On an Elliptic System of $p(x)$-Kirchhoff-Type under Neumann Boundary Condition

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Abstract. In the present paper, by using the direct variational method and the Ekeland variational principle, we study the existence of solutions for an elliptic system of $p(x)$-Kirchhoff-type under Neumann boundary condition and show the existence of a weak solution.

Keywords: variational method, $p(x)$-Kirchhoff-type equation, variable exponent Lebesgue–Sobolev space, nonlocal problems, Ekeland variational principle.

AMS Subject Classification: 35D05; 35J60; 35J70.

1 Introduction

We are concerned with the following problem

$$\begin{cases} -M_1\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f(u,v) + h_1(x) \text{ in } \Omega, \\ -M_2\left(\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx\right)\operatorname{div}\left(|\nabla v|^{p(x)-2}\nabla v\right) = g(u,v) + h_2(x) \text{ in } \Omega, \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \text{ on } \partial\Omega, \end{cases}$$

(1.1)

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $p \in C(\overline{\Omega})$ with $1 < p(x) < N$, η is the unit exterior vector on $\partial\Omega$. The system (1.1) is a generalization of a model, so-called Kirchhoff equation, introduced by Kirchhoff [17]. To be more precise, Kirchhoff established a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2\,dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

(1.2)

where ρ, P_0, h, E, L are constants, which extends the classical D'Alambert's wave equation, by considering the effects of the changes in the length of the

strings during the vibrations. A distinguish feature of the Kirchhoff equation (1.2) is that the equation contains a nonlocal coefficient $\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2\, dx$ which depends on the average $\frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2\, dx$ of the kinetic energy $\frac{1}{2}|\frac{\partial u}{\partial x}|^2$ on $[0, L]$, and hence the equation is no longer a pointwise identity.

On the other hand, the stationary counterpart of (1.2) is given by

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\, dx\right)\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

which has attracted much attention after Lions' paper [19] in which an abstract framework to the problem was given. For some interesting results we refer to [4, 5, 11]. Moreover, nonlocal boundary value problems like (1.1) can be used for modeling several physical and biological systems where u describes a process which depend on the average of itself, such as the population density [2, 6].

We want to remark that in studying the Kirchhoff-type equations establishing conditions on M and f is the key argument. In that context the following condition is typical

$$M(t) \geq m_0 > 0, \quad \text{for all } t \geq 0. \tag{1.4}$$

In [1], the authors show the following Kirchhoff problem has a positive solution

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

under the conditions (1.4) and

$$\widehat{M}(t) = \int_0^t M(s)\, ds \geq M(t)t, \quad \text{for all } t \geq 0. \tag{1.6}$$

However, according to the original meaning of the M, in the Kirchhoff equation (1.5), it should be an increasing function. Then,

$$\widehat{M}(t) < \int_0^t M(t)\, ds = M(t)t, \quad \text{for all } t > 0$$

and therefore, condition (1.6) cannot be satisfied. To overcome these difficulties, the authors assume the following:

There exists $m_1 \geq m_0$ and $t_0 > 0$ such that $M(t) = m_1$, for all $t \geq t_0$.

In [7], the authors show the following p-Kirchhoff equation has positive solutions

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p\, dx\right)^{p-1}\Delta_p u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $M(t)$ satisfies (1.4) for all $t \in \mathbb{R}^+$ and

$$\widehat{M}(t) \geq [M(t)]^{p-1}t, \quad \text{for all } t \geq 0. \tag{1.8}$$

However, the condition imposed on M is far away from the physical sense of the original Kirchhoff equation. Hence, the authors use the similar arguments used in [1]. In [8], by using Krasnoselskii's genus it is showed that (1.7) has infinitely many solutions under the conditions $At^{\alpha} \leq [M(t)]^{p-1} \leq Bt^{\alpha}$ and $Q_1 t^{q-1} \leq f(x,t) \leq Q_2 t^{q-1}$ for all $t \geq 0$ and for all $x \in \overline{\Omega}$, where A, B, α, Q_1, Q_2, q positive constants, $q \in (p, p^*)$ and $\alpha p > q$. However, under these assumptions the condition (1.4) fails.

