Existence of the Solution of Classical Smoothing Problems with Obstacles and Weights*

Evely Leetma and Peeter Oja

Institute of Mathematics, University of Tartu

J. Liivi 2, 50409 Tartu, Estonia

E-mail(*corresp.*): `Evely.Leetma@ut.ee`

E-mail: `Peeter.Oja@ut.ee`

Abstract. We show that classical smoothing problems with obstacles and weights have always the solution. These problems are considered in quite general case, namely, we allow arbitrary dimension of variable and arbitrary degree in derivative part of functional to minimize. While the existence is proved without any assumption about the uniqueness of solution, some conditions assuring the uniqueness are also analyzed.

Keywords: smoothing problems, natural spline, existence of solution.

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1 Introduction

There are several works where smoothing problems with obstacles and weights are studied (see [1, 3, 4, 5, 7] and references to research papers therein). In any of them it is assumed to be satisfied some conditions which guarantee the uniqueness of solution. Usually they are assumptions about unique solvability of polynomial interpolation problem. It is quite well known that the solutions of both smoothing problems are natural splines. The problem with weights is a linear one in nature, because it leads to the solution of a linear system with respect to spline parameters.

The problem with obstacles is more practical because obstacle values arise naturally while weights cannot be determined from given data. At the same time, the methods of solution for obstacle problems are under research (see, e.g., [3, 4, 5]).

Anyway, it is always important to answer the principal question when the solution of problem exists. In this work, as main results, we show that classical smoothing problem with obstacles and also the problem with weights have always a solution which is a natural spline. In addition, we give necessary and sufficient conditions for the uniqueness of solution. Note that, in the problem

with weights, we allow nonnegative weights, not necessarily positive ones. This generalizes somewhat the position of problem (cf. [1, 3]) and is important in establishing the equivalence of problems [1, 4].

2 Classical Smoothing Problems

For given integers r and n, $2r > n \geq 1$, let us denote by $L_2^{(r)}(\mathbb{R}^n)$ the space of functions defined on $\mathbb{R}^n$ having all partial (distributional) derivatives of order r in $L_2(\mathbb{R}^n)$, i.e.,

$$L_2^{(r)}(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R} \mid D^\alpha f \in L_2(\mathbb{R}^n), \ |\alpha| = r \right\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \geq 0$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The space $L_2^{(r)}(\mathbb{R}^n)$ is called Beppo Levi space. Define the operator

$$T : L_2^{(r)}(\mathbb{R}^n) \to L_2(\mathbb{R}^n) \times \cdots \times L_2(\mathbb{R}^n)$$

as

$$Tf = \left\{ \sqrt{\frac{r!}{\alpha!}} D^\alpha f \ \Big| \ |\alpha| = r \right\}$$

with $\alpha! = \alpha_1! \cdots \alpha_n!$. We also need the semi-inner product

$$(Tf, Tg) = \sum_{|\alpha|=r} \frac{r!}{\alpha!} \int_{\mathbb{R}^n} D^\alpha f \, D^\alpha g \, dX, \quad f, g \in L_2^{(r)}(\mathbb{R}^n),$$

and the corresponding seminorm $\|Tf\| = \sqrt{(Tf, Tf)}$.

Let $\mathcal{P}_{r-1}$ be the space of all polynomials of degree not exceeding $r - 1$. A function of the form

$$S(X) = P(X) + \sum_{i \in I} d_i G(X - X_i), \quad X \in \mathbb{R}^n, \tag{2.1}$$

with $P \in \mathcal{P}_{r-1}$,

$$\sum_{i \in I} d_i Q(X_i) = 0 \quad \forall Q \in \mathcal{P}_{r-1}, \tag{2.2}$$

I a finite set and arbitrary $X_i \in \mathbb{R}^n$, $X_i \neq X_j$ for $i \neq j$, is called a natural spline. Here G is the fundamental solution of the operator Δ^r, where Δ is the n-dimensional Laplace operator. It is known that, for n odd, $G(X) = c_{nr}\|X\|^{2r-n}$ and, for n even, $G(X) = c_{nr}\|X\|^{2r-n} \log \|X\|$ with some constants $c_{nr} > 0$ and $\|X\| = \sqrt{x_1^2 + \cdots + x_n^2}$ (see [6, p. 521]). Although G does not belong to $L_2^{(r)}(\mathbb{R}^n)$, the linear combinations of type (2.1) belong to it due to condition (2.2), thus any natural spline belongs to $L_2^{(r)}(\mathbb{R}^n)$. In case $n = 1$, a reader can easily check this himself; a proof in general case can be found, e.g., in [3, p. 21].

