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A Numerical Method for Solving Two-Dimensional Nonlinear Parabolic Problems Based on a Preconditioning Operator

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Abstract. This article considers a nonlinear system of elliptic problems, which is obtained by discretizing the time variable of a two-dimensional nonlinear parabolic problem. Since the system consists of ill-conditioned problems, therefore a stabilized, mesh-free method is proposed. The method is based on coupling the preconditioned Sobolev space gradient method and WEB-spline finite element method with Helmholtz operator as a preconditioner. The convergence and error analysis of the method are given. Finally, a numerical example is solved by this preconditioner to show the efficiency and accuracy of the proposed methods.

Keywords: Sobolev space gradient method, WEB-spline finite element method, preconditioning operator, nonlinear parabolic problems.

AMS Subject Classification: 35K55.

1 Introduction

Consider the following nonlinear parabolic problem

$$\begin{aligned} &U_t - \boldsymbol{\nabla} \cdot (f(\mathbf{x}, \nabla U)) = p(\mathbf{x}, t), \quad (\mathbf{x}, t) \in Q_T := \Omega \times (0, T),\\ &U(\mathbf{x}, t) = 0, \qquad (\mathbf{x}, t) \in \partial\Omega \times (0, T),\\ &U(\mathbf{x}, 0) = U_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$, $p \in L^2(Q_T)$ and $f : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a known function such that $\mathbf{x} \mapsto f(\mathbf{x}, \eta)$ is a bounded measurable function for any $\mathbf{x} \in \Omega$ and $\eta \mapsto f(\mathbf{x}, \eta)$ belongs to C^1 for $\eta \in \mathbb{R}^2$. In addition, Jacobian matrices $\partial_\eta f(\mathbf{x}, \eta)$ are symmetric and their eigenvalues λ, satisfy $0 < \mu_1 \leq \lambda \leq \mu_2$ where μ_1 and μ_2 are constants and independent of $(\mathbf{x}, \eta)$.

For time-dependent problems, a common method for discretizing the time variable t is the backward finite difference method with the stepsize $\Delta t = \frac{T}{n}$ where $n \in \mathbb{N}$. This leads to the following system of nonlinear elliptic problems

$$\begin{aligned} &\aleph(u_i) := -\boldsymbol{\nabla} \cdot (f(\mathbf{x}, \nabla u_i)) + q(\mathbf{x}, u_i) = g(\mathbf{x}, t_i), \quad \mathbf{x} \in \Omega, \\ &u_i(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial\Omega, \end{aligned} \tag{1.2}$$

where $u_i = u_i(\mathbf{x})$ is an approximation of $U = U(\mathbf{x}, t)$ at $t_i = i\Delta t$, $i = 1, 2, ..., n$ and

$$q(\mathbf{x}, u_i) = \frac{1}{\Delta t} u_i(\mathbf{x}), \;\; u_0(\mathbf{x}) = U_0(\mathbf{x}), \;\; g(\mathbf{x}, t_i) = p(\mathbf{x}, t_i) + \frac{1}{\Delta t} u_{i-1}(\mathbf{x}).$$

Since the r.h.s. $g(\mathbf{x}, t_i)$ includes the unknown function $u_{i-1}(\mathbf{x})$, we have to solve the equations recursively (as i increases). In addition, in order to prove the existence and uniqueness of the solution of (1.2), one can use the existence theorem in [4] for the distinct equations. To be exact, Farago and Karatson in [4] proved that if $q = q(\mathbf{x}, \xi)$ is a known bounded measurable function and C^1 with respect to the variables $\mathbf{x} \in \Omega$ and $\xi \in \mathbb{R}$, respectively, which satisfy

$$0 \leq \partial_\xi q(\mathbf{x}, \xi) \leq c_1 + c_2 |\xi|^{p-2}, \qquad p \geq 2, \tag{1.3}$$

where $c_1, c_2 \geq 0$ are constants, then for every $i = 1, 2, \ldots, n$ the nonlinear elliptic problem in general form (1.2) has a unique weak solution $u_i^* \in H_0^1(\Omega)$, such that

$$\int_\Omega (f(\mathbf{x}, \nabla u_i^*) \cdot \nabla v + \; q(\mathbf{x}, u_i^*) v) d\mathbf{x} = \int_\Omega g(\mathbf{x}, t_i) v d\mathbf{x}, \;\; v \in H_0^1(\Omega),$$

