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Zeros of the Lerch Transcendent Function*

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Abstract. We investigate the distribution of zeros of the Lerch transcendent function $\Phi(q,s,\alpha)=\sum_{n=0}^{\infty} q^n(n+\alpha)^{-s}$. We find an upper and lower estimates of zeros of the function $\Phi(q,s,\alpha)$ in any rectangle $\{s : \sigma_1 < \text{Re } s < \sigma_2 \le 1.73\ldots, 0 < \text{Im } s \le T\}$. Further we are interested in a computer calculations concerning the zeros of $\Phi(q,s,\alpha)$ in $\{s : \text{Re } s > 1, 0 < \text{Im } s \le 1000\}$.

Keywords: polylogarithm, Lerch transcendent, zero distribution.

AMS Subject Classification: 11M41.

1 Introduction

Let $s=\sigma+it$ denotes a complex variable. The Lerch transcendent function is the analytic continuation of the series

$$\Phi(q,s,\alpha)=\sum_{n=0}^{\infty} q^n(n+\alpha)^{-s},$$

which converges for any real number $\alpha>0$ if q and s are complex numbers with either $|q|<1$, or $|q|=1$ and $\sigma>1$. Here we consider $\Phi(q,s,\alpha)$ as a function of s with the parameters $q\in\mathbb{C}$, $0<|q|\le 1$, and $0<\alpha\le 1$. Special cases include the Riemann zeta-function $\zeta(s)=\Phi(1,s,1)$, the Hurwitz zeta-function $\zeta(s,\alpha)=\Phi(1,s,\alpha)$, the polylogarithm function $Li_s(q)=q\Phi(q,s,1)$, and the Lerch zeta-function $L(\lambda,\alpha,s)=\Phi(\exp(2\pi i\lambda),s,\alpha)$.

The Riemann zeta-function has no zeros in the right-half-plane $\sigma\ge 1$. In the left-half-plane $\sigma\le 0$ it has only trivial zeros at even negative integers. The famous Riemann hypothesis (RH) asserts that the remaining, nontrivial, zeros lie on the critical line $\sigma=1/2$.

The Hurwitz zeta-function $\zeta(s,\alpha)$ has infinitely many zeros in $1<\sigma<1+\alpha$ if α is transcendental or rational $\ne 1/2, 1$ (Davenport and Heilbronn [2]). This

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result was extended by Cassels [1] for α algebraic irrational. Let $1/2 < \sigma_1 < \sigma_2 < 1$. Then Voronin [16] (for rational $\alpha \neq 1/2, 1$) and Gonek [10] (for transcendental α) proved that the number of zeros of $\zeta(s, \alpha)$ in the rectangle $\sigma_1 < \sigma < \sigma_2$, $0 < t < T$ is approximately equal to T for sufficiently large T. Gonek [11] also showed that for certain values of α the proportion of zeros of $\zeta(s, \alpha)$ on $\sigma = 1/2$ is definitely less than 1. In the complex s-plane trajectories of zeros $\rho = \rho(\alpha)$ of the Hurwitz zeta function were considered in [8] and [9]. Based on these trajectories, the classification of nontrivial zeros of the Riemann zeta function were introduced. For the zero distribution of the Lerch zeta-function see [4, 5, 6, 7, 12].

Fornberg and Kölbig [3] investigated trajectories of zeros $\rho = \rho(q)$ of the polylogarithm function $Li_s(q)$ for real q with $|q| < 1$. They found that some trajectories tend towards the zeros of $\zeta(s)$ as $q \to -1$, and approach these zeros closely as $q \to 1 - \delta$ for small but finite $\delta > 0$. However, the later trajectories appear to descend to the point $s = 1$ as $\delta \to 0$. Both, for $q \to -1$ and $q \to 1$, there are trajectories which do not tend towards zeros of $\zeta(s)$.