The generalization of Kirchhoff equations to $p(x)$-Kirchhoff equation is a quite new topic, so there exists only a few paper [10]. In [10], Dai and Hao study $p(x)$-Kirchhoff equation

$$\begin{cases} -M\Big(\int_{\Omega} \dfrac{|\nabla u|^{p(x)}}{p(x)}\, dx \Big) \operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

and show the existence and multiplicity of solutions of (1.9), where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition, $M(t)$ satisfies (1.4) and

$$\exists\, 0 < \mu < 1 \text{ such that } \widehat{M}(t) \geq (1 - \mu)M(t)t. \tag{1.10}$$

Even under the constant exponent case, condition (1.10) is weaker than condition (1.8). It forces the authors to deal with more potential functions than [7]; for example, they deal with increasing function $M(t) = a + bt$.

In recent years, an increasing attention has been paid to the study of differential equations and variational problems with nonstandard growth conditions. The main interest in studying such problems arises from the presence of the $p(x)$-Laplace operator $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. The $p(x)$-Laplace operator is a generalization of p-Laplace operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ obtained in the case when $p(x) \equiv p$ (constant). It is well known that, comparing with the p-Laplace operator, the $p(x)$-Laplace operator possesses more complicated nonlinear properties, for example, it is inhomogeneous and usually it does not have the so-called first eigenvalue, since the infimum of its principle eigenvalue is zero. This causes many problems, some classical theories and methods, such as the Lagrange multiplier theorem and the theory of Sobolev spaces, are not applicable.

The nonlinear problems involving the $p(x)$-Laplace operator are extremely attractive because they can be used to model dynamical phenomenons which arise from the study of electrorheological fluids or elastic mechanics. Problems with variable exponent growth conditions also appear in the modelling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium [3, 16, 20, 21].

In the present paper, we consider an elliptic system of $p(x)$-Kirchhoff-type equation (1.1) and when doing this, we don't apply any kind of mostly-used constraints, i.e., growth or asymptotic conditions or Ambrosetti-Rabinowtz condition etc., to nonlinearities $f(x,u)$ and $g(x,u)$. Instead, using the theory of variable exponent Lebesgue–Sobolev spaces and the Ekeland variational principle, we establish the existence of a weak solution. As far as we are concerned, system of $p(x)$-Kirchhoff-type equation has not been investigated yet.

Nevertheless, in [9] Corrêa and Nascimento investigated an elliptic system of p-Kirchhoff-type equation of (1.1). The present paper can be regarded as a generalization of [9]. However, there are some differences; for example, we assume that $\alpha > 0$, hence for $\alpha = 1$ our condition (**M**) (see the main results) coincides with the following condition:

(M^*) $M_1, M_2 : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions and there is a positive constant m_0 such that $M_1(t), M_2(t) \geq m_0 > 0$, for all $t \geq 0$,

which is assumed in [9]. Moreover, when $\alpha \in (0, 1)$ the Kirchhoff functions M_1, M_2 may be singular at $t = 0$ in (**M**) while it is not the case for (M^*).

2 Preliminaries

Firstly, we state some basic properties of the variable exponent Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ (for details, see [14, 15, 18]).

Set $L_+^\infty(\Omega) = \{p;\ p \in L^\infty(\Omega),\ \operatorname{ess\,inf} p(x) > 1,\ \forall x \in \Omega\}$. For any $p \in L_+^\infty(\Omega)$, denote

$$1 < p^- := \operatorname*{ess\,inf}_{x \in \Omega} p(x) \leq p(x) \leq p^+ := \operatorname*{ess\,sup}_{x \in \Omega} p(x) < \infty.$$

Let $p \in L_+^\infty(\Omega)$. We define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{u \mid u : \Omega \to \mathbb{R} \text{ is measurable, } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\},$$

then $L^{p(x)}(\Omega)$ endowed with the norm

$$|u|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|u(x)/\lambda\right|^{p(x)}\,dx \leq 1\right\},$$

becomes a Banach space. The modular of $L^{p(x)}(\Omega)$ is the mapping $\rho(u) :$ $L^{p(x)}(\Omega) \to \mathbb{R}$ which is defined by

$$\rho(u) = \int_\Omega |u(x)|^{p(x)}\,dx, \quad \forall u \in L^{p(x)}(\Omega).$$

Proposition 1. [14, 18] *If $u, u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, \ldots$, then the following statements are equivalent*:

(i) $\lim\limits_{n \to \infty} |u_n - u|_{p(x)} = 0$;   (ii) $\lim\limits_{n \to \infty} \rho(u_n - u) = 0$;

(iii) $u_n \to u$ *in measure in Ω and* $\lim\limits_{n \to \infty} \rho(u_n) = \rho(u)$.

