u_0) \leq \varphi(u_0 + h) \quad \text{for all } h \in C_0^1(\overline{\Omega}) \text{ with } \|h\|_{C_0^1(\overline{\Omega})} \leq r,$$

then $u_0 \in C_0^1(\overline{\Omega})$ *is a local minimizer of* φ *in the* $W_0^{1,p(x)}(\Omega)$ *topology, i.e., there exists* $k > 0$ *such that*

$$\varphi(u_0) \leq \varphi(u_0 + h) \quad \text{for all } h \in W_0^{1,p(x)}(\Omega) \text{ with } \|h\|_{W_0^{1,p(x)}(\Omega)} \leq k.$$

Proof. By assumption, there exists $r_0 > 0$ such that

$$\varphi(u_0) \leq \varphi(u_0 + h), \quad \text{for all } h \in C_0^1(\overline{\Omega}),\ \|h\|_{C_0^1(\overline{\Omega})} \leq r_0.$$

Choose $h \in C_0^1(\overline{\Omega})$, then for small enough $t > 0$ we have $\varphi(u_0) \leq \varphi(u_0 + th)$, hence,

$$0 \leq \varphi^\circ(u_0; h). \tag{3.1}$$

Since $C_0^1(\overline{\Omega})$ is dense in $W_0^{1,p(x)}(\Omega)$ and φ° is continuous, (3.1) holds for all $h \in W_0^{1,p(x)}(\Omega)$, so $0 \in \partial\varphi(u_0)$. For any $u \in W_0^{1,p(x)}(\Omega)$, define

$$\beta(x) = \frac{\alpha(x)}{\alpha(x) - 1},$$
$$N(u) = \big\{w \in L^{\beta(x)}(\Omega)\colon w(x) \in \partial F\big(x, u(x)\big) \text{ for a.e. } x \in \Omega\big\}.$$

Then, there exists $v_0 \in N(u_0)$ such that

$$-\mathrm{div}\big(|\nabla u_0|^{p(x)-2}\nabla u_0\big) = v_0. \tag{3.2}$$

From (3.2) , we infer that

$$\begin{cases} -\mathrm{div}(|\nabla u_0(x)|^{p(x)-2}\nabla u_0(x)) = v_0(x), & \text{a.e. } x \in \Omega, \\ u_0|_{\partial\Omega} = 0. \end{cases} \tag{3.3}$$

From (3.3) and nonlinear regularity (see Fan and Zhao [6, Theorem 4.1] and Fan [4, Theorem 1.2]), we have $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, 1)$.

Now, we argue by contradiction. Suppose that u_0 is not a local minimizer of φ in the $W_0^{1,p(x)}(\Omega)$. Exploiting the compact embedding of $W_0^{1,p(x)}(\Omega)$ into $L^{p(x)}(\Omega)$, we can easily check that φ is sequentially weakly lower semicontinuous.

Define $G(u) = \int_\Omega \frac{|\nabla u - \nabla u_0|^{p(x)}}{p(x)}\, dx$, $\forall u \in W_0^{1,p(x)}(\Omega)$.

Fix $\varepsilon \in (0, 1]$, see $\overline{B}_\varepsilon = \{u \in W_0^{1,p(x)}(\Omega)\colon G(u) \leq \varepsilon\}$. Then $\overline{B}_\varepsilon$ is weakly compact, closet convex subset of $W_0^{1,p(x)}(\Omega)$ and is a neighbourhood of u_0 in $W_0^{1,p(x)}(\Omega)$. From the Weierstrass theorem we can find $u_\varepsilon \in \overline{B}_\varepsilon$ such that

$$\varphi(u_\varepsilon) = \inf_{u \in \overline{B}_\varepsilon} \varphi(u) < \varphi(u_0).$$

By the nonsmooth multiplier rule of Clarke [3] (Theorem 1 and Proposition 13), we can find $\lambda_\varepsilon \le 0$ such that $\lambda_\varepsilon G'(u_\varepsilon) \in \partial\varphi(u_\varepsilon)$. Hence,

$$A(u_\varepsilon) - w_\varepsilon = \lambda_\varepsilon A(u_\varepsilon - u_0), \quad w_\varepsilon \in N(u_\varepsilon).$$

That is,

$$-\mathrm{div}\bigl\{|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon - \lambda_\varepsilon|\nabla u_\varepsilon - \nabla u_0|^{p(x)-2}(\nabla u_\varepsilon - \nabla u_0)\bigr\} = w_\varepsilon. \tag{3.4}$$