where $H_0^1(\Omega) = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$. Obviously, $q(\mathbf{x}, u_i) = \frac{1}{\Delta t} u_i(\mathbf{x})$ satisfies (1.3) with $c_1 = \frac{1}{\Delta t}$ and $c_2 = 0$, so the above result is valid for the nonlinear elliptic problem (1.2). To be exact, we can write

$$\langle F(u_i^*), v \rangle = \int_\Omega g(\mathbf{x}, t_i) v d\mathbf{x}, \;\; v \in H_0^1(\Omega),$$

where

$$\langle F(u_i^*), v \rangle := \int_\Omega \left(f(\mathbf{x}, \nabla u_i^*) \cdot \nabla v + \; \frac{1}{\Delta t} u_i^* \, v \right) d\mathbf{x},$$

w.r.t. the new energy norm

$$\|u\|_H^2 := \int_\Omega \left(|\nabla u|^2 + \frac{1}{\Delta t} u^2 \right) d\mathbf{x},$$

induced by the Helmholtz operator $Su := -\Delta u + \frac{1}{\Delta t}u$.

It should be mentioned that the condition number of each equation in (1.2) is infinite, i.e., $\mathrm{cond}(\aleph) = \infty$ (see Appendix). Therefore, using discretization methods such as backward finite difference method for discretizing the time or finite element method for discretizing the space leads to an unbounded condition number, as space or time discretization is refined [4].

In order to improve the condition number, we can use preconditioning operator in iterative methods such as Sobolev space gradient method, conjugate gradient method, Newton-like methods and so on. In most cases, finding a suitable preconditioner is the fundamental part of these iterative methods (see [1, 2, 3, 5, 10, 12]). Farago and Karatson in [4] provide an overview of existing preconditioned iterative methods, especially on nonlinear elliptic problems. Although the works on preconditioned iterative methods are limited to boundary value problems of elliptic type, it is also likely of interest also for other related problems such as variational inequality problems, and parabolic ones. The main contribution of this paper is to apply a preconditioned iterative method to a class of nonlinear parabolic problems (1.1), and analyze the role of the time variable t in this iterative method. The method is based on coupling the Sobolev space gradient method with Helmholtz preconditioner as a preconditioned iterative method and WEB-spline (Weighted Extended B-spline) finite element method as a mesh free method which reduces the order of system. In this paper, instead of Laplacian preconditioner (a common preconditioner in iterative methods), we apply Helmholtz preconditioner. Because, if we apply Laplacian preconditioner, can only control the instability due to space discretization. In other words, when the time discretization is refined, the instability is still remained in the problem, thereby slowing down the rate of convergence for the preconditioned iterative method. But, by applying the Helmholtz preconditioner, the upper bound of condition number is independent of Δt and the instability due to both time and space discretizations is controlled.

Moreover, we apply WEB-spline basis in finite element method. Because, if we apply the standard finite element method, we have to use the FEM with subspaces belonging to $H^2(\Omega) \cap H_0^1(\Omega)$ to prove the convergence of the preconditioned iterative method. This leads to apply the standard full quantic finite element method with polynomials of degree 5 with 21 unknown coefficients for each triangle. Although this method is stable and convergent, but one should solve a large system of equations for each iteration. So, to deal with this difficulty, instead of FEM, WEB-spline finite element method was applied in [13]. The WEB-spline basis belongs to $H^2(\Omega) \cap H_0^1(\Omega)$, has been considered as test functions in FEM and consequently reduced the order of the system. Some advantages of WEB-spline finite element method are as follows. No mesh generation is required, the uniform grid is ideally suited for parallelization and multigrid techniques, accurate approximations are possible with relatively low-dimensional subspaces, smoothness and approximation order can be chosen arbitrarily, hierarchical bases permit adaptive refinement (see [6, 7, 8, 9, 11]).

The paper is organized as follows. In next section, we briefly describe WEB-spline basis by using the notations in [6]. The details of the proposed preconditioned iterative method with Helmholtz preconditioner are given in Section 3. In addition, the convergence conditions and error analysis of the method are studied in Section 4. Finally, a numerical example with numerical error bound is given in Section 5.

2 WEB-spline basis

Following the notations in [6, 11, 13], we give a brief description of WEB-spline basis functions.