For given sets of indexes I_0, I_1, $I_0 \cap I_1 = \emptyset$, $I_0 \cup I_1 = I$, weights $w_i \geq 0$, $i \in I_1$, pairwise distinct points $X_i \in \mathbb{R}^n$, $i \in I$, and values $z_i \in \mathbb{R}$, $i \in I$, define

$$\Omega_0 = \{ f \in L_2^{(r)}(\mathbb{R}^n) \mid f(X_i) = z_i, \ i \in I_0 \}.$$

We consider the minimization problem

$$\min_{f \in \Omega_0} \left(\|Tf\|^2 + \sum_{i \in I_1} w_i |f(X_i) - z_i|^2 \right) \tag{2.3}$$

as the classical smoothing problem with weights.

For given I_0, I_1, I and X_i as above, values $z_i \in \mathbb{R}$, $i \in I_0$, and obstacles α_i, β_i, $\alpha_i < \beta_i$, $i \in I_1$ (it may be that $\alpha_i = -\infty$ or $\beta_i = \infty$ for some $i \in I_1$), define

$$\Omega_{\alpha\beta} = \{f \in L_2^{(r)}(\mathbb{R}^n) \mid f(X_i) = z_i, \ i \in I_0, \ \alpha_i \leq f(X_i) \leq \beta_i, \ i \in I_1\}.$$

Then we consider the minimization problem

$$\min_{f \in \Omega_{\alpha\beta}} \|Tf\|^2 \tag{2.4}$$

as the classical smoothing problem with obstacles.

3 Existence of the Solution of the Problem with Obstacles

Let $\Omega_{\alpha\beta}^S$ be the set of natural splines S of the form (2.1) such that $S \in \Omega_{\alpha\beta}$. Consider the auxiliary problem

$$\min_{S \in \Omega_{\alpha\beta}^S} \|TS\|^2, \tag{3.1}$$

which is actually the restriction of problem (2.4) to the space of natural splines. We will use the following result of characterization (see, e.g., [3, p. 66]).

Lemma 1. *A natural spline $S \in \Omega_{\alpha\beta}^S$ is a solution of problem* (3.1) *if and only if the coefficients d_i, $i \in I_1$, of S satisfy the conditions*

$$\begin{aligned}
d_i &= 0, & \text{if } \alpha_i < S(X_i) < \beta_i, \\
(-1)^r d_i &\geq 0, & \text{if } S(X_i) = \alpha_i, \\
(-1)^r d_i &\leq 0, & \text{if } S(X_i) = \beta_i.
\end{aligned} \tag{3.2}$$

Although in [3] the result is presented under some assumptions about the unique solvability of polynomial interpolation, the assertion is valid without these assumptions.

Proposition 1. *For any data, problem* (2.4) *has a solution.*

Proof. Consider the problem (3.1) with the data from problem (2.4). Choose a basis $Q_1, \ldots, Q_p$ in the space $\mathcal{P}_{r-1}$ and define

$$D = \left\{ d \in \mathbb{R}^{|I|} \ \Big| \ \sum_{i \in I} d_i Q_j(X_i) = 0, \ j = 1, \ldots, p \right\}.$$

It is known (see, e.g., [3, p. 22]) that for all $f \in L_2^{(r)}(\mathbb{R}^n)$ and any natural spline S of the form (2.1) it holds

$$(TS, Tf) = (-1)^r \sum_{i \in I} d_i f(X_i). \tag{3.3}$$

By this property, the functional to minimize in (3.1) takes the form

$$\|TS\|^2 = (TS, TS) = (-1)^r \sum_{i \in I} d_i S(X_i)$$

$$= (-1)^r \sum_{i \in I} d_i \left(\sum_{j=1}^{p} c_j Q_j(X_i) + \sum_{j \in I} d_j G(X_i - X_j) \right)$$

$$= (-1)^r \sum_{i,j \in I} d_i d_j G(X_i - X_j), \quad d \in D. \tag{3.4}$$

It can be considered as a quadratic functional $(c, d) \to \Phi(d)$ with $c \in \mathbb{R}^p$ as vector of coefficients in polynomial part of S and $d \in D \subset \mathbb{R}^I$ (we have seen that $\|TS\|^2$ does not depend on c). Thus, problem (3.1) reduces to a problem of quadratic programming subject to linear constraints for c and d which are determined by the conditions $S \in \Omega_{\alpha\beta}$. The functional Φ is bounded from below as $\Phi(d) \geq 0$ for all $d \in D$. The feasible set $\Omega_{\alpha\beta}^S$ is nonempty since it contains a natural spline interpolant satisfying conditions

$$\sum_{j=1}^{p} c_j Q_j(X_i) + \sum_{j \in I} d_j G(X_i - X_j) = z_i, \quad i \in I_0,$$

$$\sum_{j=1}^{p} c_j Q_j(X_i) + \sum_{j \in I} d_j G(X_i - X_j) = \gamma_i, \quad i \in I_1, \ \alpha_i \leq \gamma_i \leq \beta_i,$$

$$\sum_{i \in I} d_i Q_j(X_i) = 0, \quad j = 1, \ldots, p.$$

It is important to note that the natural spline interpolant always exists because the radial basis function G has necessary properties (see [8, pp. 113, 117]). Now we use the fact that the minimization problem of quadratic programming with linear constraints has the solution if the functional is bounded from below on nonempty feasible set (see [2, p. 111]). This proves that problem (3.1) has the solution S_*.