Proposition 2. [14, 18] *If $u, u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, \ldots$, then we have*

(i) $|u|_{p(x)} < 1\ (= 1;\ > 1) \iff \rho(u) < 1\ (= 1;\ > 1)$;

(ii) $|u|_{p(x)} > 1 \implies |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$;

(iii) $|u|_{p(x)} < 1 \implies |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$;

(iv) $\quad \lim_{n \to \infty} |u_n|_{p(x)} = 0 \iff \lim_{n \to \infty} \rho(u_n) = 0;$

(v) $\quad \lim_{n \to \infty} |u_n|_{p(x)} \to \infty \iff \lim_{n \to \infty} \rho(u_n) \to \infty.$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

Proposition 3. [14, 18] *If $1 < p^-$ and $p^+ < \infty$, then the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 4. [14, 18] *Let $\Omega \subset \mathbb{R}^N$ be bounded and $p \in C(\overline{\Omega})$. If $q \in C(\overline{\Omega})$ and $1 \leq q(x) < p^*(x) = Np(x)/(N - p(x))$ for any $x \in \overline{\Omega}$, then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.*

Proposition 5. [14, 18] *If $p \in L^\infty(\Omega)$, the conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left| \int_\Omega uv\, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p^-)'} \right) |u|_{p(x)} |v|_{p'(x)}.$$

3   The Main Results

In the sequel, for the simplicity of notation, we write $X := W^{1,p(x)}(\Omega) \times W^{1,p(x)}(\Omega)$ and $\widehat{\rho}(\nabla u) := \int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}\, dx$.

DEFINITION 1. We say that $(u, v) \in X$ is a weak solution of (1.1) if

$$M_1\big(\widehat{\rho}(\nabla u)\big) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi\, dx = \int_\Omega \big(f(u, v) + h_1(x)\big)\varphi\, dx,$$

$$M_2\big(\widehat{\rho}(\nabla v)\big) \int_\Omega |\nabla v|^{p(x)-2} \nabla v \nabla \psi\, dx = \int_\Omega \big(g(u, v) + h_2(x)\big)\psi\, dx,$$

for any $(\varphi, \psi) \in X$.

We will study problem (1.1) under the following assumptions:

(**M**): $M_1, M_2 : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions. There exist real numbers $m_0 > 0$ and $\alpha > 0$ such that $M_1(t), M_2(t) \geq m_0 t^{\alpha-1}$, for all $t > 0$. Note that the Kirchhoff functions M_1, M_2 may be singular at $t = 0$ for $\alpha \in (0, 1)$;

(**F$_1$**): There is a C^1-function $F : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial F}{\partial u}(u, v) = f(u, v) \quad \text{and} \quad \frac{\partial F}{\partial v}(u, v) = g(u, v),$$

for all $(u, v) \in \mathbb{R}^2$;

($\mathbf{F_2}$): There is $k > 0$ such that $F(u, v) = F(u + k, v + k)$, for all $(u, v) \in \mathbb{R}^2$;

(**h**): $h_1, h_2 \in L^{q(x)}(\Omega)$, $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, $1 < q(x) < p^*(x)$ such that

$$\int_\Omega h_1(x)\, dx = \int_\Omega h_2(x)\, dx = 0.$$

The energy functional corresponding to problem (1.1) is defined as $I : X \to \mathbb{R}$,

$$I(u, v) = \widehat{M_1}\big(\widehat{\rho}(\nabla u)\big) + \widehat{M_2}\big(\widehat{\rho}(\nabla v)\big)$$
$$- \int_\Omega F(u, v)\, dx - \int_\Omega h_1(x) u\, dx - \int_\Omega h_2(x) v\, dx$$

for all $(u, v) \in X$, where $\widehat{M_i}(t) = \int_0^t M_i(s)\, ds$, $i = 1, 2$. In a standard way, it can be proved that $I \in C^1(X \backslash \{(0,0)\}, \mathbb{R})$. Moreover, for every $(u, v) \in X \backslash \{(0,0)\}$, the derivative of I is given by

$$\langle I'(u, v), (\phi, \omega)\rangle = M_1\big(\widehat{\rho}(\nabla u)\big) \int_\Omega |\nabla u|^{p(x)-2}\, \nabla u \nabla \phi\, dx + M_2\big(\widehat{\rho}(\nabla v)\big)$$
$$\times \int_\Omega |\nabla v|^{p(x)-2}\, \nabla v \nabla \omega\, dx - \int_\Omega \big(f(u, v) + h_1(x)\big)\phi\, dx - \int_\Omega \big(g(u, v) + h_2(x)\big)\omega\, dx$$

for any $(\phi, \omega) \in X$. Thus $(u, v) \in X \backslash \{(0,0)\}$ is a weak solution of (1.1) if and only if (u, v) is a nontrivial critical point of I.