Next we consider the zeros of $\Phi(q, s, \alpha)$ for $0 < \alpha < 1$ and $q \in \mathbb{C}$, $0 < |q| < 1$. Let $N_\Phi(\sigma_1, \sigma_2, T) = N_\Phi(\sigma_1, \sigma_2, T, q, \alpha)$ denote the number of zeros of $\Phi(q, s, \alpha)$ in the region $\{s \colon \sigma_1 < \operatorname{Re} s < \sigma_2,\ 0 < \operatorname{Im} s \leq T\}$. Let $\sigma_0 = \sigma_0(q, \alpha)$ be a real number defined by the equality

$$\sum_{n=1}^{\infty} \frac{|q|^n}{(n/\alpha + 1)^{\sigma_0}} = 1.$$

It is easy to see that $\sigma_0 \leq c = 1.73\ldots$, where $\zeta(c) = \sum_{n=1}^{\infty} n^{-c} = 2$, and that σ_0 can take any value between $-\infty$ and c.

Theorem 1. *Let $q \in \mathbb{C}$, $0 < |q| < 1$. Let $0 < \alpha < 1$ be a transcendental number. Then we have that, for any fixed strip $\sigma_1 < \sigma < \sigma_2 \leq \sigma_0$,*

$$T \ll N_\Phi(\sigma_1, \sigma_2, T) \ll T$$

and $\Phi(q, s, \alpha)$ has no zeros for $\sigma > \sigma_0$.

The theorem is proved in Section 3.

Wiener and Wintner [17, Section 4] pointed to a possible relationship between the behaviour of the zeros in the right-half-plane $\sigma > 1$ of the polylogarithm function and the Riemann Hypothesis. They proved that the Riemann Hypothesis is true if there exists a number $0 < \varepsilon < 1$ such that $\sum_{n=1}^{\infty} q^n n^{-s} \neq 0$ for $\sigma > 1$ and $1 - \varepsilon < q < 1$. However, Montgomery [13] pointed that the polylogarithm function $Li_s\left(e^{-1/N}\right)$ has zeros in the region $\sigma > 1$ for all sufficiently large integers N, making Wiener and Winter theorem vacuous. Theorem 1 shows that the Lerch transcendent function $\Phi(q, s, \alpha)$ also has zeros in the region $\sigma > 1$ for $0.92 < q < 1$ and transcendental α, $1/2 < \alpha < 1$. In the next section, we try to find explicit zeros in $\sigma > 1$. We see that it is relatively easy to find zeros if $\alpha \neq 1$. In the case $\alpha = 1$ the zeros in the right half-lane, $\sigma > 1$ currently are out of reach.

2 Calculations

The calculations of this section were done with programme MATHEMATICA. To calculate the number N of zeros of $\varPhi(q,s,\alpha)$ inside the contour $\varGamma$ we have used the well known formula

$$N = \frac{1}{2\pi i}\int_{\varGamma} \frac{(\varPhi(q,s,\alpha))'_s}{\varPhi(q,s,\alpha)}\, ds.$$

If the interior of the contour $\varGamma$ contains one zero ρ, then we find this zero using the following expression

$$\rho = \frac{1}{2\pi i}\int_{\varGamma} s\frac{(\varPhi(q,s,\alpha))'_s}{\varPhi(q,s,\alpha)}\, ds.$$

The zero ρ can be adjusted by MATHEMATICA command *FindRoot*.

Let $R = \{s\colon\ \operatorname{Re} s > 1,\ 0 < \operatorname{Im} s \le 1000\}$. In Table 1, we present the number of zeros of function $\varPhi(q,s,\alpha)$ for chosen q and α in the region R. For example, we see that $\varPhi(0.99,s,0.9)$ has 34 zeros in R. In Table 1, the last column describes zeros of the Hurwitz zeta-function, and the last row describes zeros of the polylogarithm function. In view of Montgomery's result [13] we expect that $\varPhi(q,s,1)$ has zeros in $\sigma > 1$ for $q \ge 0.9$. If so, then Table 1 possibly indicates the different behaviour of zeros of $\varPhi(q,s,\alpha)$ in $\sigma > 1$ dependently on $\alpha = 1$ or $\alpha \ne 1$.

Table 1. Number of zeros of the function $\varPhi(q,s,\alpha)$ in the region R.