Define $A_\varepsilon : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ and $B_\varepsilon : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} A_\varepsilon(x,\eta) &= \frac{1}{1-\lambda_\varepsilon}|\eta|^{p(x)-2}\eta - \frac{\lambda_\varepsilon}{1-\lambda_\varepsilon}|\eta - \nabla u_0|^{p(x)-2}(\eta - \nabla u_0),\\ B_\varepsilon(x,\eta) &= \frac{1}{1-\lambda_\varepsilon}w_\varepsilon, \end{aligned}$$

where $w_\varepsilon \in \partial F(x,\eta)$. Then u_ε is a solution of the following problem:

$$\begin{cases} -\mathrm{div}A_\varepsilon(x,\nabla u) = B_\varepsilon(x,u), & \text{a.e. } x \in \Omega,\\ u|_{\partial\Omega} = 0. \end{cases} \tag{3.5}$$

We can verify that A_ε and B_ε satisfy the following conditions:

$$A_\varepsilon(x,\eta)\eta \ge c_1|\eta|^{p(x)} - c_2, \quad \forall x \in \overline{\Omega},\ \eta \in \mathbb{R}^N, \tag{3.6}$$

$$\bigl|A_\varepsilon(x,\eta)\bigr| \le c_3|\eta|^{p(x)-1} + c_4, \quad \forall x \in \overline{\Omega},\ \eta \in \mathbb{R}^N, \tag{3.7}$$

$$\bigl|B_\varepsilon(x,\eta)\bigr| \le c_5 + c_6|\eta|^{\alpha(x)-1}, \quad \forall x \in \overline{\Omega},\ \eta \in \mathbb{R}^N, \tag{3.8}$$

where c_i are positive constants independent of $\varepsilon \in (0,1)$.

Since the verification of (3.7) and (3.8) is simple, we only give the proof of (3.6) here. By the definition of $A_\varepsilon(x,\eta)$,

$$\begin{aligned} A_\varepsilon(x,\eta)\eta &= \frac{1}{1-\lambda_\varepsilon}\bigl[\bigl(|\eta|^{p(x)-2}\eta - \lambda_\varepsilon|\eta|^{p(x)-2}\eta\bigr)\\ &\qquad - \lambda_\varepsilon\bigl(|\eta - \nabla u_0|^{p(x)-2}(\eta - \nabla u_0) - |\eta|^{p(x)-2}\eta\bigr)\bigr]\eta\\ &= \frac{1}{1-\lambda_\varepsilon}\bigl[(1-\lambda_\varepsilon)\bigl(|\eta|^{p(x)} - \lambda_\varepsilon I\bigr)\bigr], \end{aligned}$$

where $I = (|\eta - \nabla u_0|^{p(x)-2}(\eta - \nabla u_0) - |\eta|^{p(x)-2}\eta)\eta$.

Note that,

$$\bigl||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\bigr| \le \begin{cases} c_0|\xi - \eta|(|\xi| + |\eta|)^{p-2}, & \text{if } p \ge 2,\\ c_0|\xi - \eta|^{p-1}, & \text{if } 1 < p < 2, \end{cases} \qquad \text{see Azorero [1].}$$

So, when $p(x) \ge 2$, we have

$$\begin{aligned} |I| &= \bigl|\bigl(|\eta - \nabla u_0|^{p(x)-2}(\eta - \nabla u_0) - |\eta|^{p(x)-2}\eta\bigr)\bigr||\eta|\\ &\le c_0|\nabla u_0|\bigl(|\eta - \nabla u_0| + |\eta|\bigr)^{p(x)-2}|\eta|\\ &\le C|\nabla u_0|\bigl(|\eta|^{p(x)-2} + |\nabla u_0|^{p(x)-2}\bigr)|\eta|\\ &\le C|\eta|^{p(x)-1} + C \le \frac{1}{2}|\eta|^{p(x)} + C, \end{aligned}$$

and when $p(x) < 2$,

$$|I| = \big|\big(|\eta - \nabla u_0|^{p(x)-2}(\eta - \nabla u_0) - |\eta|^{p(x)-2}\eta\big)\big||\eta| \quad \le c_0|\nabla u_0|^{p(x)-1}|\eta|$$
$$\le C|\eta| \le \frac{1}{2}|\eta|^{p(x)} + C,$$

where C is a generic positive constant independent of ε.