Our main result is the following theorem:

Theorem 1. *Suppose* (**M**)*,* ($\mathbf{F_1}$)*,* ($\mathbf{F_2}$) *and* (**h**) *hold. Then, problem* (1.1) *has a weak solution* $(u_0, v_0) \in X$ *if I is differentiable at* (u_0, v_0).

By the same idea developed by Fan in [13], we can split the space $W^{1,p(x)}(\Omega)$ in the following way. Define

$$W_0 = \Big\{w \in W^{1,p(x)}(\Omega) \colon \int_\Omega w\, dx = 0\Big\}.$$

For $w \in W^{1,p(x)}(\Omega)$, denote $\overline{w} = \frac{1}{|\Omega|} \int_\Omega w\, dx$ and $\widetilde{w} = w - \overline{w}$. Then $w = \widetilde{w} + \overline{w}$, where $\overline{w} \in \mathbb{R}$ and $\widetilde{w} \in W_0$. So, $W^{1,p(x)}(\Omega) = W_0 \oplus \mathbb{R}$. W_0 is a closed linear subspace of $W^{1,p(x)}(\Omega)$ with codimension 1.

Proposition 6 [Poincaré inequality]. [13] *Let* $\Omega \subset \mathbb{R}^N$ *be bounded and smooth. There exists a positive constant c such that*

$$|w|_{p(x)} \le c\, |\nabla w|_{p(x)}, \quad \forall w \in W_0.$$

Proof. We will give a similar proof to Fan's stated in [13]. Arguing by contradiction, assume that there exists a sequence $\{w_n\} \subset W_0$ such that $|w_n|_{p(x)} \ge n\, |\nabla w_n|_{p(x)}$. Without loss of generality we may assume $|w_n|_{p(x)} = 1$, then $|\nabla w_n|_{p(x)} \le \frac{1}{n}$. We may assume, taking a subsequence if necessary, that

$w_n \rightharpoonup w_0$ in $W^{1,p(x)}(\Omega)$, $w_n \to w_0$ in $L^{p(x)}(\Omega)$ and $w_n(x) \to w_0(x)$ for a.e. $x \in \Omega$. So we have that $|w_0|_{p(x)} = 1$, $\int_\Omega w_0 \, dx = \lim_{n\to\infty} \int_\Omega w_n \, dx = 0$, and $\int_\Omega |\nabla w_0|^{p(x)} \, dx \le \lim_{n\to\infty} \int_\Omega |\nabla w_n|^{p(x)} \, dx = 0$. It follows that $w_0 \in W_0$ and $\nabla w_0 = 0$, consequently $w_0 = 0$, which contradicts $|w_0|_{p(x)} = 1$. $\square$

To get the proof of Theorem 1, we will use the following version of the Ekeland variational principle.

Proposition 7. [12] *Let E be a Banach space and let $\Psi : E \to \mathbb{R}$ be a C^1-function which is bounded from below. Then, for any $\epsilon > 0$, there exists $w_\epsilon \in E$ such that*

$$\Psi(w_\epsilon) \le \inf_E \Psi + \epsilon \quad and \quad \left\|\Psi'(w_\epsilon)\right\|_{E^*} \le \epsilon.$$

Lemma 1. *The functional I is bounded from below.*

Proof. At the beginning, we will show that the energy functional I is well defined. To obtain this, it is enough to verify that the functional $J : X \to \mathbb{R}$, $J(u,v) = \int_\Omega F(u,v) \, dx$, is well defined. Since F is continuous on $[0,k] \times [0,k]$ and $F(u,v) = F(u+k,v+k)$ for all $(u,v) \in \mathbb{R}^2$, we obtain $|F(u,v)| \le c_1$ for all $(u,v) \in \mathbb{R}^2$. Hence,

$$\left|J(u,v)\right| \le \int_\Omega \left|F(u,v)\right| dx \le c_1|\Omega|, \quad \text{for all } (u,v) \in X,$$

i.e., J is well defined, where $|\Omega|$ is the Lebesgue measure of Ω. Now, we will show that I is bounded from below. Let $(u,v) \in X$. Then u and v can be written as $u = \tilde{u} + \overline{u}$ and $v = \tilde{v} + \overline{v}$, where $\overline{u}, \overline{v} \in \mathbb{R}$ and $\tilde{u}, \tilde{v} \in W_0$ with $\int_\Omega \tilde{u} \, dx = \int_\Omega \tilde{v} \, dx = 0$. Therefore,