α \ q	**0.9**	**0.95**	**0.99**	**1**
0.9	2	8	34	40
0.95	4	10	37	46
0.99	14	27	41	45
1	0	0	0	0

In Table 2, we present zeros of functions $\varPhi(0.9,s,0.9)$, $\varPhi(0.9,s,0.95)$, $\varPhi(0.9,s,0.99)$. In this table numbers were rounded up to two decimal places.

3 Proof of Theorem 1

First we formulate theorems of Kronecker and Rouché (see Tichmarsh [15, Section 8.3] and Tichmarsh [14, Section 3.42]).

Lemma 1 [Kronecker's theorem]. *Let $a_1, a_2, \dots, a_N$ be linearly independent real numbers, i.e. numbers such that relation $\lambda_1 a_1 + \dots + \lambda_N a_N = 0$ is possible only if $\lambda_1 = \dots = \lambda_N = 0$. Let $b_1, \dots, b_N$ be any real numbers, and ε a given positive number. Then we can find a number t and integers $x_1, \dots, x_N$ such that $|ta_n - b_n - x_n| < \varepsilon$, $n = 1, \dots, N$.*

Table 2. Coordinates of zeros of the function $\Phi(q, s, \alpha)$ in the region R.

	$\Phi(0.9, s, 0.9)$	$\Phi(0.9, s, 0.95)$	$\Phi(0.9, s, 0.99)$
1	$1.02 + 550.55i$	$1.07 + 108.39i$	$1.05 + 480.29i$
2	$1.02 + 609.75i$	$1.01 + 135.21i$	$1.11 + 525.79i$
3	–	$1.09 + 169.68i$	$1.08 + 588.57i$
4	–	$1.07 + 196.67i$	$1.06 + 616.03i$
5	–	–	$1.11 + 651.27i$
6	–	–	$1.11 + 696.71i$
7	–	–	$1.13 + 724.38i$
8	–	–	$1.05 + 759.64i$
9	–	–	$1.15 + 787.05i$
10	–	–	$1.02 + 805.00i$
11	–	–	$1.12 + 849.96i$
12	–	–	$1.17 + 895.31i$
13	–	–	$1.09 + 958.10i$
14	–	–	$1.00 + 985.50i$

Lemma 2 [Rouché's theorem]. *Suppose that $f(s)$ and $g(s)$ are analytic functions inside and on a regular closed curve γ, and that $|f(s)| > |g(s)|$ for all $s \in \gamma$. Then $f(s) + g(s)$ and $f(s)$ have the same number of zeros inside γ.*

The next lemma will be useful in the proof of Theorem 1.

Lemma 3. *Let $q \in \mathbb{C}$, $0 < |q| < 1$, and $0 < \alpha < 1$ be a transcendental number. Let σ' be a real number. Let $a(n)$ be a sequence of complex numbers such that $|a(n)| = 1$. Let $\Phi_a(q, s, \alpha) = \sum_{n=0}^{\infty} a(n) q^n (n + \alpha)^{-s}$. Then for any $\varepsilon > 0$ there exist $\tau \in \mathbb{R}$ such that*

$$\left|\Phi(q, s + i\tau, \alpha) - \Phi_a(q, s, \alpha)\right| < \varepsilon$$

for $\operatorname{Re} s \geq \sigma'$.

Proof. The Dirichlet series of the Lerch transcendent function converges absolutely for any s if $|q| < 1$. Therefore, for given σ' there is a positive integer N such that, for any real number u and $\sigma \geq \sigma'$,

$$\left|\sum_{n=N+1}^{\infty} \frac{q^n}{(n + \alpha)^{s+iu}} - \sum_{n=N+1}^{\infty} \frac{q^n a(n)}{(n + \alpha)^{s}}\right| \leq 2 \sum_{n=N+1}^{\infty} \frac{|q|^n}{(n + \alpha)^{\sigma}} < \frac{\varepsilon}{2}. \tag{3.1}$$

Let $A = \sum_{n=0}^{N} |q|^n / (n + \alpha)^{\sigma'}$. There is a sequence of real numbers $b(n)$ such that $e^{-2\pi i b(n)} = a(n)$. The numbers $\log(n + \alpha)$ are linearly independent over $\mathbb{Q}$ since α is the transcendental number. By Kronecker's theorem (Lemma 1), there exist a real number τ and integers x_n such that

$$\left|\frac{\tau \log(n + \alpha)}{2\pi} - b(n) - x_n\right| < \frac{\varepsilon}{8\pi A}.$$