We will show that S_* is also a solution of problem (2.4). Again, for all $f \in L_2^{(r)}(\mathbb{R}^n)$, using (3.3) we have

$$\|Tf\|^2 - \|TS_*\|^2 = \|Tf - TS_*\|^2 + 2(TS_*, Tf - TS_*)$$

$$= \|Tf - TS_*\|^2 + 2(-1)^r \sum_{i \in I} d_i \left(f(X_i) - S_*(X_i) \right). \tag{3.5}$$

Now, taking $f \in \Omega_{\alpha\beta}$, for $i \in I_0$, it holds $f(X_i) = S_*(X_i) = z_i$; for $i \in I_1$ we use Lemma 1 and get that in case $\alpha_i < S_*(X_i) < \beta_i$ it holds $d_i = 0$, if $S_*(X_i) = \alpha_i$ then $f(X_i) - S_*(X_i) \geq 0$ and $(-1)^r d_i \geq 0$, if $S_*(X_i) = \beta_i$ then $f(X_i) - S_*(X_i) \leq 0$ and $(-1)^r d_i \leq 0$, which means that the sum term in (3.5) is nonnegative and $\|Tf\| \geq \|TS_*\|$. $\square$

Let us remark that we did not need any assumptions about uniqueness of solution in Proposition 1. The uniqueness yields the following condition:

$$P \in \mathcal{P}_{r-1}, \quad P(X_i) = 0, \quad i \in I \quad \Rightarrow \quad P = 0. \tag{3.6}$$

Indeed, assuming the contrary there is $P_0 \in \mathcal{P}_{r-1}$, $P_0(X_i) = 0$, $i \in I$, but $P_0 \neq 0$. For a solution S of problem (2.4), by Lemma 1, the conditions (3.2) are satisfied and this also holds for $S + P_0$ because of $d_i(S + P_0) = d_i(S)$, $i \in I_1$. By Lemma 1, $S + P_0$ is a solution of problem (2.4), which contradicts to the assumption about uniqueness of solution.

4   Existence of the Solution of the Problem with Weights

In this section we will prove the existence of the solution of problem (2.3). We need

Lemma 2. *A natural spline of the form* (2.1) *is a solution of problem* (2.3) *if and only if it satisfies the conditions*

$$
(-1)^r d_i + w_i S(X_i) = w_i z_i, \quad i \in I_1,
$$
$$
S(X_i) = z_i, \quad i \in I_0. \tag{4.1}
$$

Proof. Let S be a natural spline satisfying (4.1). Each element in Ω_0 may be represented as $S + h$, where $h \in L_2^{(r)}(\mathbb{R}^n)$ and $h(X_i) = 0$, $i \in I_0$. Denote by F the functional to minimize in (2.3). Then, using (3.3), we have

$$
F(S + h) = \|T(S + h)\|^2 + \sum_{i \in I_1} w_i |S(X_i) + h(X_i) - z_i|^2
$$
$$
= \|TS\|^2 + 2(TS, Th) + \|Th\|^2 + \sum_{i \in I_1} w_i |S(X_i) - z_i|^2
$$
$$
+ 2 \sum_{i \in I_1} w_i (S(X_i) - z_i) h(X_i) + \sum_{i \in I_1} w_i |h(X_i)|^2
$$
$$
= F(S) + 2 \sum_{i \in I_1} \left((-1)^r d_i(S) + w_i S(X_i) - w_i z_i \right) h(X_i)
$$
$$
+ \|Th\|^2 + \sum_{i \in I_1} w_i |h(X_i)|^2. \tag{4.2}
$$

Hence, according to (4.1), we see that $F(S + h) \geq F(S)$, which means that S is a solution of problem (2.3).