$$I(u,v) = \widehat{M_1}\big(\widehat{\rho}(\nabla u)\big) + \widehat{M_2}\big(\widehat{\rho}(\nabla v)\big) - \int_\Omega F(\tilde{u}+\overline{u}, \tilde{v}+\overline{v}) \, dx$$

$$- \int_\Omega h_1(x)(\tilde{u}+\overline{u}) \, dx - \int_\Omega h_2(x)(\tilde{v}+\overline{v}) \, dx \ge \widehat{M_1}\big(\widehat{\rho}(\nabla u)\big) + \widehat{M_2}\big(\widehat{\rho}(\nabla v)\big)$$

$$- c_1|\Omega| - \int_\Omega h_1(x)\tilde{u} \, dx - \overline{u}\int_\Omega h_1(x) \, dx - \int_\Omega h_2(x)\tilde{v} \, dx - \overline{v}\int_\Omega h_2(x) \, dx.$$

If we consider that $h_1, h_2 \in L^{q(x)}(\Omega)$; $\tilde{u}, \tilde{v} \in L^{p(x)}(\Omega)$ and use Proposition 5, it follows

$$I(u,v) \ge \widehat{M_1}\big(\widehat{\rho}(\nabla u)\big) + \widehat{M_2}\big(\widehat{\rho}(\nabla v)\big) - c_1|\Omega| - |h_1|_{q(x)} |\tilde{u}|_{p(x)} - |h_2|_{q(x)} |\tilde{v}|_{p(x)}.$$

From the assumption (**M**) and Proposition 6, we have

$$I(u,v) \ge m_0 \int_0^{\frac{1}{p^+}\rho(\nabla u)} s^{\alpha-1} \, ds + m_0 \int_0^{\frac{1}{p^+}\rho(\nabla v)} s^{\alpha-1} \, ds$$

$$- c_3 |\nabla\tilde{u}|_{p(x)} - c_4 |\nabla\tilde{v}|_{p(x)} - c_2$$

$$= \frac{m_0}{\alpha(p^+)^\alpha} \big[(\rho(\nabla u))^\alpha + (\rho(\nabla v))^\alpha \big] - c_3 |\nabla\tilde{u}|_{p(x)} - c_4 |\nabla\tilde{v}|_{p(x)} - c_2.$$

By Proposition 2, we have

$$I(u,v) \geq \frac{m_0}{\alpha(p^+)^\alpha} \left[\left(\rho(\nabla u)\right)^\alpha + \left(\rho(\nabla v)\right)^\alpha \right] - c_3 \max\left\{ \left(\rho(\nabla \widetilde{u})\right)^{\frac{1}{p^-}}, \left(\rho(\nabla \widetilde{u})\right)^{\frac{1}{p^+}} \right\}$$

$$- c_4 \max\left\{ \left(\rho(\nabla \widetilde{v})\right)^{\frac{1}{p^-}}, \left(\rho(\nabla \widetilde{v})\right)^{\frac{1}{p^+}} \right\}. \tag{3.1}$$

So, I is bounded from below. $\square$

Remark 1. By Lemma 1, it is easy to see that the main difference between the Dirichlet and Neumann problem is related to noncoerciveness of the energy functional corresponding to Neumann problem. So, in that context, Poincaré inequality plays a key role.

Proof of Theorem 1. Since $I \in C^1(X, \mathbb{R})$ is weakly lower semi-continuous and bounded from below, from Ekeland variational principle we have $(u_n, v_n) \subset X$ such that

$$I(u_n, v_n) \to \inf_X I \quad \text{and} \quad I'(u_n, v_n) \to 0. \tag{3.2}$$

By decomposition of $W^{1,p(x)}(\Omega)$, for each $n \in \mathbb{N}$, we can write any $(u_n, v_n) \in X$ as $u_n = \widetilde{u}_n + \overline{u}_n$ and $v_n = \widetilde{v}_n + \overline{v}_n$, where $\overline{u}_n, \overline{v}_n \in \mathbb{R}$ and $\widetilde{u}_n, \widetilde{v}_n \in W_0$ with $\int_\Omega \widetilde{u}_n \, dx = \int_\Omega \widetilde{v}_n \, dx = 0$. Moreover, from (3.2) we have $|I(u_n, v_n)| \leq c_5$. Thus, if we use (3.1), it follows that