Thus we have

$$A_\varepsilon(x,\eta)\eta \ge |\eta|^{p(x)} - \frac{\lambda_\varepsilon}{1-\lambda_\varepsilon}|I| \ge \frac{1}{2}|\eta|^{p(x)} - C.$$

It follows from Theorem 4.1 in [6] that $u_\varepsilon \in L^\infty(\Omega)$ and $|u_\varepsilon|_{L^\infty(\Omega)} \le C$, because $\|u_\varepsilon\|_{W_0^{1,p(x)}(\Omega)}$ is bounded uniformly for $\varepsilon \in (0,1)$, where C is a positive constant independent of ε. Below we shall prove that $\|u_\varepsilon\|_{C_0^{1,\alpha}(\Omega)} \le C$ for some $\alpha \in (0,1)$ by using Theorem 1.2 in [4] in the following two cases, respectively.

Case (I): $\lambda_\varepsilon \in [-1,0]$.

Noting that u_0 satisfies the equation, $-\mathrm{div}(|\nabla u_0|^{p(x)-2}\nabla u_0) = v_0$, where $v_0 \in L^{\beta(x)}(\Omega)$ and $v_0(x) \in \partial F(x,u_0(x))$ for almost all $x \in \Omega$. Equation (3.4) is equivalent to the following equation:

$$\begin{aligned}&-\mathrm{div}\big\{|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon - \lambda_\varepsilon|\nabla u_\varepsilon - \nabla u_0|^{p(x)-2}(\nabla u_\varepsilon - \nabla u_0)\\ &\quad - \lambda_\varepsilon|\nabla u_0|^{p(x)-2}\nabla u_0)\big\} = w_\varepsilon - \lambda_\varepsilon v_0.\end{aligned}$$

Define $\overline{B_\varepsilon}(x,\eta) = w_\varepsilon - \lambda_\varepsilon v_0$,

$$\overline{A_\varepsilon}(x,\eta) = |\eta|^{p(x)-2}\eta - \lambda_\varepsilon|\eta - \nabla u_0|^{p(x)-2}(\eta - \nabla u_0) - \lambda_\varepsilon|\nabla u_0|^{p(x)-2}\nabla u_0.$$

Then u_ε is a solution of the following problem:

$$\begin{cases}-\mathrm{div}\overline{A_\varepsilon}(x,\nabla u) = \overline{B_\varepsilon}(x,u), \quad \text{a.a. } x \in \Omega,\\ u|_{\partial\Omega} = 0.\end{cases}$$

We can prove that, for $x, y \in \overline{\Omega}$, $\eta \in \mathbb{R}^N\backslash\{0\}$, $\xi \in \mathbb{R}^N$, $t \in \mathbb{R}$, the following estimations hold:

$$\begin{cases}\overline{A_\varepsilon}(x,0) = 0, \quad \displaystyle\sum_{i,j=1}^n \frac{\partial(\overline{A_\varepsilon})_j}{\partial\eta_i}(x,\eta)\xi_i\xi_j \ge c_7|\eta|^{p(x)-2}|\xi|^2,\\ \displaystyle\sum_{i,j=1}^n \left|\frac{\partial(\overline{A_\varepsilon})_j}{\partial\eta_i}(x,\eta)\right||\eta| \le c_8\big(1 + |\eta|^{p(x)-1}\big),\\ \big|\overline{B_\varepsilon}(x,\eta)\big| \le c_9 + c_{10}|t|^{\alpha(x)-1} \le c_{11} + c_{12}|t|^{\alpha(x)},\end{cases} \tag{3.9}$$

and for sufficiently small $\delta > 0$, there exists a positive constant c_δ, depending on p^+, p^- and δ, but independent of $\lambda_\varepsilon \in [-1,0]$, such that

$$\big|\overline{A_\varepsilon}(x,\eta) - \overline{A_\varepsilon}(y,\eta)\big| \le c_\delta|x-y|^\beta\big(1 + |\eta|^{p_0-1+\delta}\big), \tag{3.10}$$

where $p_0 = \max\{p(x), p(y)\}$.