Let S be a solution of problem (2.3). We have to show that (4.1) holds. Suppose there is an index $k \in I_1$ such that $(-1)^r d_k + w_k S(X_k) \neq w_k z_k$. Take $h \in L_2^{(r)}(\mathbb{R}^n)$ such that $h(X_k) = 1$ and $h(X_i) = 0$, $i \in I_0 \cup (I_1 \setminus \{k\})$ (e.g., we can take a natural spline interpolant). Let $h_\delta = \delta h$, $\delta \neq 0$, $\delta \to 0$. Clearly, $S + h_\delta \in \Omega_0$. Then, according to (4.2), we have

$$
F(S + h_\delta) = F(S) + 2\left((-1)^r d_k + w_k S(X_k) - w_k z_k \right) \delta + \delta^2 \|Th\|^2 + w_k \delta^2.
$$

Choosing δ so that $\left((-1)^r d_k + w_k S(X_k) - w_k z_k \right) \delta < 0$, we get for sufficiently small δ a contradiction $F(S + h_\delta) < F(S)$.   □

Proposition 2. *For any data, problem* (2.3) *has a solution.*

Proof. Consider the problem

$$\min_{S \in \Omega_0^S} \left(\|TS\|^2 + \sum_{i \in I_1} w_i |S(X_i) - z_i|^2 \right) \tag{4.3}$$

where Ω_0^S denotes the set of natural splines belonging to Ω_0. Taking into account the spline representation (2.1) with coefficients c and d together with (3.4) we have

$$\|TS\|^2 + \sum_{i \in I_1} w_i |S(X_i) - z_i|^2 = (-1)^r \sum_{i,j \in I} d_i d_j G(X_i - X_j)$$

$$+ \sum_{i \in I_1} w_i \left| \sum_{j=1}^{p} c_j Q_j(X_i) + \sum_{j \in I} d_i G(X_i - X_j) - z_i \right|^2.$$

This is a quadratic functional $(c, d) \to \Phi(c, d)$ bounded from below (namely $\Phi(c, d) \geq 0$) on a nonempty feasible set of (c, d) corresponding to $\Omega_0^S \neq \emptyset$ (note that $\Omega_{\alpha\beta}^S \subset \Omega_0^S$ and we have seen that $\Omega_{\alpha\beta}^S \neq \emptyset$). Thus, the problem (4.3) has a solution.

We claim that the solution S of problem (4.3) satisfies conditions (4.1). If this would not be the case, we could repeat second part of the proof of Lemma 2 and get a contradiction (note, that in this proof h_δ may be taken as natural spline).

Finally, we use Lemma 2 and obtain that the solution S of problem (4.3) is a solution of problem (2.3). □

We note that Proposition 2 is established without any assumption about uniqueness of solution.

Denote $I_1^+ = \{i \in I_1 \mid w_i \neq 0\}$ and consider the following condition about polynomial interpolation:

$$P \in \mathcal{P}_{r-1}, P(X_i) = 0, \quad i \in I_0 \cup I_1^+ \quad \Rightarrow \quad P = 0. \tag{4.4}$$

Proposition 3. *The solution of problem* (2.3) *is unique if and only if the condition* (4.4) *is satisfied.*

Proof. Let S be a solution of problem (2.3) and let (4.4) do not hold, i.e., there is $P_0 \in \mathcal{P}_{r-1}$ such that $P_0(X_i) = 0$, $i \in I_0 \cup I_1^+$, but $P_0 \neq 0$. Then, as S satisfies (4.1) by Lemma 2, $S + P_0$ is a solution of problem (2.3), different from S. We have shown that condition (4.4) is necessary for the uniqueness of solution in problem (2.3).

Suppose now that condition (4.4) is satisfied. We will show that the homogeneous system corresponding to (4.1) has only trivial solution. Let S_0 be a natural spline such that

$$(-1)^r d_i(S_0) + w_i S_0(X_i) = 0, \quad i \in I_1,$$
$$S_0(X_i) = 0, \quad i \in I_0.$$

If $d_i(S_0) \neq 0$, then $w_i \neq 0$ and $S_0(X_i) = -(-1)^r \frac{d_i(S_0)}{w_i}$. From (3.3) with $S = f = S_0$, we obtain

$$0 \leq (TS_0, TS_0) = (-1)^r \sum_{i \in I} d_i(S_0) S_0(X_i)$$

$$= (-1)^r \sum_{i \in I_1,\, d_i(S_0) \neq 0} d_i(S_0) S_0(X_i)$$

$$= -(-1)^{2r} \sum_{i \in I_1,\, d_i(S_0) \neq 0} \frac{d_i^2(S_0)}{w_i},$$

which implies that $d_i(S_0) = 0$ for all $i \in I_1$ and $\|TS_0\| = 0$. This gives (see [8, p. 162]) that $S_0 \in \mathcal{P}_{r-1}$. Since $w_i S_0(X_i) = 0$, $i \in I_1$, and $S_0(X_i) = 0$, $i \in I_0$, we have $S_0(X_i) = 0$, $i \in I_0 \cup I_1^+$, and (4.4) implies $S_0 = 0$. Thus, the solution of system (4.1) is unique and, by Lemma 2, such is the solution of (2.3). □

Note that condition (4.4) implies (3.6) but, in general, the inverse does not hold.

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