$$c_6 \leq I(u_n, v_n) \leq c_5.$$

This means the sequences $\{\rho(\nabla \widetilde{u}_n)\}$ and $\{\rho(\nabla \widetilde{v}_n)\}$ are bounded. Using Proposition 2 and Proposition 6, we obtain that $\{\rho(\widetilde{u}_n)\}$ and $\{\rho(\widetilde{v}_n)\}$ are also bounded. Thus, $\{\widetilde{u}_n\}$ and $\{\widetilde{v}_n\}$ are bounded sequences in $W^{1,p(x)}(\Omega)$. In addition, if we choose $\overline{u}_n, \overline{v}_n$ in the interval $[0, k]$, $u_n = \widetilde{u}_n + \overline{u}_n$ and $v_n = \widetilde{v}_n + \overline{v}_n$, for each $n \in \mathbb{N}$, we can see that $\{\widetilde{u}_n\}$ and $\{\widetilde{v}_n\}$ are bounded sequences in $W^{1,p(x)}(\Omega)$. So, for convenient subsequences, we have $u_n \rightharpoonup u_0$ and $v_n \rightharpoonup v_0$ in $W^{1,p(x)}(\Omega)$, and therefore

$$\int_\Omega h_1 u_n \, dx \to \int_\Omega h_1 u_0 \, dx \quad \text{and} \quad \int_\Omega h_2 v_n \, dx \to \int_\Omega h_2 v_0 \, dx.$$

Using compact embedding, i.e., Proposition 4, we have

$$u_n \to u_0 \quad \text{and} \quad v_n \to v_0 \quad \text{in } L^{q(x)}(\Omega),$$

and up to subsequences,

$$u_n(x) \to u_0(x) \quad \text{and} \quad v_n(x) \to v_0(x) \quad \text{a.e. } x \in \Omega.$$

Moreover, by continuity of F, it follows

$$F\big(u_n(x), v_n(x)\big) \to F\big(u_0(x), v_0(x)\big) \quad \text{a.e. } x \in \Omega.$$

Since $|F(u_n(x), v_n(x))| \leq c$ for all $n \in \mathbb{N}$ and a.e. $x \in \Omega$, by help of the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega F(u_n, v_n) \, dx \to \int_\Omega F(u_0, v_0) \, dx.$$

From (3.2) we know that

$$\inf_X I = \lim I(u_n, v_n) = \lim \Big[\widehat{M_1}\big(\widehat{\rho}(\nabla u_n)\big) + \widehat{M_2}\big(\widehat{\rho}(\nabla v_n)\big)$$

$$- \int_\Omega F(u_n, v_n)\, dx - \int_\Omega h_1(x) u_n\, dx - \int_\Omega h_2(x) v_n\, dx \Big].$$

Finally, we have to show that $\{\widehat{M_1}(\widehat{\rho}(\nabla u_n))\}$ and $\{\widehat{M_2}(\widehat{\rho}(\nabla v_n))\}$ are convergent. Considering that $u_n \rightharpoonup u_0$ and $v_n \rightharpoonup v_0$ in $W^{1,p(x)}(\Omega)$ and using Fatou's Lemma, we get

$$\widetilde{\rho}(\nabla u_0) \le \liminf \widetilde{\rho}(\nabla u_n) \quad \text{and} \quad \widetilde{\rho}(\nabla v_0) \le \liminf \widetilde{\rho}(\nabla v_n).$$

Since $\widehat{M_1}$ and $\widehat{M_2}$ are continuous and increasing, we obtain

$$\widehat{M_1}\big(\widehat{\rho}(\nabla u_0)\big) \le \lim \widehat{M_1}\big(\widehat{\rho}(\nabla u_n)\big),$$

and similarly

$$\widehat{M_2}\big(\widehat{\rho}(\nabla v_0)\big) \le \lim \widehat{M_2}\big(\widehat{\rho}(\nabla v_n)\big).$$

Eventually,

$$\inf_X I \ge \widehat{M_1}\big(\widehat{\rho}(\nabla u_0)\big) + \widehat{M_2}\big(\widehat{\rho}(\nabla v_0)\big)$$

$$- \int_\Omega F(u_0, v_0)\, dx - \int_\Omega h_1(x) u_0\, dx - \int_\Omega h_2(x) v_0\, dx = I(u_0, v_0),$$

which means $I(u_0, v_0) = \inf_X I$. Hence, $(u_0, v_0) \in X$ is a weak solution of problem (1.1) if I is differentiable at (u_0, v_0). The proof is completed. $\square$

Acknowledgement

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