In view of the inequality $|e^z - 1| \le 2|z|$, where $|z| < 1$, we obtain

$$\left|(n+\alpha)^{-i\tau} - a(n)\right| = \left|e^{-2\pi i(\tau \log(n+\alpha)/2\pi - b(n) - x_n)} - 1\right| < \frac{\varepsilon}{2A}.$$

By above we see that there is τ such that, for $\operatorname{Re} s \ge \sigma'$,

$$\left|\sum_{n=0}^{N} \frac{q^n}{(n+\alpha)^{s+i\tau}} - \sum_{n=0}^{N} \frac{q^n a(n)}{(n+\alpha)^s}\right| \le \sum_{n=0}^{N} \frac{|q|^n}{(n+\alpha)^{\sigma'}} \left|(n+\alpha)^{-i\tau} - a(n)\right| < \frac{\varepsilon}{2}.$$

This and inequality (3.1) in view of triangle inequality, prove Lemma 3. □

Proof of Theorem 1. For fixed q and α the function $\Phi(q,s,\alpha)$ is bounded in any right half-plane, of complex numbers. This together with Theorem 9.62 of Titchmarsh [14] give the bound

$$N_\Phi(\sigma_1, \sigma_2, T) \ll T.$$

Further, if the strip $\sigma_1 < \sigma < \sigma_2$ contains a zero of $\Phi(q,s,\alpha)$ then, arguing as in Lemma 1 of [4], we get the bound

$$N_\Phi(\sigma_1, \sigma_2, T) \gg T.$$

Next we will show that the function $\Phi(q,s,\alpha)$ has a zero in the strip $\sigma_1 < \sigma < \sigma_2$. We consider an auxiliary function $\Phi_a(q,\sigma,\alpha) = \sum_{n=0}^{\infty} a(n) q^n (n+\alpha)^{-\sigma}$. For fixed σ, q and α, let V be a set of values taken by $\Phi_a(q,\sigma,\alpha)$ for independent $a(0)$, $a(1)$, …, where $a(n) \in \mathbb{C}$ and $|a(n)| = 1$. If $\sigma < \sigma_0$, then by Tichmarsh [15, Section 11.5, p. 297] we see that

$$V = \left\{ z\colon |z| \le \sum_{n=0}^{\infty} |q|^n (n+\alpha)^{-\sigma} \right\}.$$

Thus for $\sigma_1 < \sigma' < \sigma_2$, q, and α there is a sequence $a(1), a(2), \ldots$, such that $\Phi_a(q,\sigma',\alpha) = 0$.

Let $0 < \varepsilon' < \min(\sigma' - \sigma_1, \sigma_2 - \sigma')$ be such that $\Phi_a(q,s,\alpha) \ne 0$ for $|s - \sigma'| = \varepsilon'$. Let

$$\varepsilon = \min_{|s-\sigma'|=\varepsilon'} \left|\Phi_a(q,s,\alpha)\right|.$$

By Lemma 3 there is a real shift τ such that

$$\left|\Phi(q, s+i\tau, \alpha) - \Phi_a(q,s,\alpha)\right| < \varepsilon$$

for $\operatorname{Re} s \ge \sigma_1$. Hence Rouché's theorem gives that $\Phi(q,s,\alpha)$ has a zero in the disk $|s - \sigma' - i\tau| < \varepsilon'$, which is contained in the strip $\sigma_1 < \sigma < \sigma_2$. By this Theorem 1 is proved. □

4 Conclusions

Let $0 < q < 1$ and $1/2 < \alpha \le 1$. We expect that the Lerch transcendent function $\Phi(q,s,\alpha)$ has zeros in $\operatorname{Re} s > 1$, if q is sufficiently near to 1. For $\alpha = 1$ this is due to Montgomery [13]. Here we prove the case when α is a transcendental number. However, computer calculations indicate the different behaviour of zeros of $\Phi(q,s,\alpha)$ in $\operatorname{Re} s > 1$ dependently on $\alpha = 1$ or $\alpha \ne 1$.

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