The proof of (3.9) is immediate (see [1]), here we only prove (3.10). It follows from $p \in C^{0,\beta}(\overline{\Omega})$ that

$$|\eta|^{p(x)-2}\eta - |\eta|^{p(y)-2}\eta \leq \big||\eta|^{p(x)-1} - |\eta|^{p(y)-1}\big| \\ \leq c|x-y|^{\beta}\big(1+|\eta|^{p_0-1}\big)\big(1+\big|\log|\eta|\big|\big).$$

For given $\delta \in (0, p^- - 1)$, by $\lim_{t\to+\infty} \frac{\log t}{t^{\delta}} = 0$ and $\lim_{t\to 0^+} \frac{\log t}{t^{-\delta}} = 0$, we have that there exists a positive constant $c(\delta)$, depending only on δ, such that $|\log|\eta|| \leq c(\delta) + |\eta|^{\delta} + |\eta|^{-\delta}$ for every $\eta \in \mathbb{R}^N$. Thus we obtain

$$\big||\eta|^{p(x)-2}\eta - |\eta|^{p(y)-2}\eta\big| \leq c_\delta|x-y|^{\beta}\big(1+|\eta|^{p_0-1+\delta}\big).$$

Similarly, we have

$$\begin{aligned}&\big||\eta-\nabla u_0|^{p(x)-2}(\eta-\nabla u_0) - |\eta-\nabla u_0|^{p(y)-2}(\eta-\nabla u_0)\big| \\ &\quad\leq c_\delta|x-y|^{\beta}\big(1+|\eta-\nabla u_0|^{p_0-1+\delta}\big) \\ &\quad\leq \overline{c_\delta}|x-y|^{\beta}\big(1+|\eta|^{p_0-1+\delta}\big)\end{aligned}$$

and

$$\big||\nabla u_0|^{p(x)-2}(\nabla u_0) - |\nabla u_0|^{p(y)-2}(\nabla u_0)\big| \leq c_\delta|x-y|^{\beta}\big(1+|\nabla u_0|^{p_0-1+\delta}\big) \\ \leq \overline{\overline{c_\delta}}|x-y|^{\beta}.$$

Hence, noting that $|\lambda_\varepsilon| \leq 1$, we see that (3.10) is true.

Case (II): $\lambda_\varepsilon < -1$.

From (3.4), we have

$$-\mathrm{div}\bigg(-\frac{1}{\lambda_\varepsilon}|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon + |\nabla u_\varepsilon - \nabla u_0|^{p(x)-2}(\nabla u_\varepsilon - \nabla u_0)\bigg) = -\frac{1}{\lambda_\varepsilon}w_\varepsilon.$$

Note that

$$-\mathrm{div}\bigg(\frac{1}{\lambda_\varepsilon}|\nabla u_0|^{p(x)-2}\nabla u_0\bigg) = \frac{1}{\lambda_\varepsilon}v_0.$$

So,

$$\begin{aligned}-\,\mathrm{div}\bigg[&|\nabla u_\varepsilon - \nabla u_0|^{p(x)-2}(\nabla u_\varepsilon - \nabla u_0) - \frac{1}{\lambda_\varepsilon}|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \\ &+ \frac{1}{\lambda_\varepsilon}|\nabla u_0|^{p(x)-2}\nabla u_0\bigg] = -\frac{1}{\lambda_\varepsilon}(w_\varepsilon - v_0).\end{aligned}$$

Then u_ε is a solution of the following problem:

$$\begin{cases}-\mathrm{div}\widetilde{A_\varepsilon}(x,\nabla u) = \widetilde{B_\varepsilon}(x,u), \quad \text{a.a. } x \in \Omega,\\ u|_{\partial\Omega} = 0,\end{cases}$$

where $\widetilde{B_\varepsilon}(x,t) = -\frac{1}{\lambda_\varepsilon}(w_\varepsilon - v_0)$,

$$\widetilde{A_\varepsilon}(x,\eta) = |\eta - \nabla u_0|^{p(x)-2}(\eta - \nabla u_0) - \frac{1}{\lambda_\varepsilon}|\eta|^{p(x)-2}\eta + \frac{1}{\lambda_\varepsilon}|\nabla u_0|^{p(x)-2}\nabla u_0.$$

Analogously to the case (I), we can prove that $\widetilde{A_\varepsilon}$ and $\widetilde{B_\varepsilon}$ satisfy the corresponding conditions (3.9).

From the analysis in case (I) and case (II), we know that Theorem 1.2 in [4] is available. Hence $u_\varepsilon \in C_0^{1,\alpha}(\overline{\Omega})$ and $\|u_\varepsilon\|_{C_0^{1,\alpha}(\overline{\Omega})} \le C$, that is $u_\varepsilon \to u_0$ in $C^1(\overline{\Omega})$ (by $C_0^{1,\alpha}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$ compact embedding) as $\varepsilon \to 0$. The proof of Theorem 1 is complete. □

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