B-spline tensor product is an extension of B-spline in higher dimensions. So, in order to make a bivariate B-spline of degree d, denoted by $b^d_{\mathbf{k},h}$, the tensor product of one-dimensional B-splines is used as follows

$$b^d_{\mathbf{k},h}(x,y) = b^d_{k_1,h}(x) \otimes b^d_{k_2,h}(y), \qquad \mathbf{k} = (k_1,k_2) \in K, \tag{2.1}$$

in which $b^d_{\ell,h}(x) = b^d(x/h-\ell)$, $b^d(x) = \frac{x}{d}\, b^{d-1}(x) + \frac{d+1-x}{d}\, b^{d-1}(x-1)$, $(d = 2,3,\ldots)$ and

$$b^1(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2-x, & 1 \le x \le 2, \\ 0, & \text{otherwise}. \end{cases}$$

Here, the set K includes all indices $\mathbf{k}$ such that for some $\mathbf{x} = (x,y) \in \Omega$, $b^d_{\mathbf{k},h}(\mathbf{x}) \neq 0$ and h is increment of the variables x and y. We note that the support of $b^d_{\mathbf{k},h}$ is $[k_1, k_1+d+1]h \times [k_2, k_2+d+1]h$. The bivariate B-spline basis (2.1) is divided into two parts, inner and outer B-splines [6]. The B-spline $b^d_{\mathbf{k},h}$ is considered as inner B-spline only when its support contains at least one grid cell inside Ω, otherwise that B-spline is considered as outer B-spline. Having the above mentioned notations in mind, we can split the set of K indices into subsets: inner B-splines, I, and outer B-splines, J. In other words $K = I \cup J$, see [6, 11, 13].

Because of the two following deficiencies, using finite element method with B-spline basis functions seems to be impossible. The B-spline basis functions do not satisfy the essential boundary conditions and since each outer B-spline has a small support in Ω, the condition number of Galerkin matrix might be extremely large [6, 11]. To overcome the first deficiency, the *WEB-method* is applied [8], which uses weighted B-splines on regular grids as basis functions. In other words, this problem can be solved by multiplying $b^d_{\mathbf{k},h}$ and a smooth distance function $\omega(\mathbf{x}) \asymp dist(\mathbf{x}, \partial\Omega)$, $(\mathbf{x} \in \Omega)$. This choice of shape functions provides optimal approximation order with a minimal number of parameters and lives up to the de facto standard in CAD/CAM systems, thus providing a natural link between geometry description and finite element simulations [8]. According to [6] the best option is a signed weight function which is defined thoroughly as a continuous function and is positive on Ω and negative on its complement $\bar{\Omega}$. Rvachev method (or R-function) is a method which can be used in numerical methods properly. For example, this method for a problem with homogeneous Dirichlet boundary conditions and rectangular domain $\Omega =$

$\{(x, y) \,|0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$, leads to the following weight function

$$\omega(x, y) = 2 - \sqrt{(1-x)^2 + (1-y)^2} - \sqrt{x^2 + y^2}$$
$$- \left(\left(2 - x - \sqrt{(1-x)^2 + (1-y)^2} - y\right)^2 + \left(x + y - \sqrt{x^2 + y^2}\right)^2 \right)^{\frac{1}{2}}. \quad (2.2)$$

For the second deficiency, that is to control the unstable outer B-splines, we join them appropriately with the inner B-splines by the coefficients $e_{\mathbf{ij}}$ suggested by the following definition.

Definition 1. [6, Page 48] For an outer index $\mathbf{j} \in J$, let $I(\mathbf{j})=\ell+\{0, \ldots, d\}^2 \subset I$ be a two-dimensional array of inner indices closet to $\mathbf{j}$, assuming that h is small enough so that such an array exists. Then

$$e_{\mathbf{ij}} = \prod_{\nu=1}^{2} \prod_{u=0, \ell_\nu+\mu\neq i_v}^{d} \frac{j_\nu - \ell_v - \mu}{i_\nu - \ell_v - \mu}$$

are the values of the Lagrange polynomials associated with $I(\mathbf{j})$.

Consequently, for $\mathbf{i} \in I$, the WEB-spline $B_{\mathbf{i}}$ is constructed by

$$B_{\mathbf{i}} = \frac{\omega}{\omega(x_{\mathbf{i}})}(b_{\mathbf{i}} + \sum_{\mathbf{j}\in J} e_{\mathbf{ij}} b_{\mathbf{j}}), \quad (b_{\mathbf{k}} = b^d_{\mathbf{k},h}),$$

in which $x_{\mathbf{i}}$ denotes the center of a grid cell in support $b_{\mathbf{i}}$ which is completely inside Ω.

According to [6, 8, 11], we give some features of $B_{\mathbf{i}}$:

1. Because of the linear independence of B-splines, WEB-splines are linearly independent, too.

2. The factor $\omega/\omega(x_{\mathbf{i}})$ causes the WEB-splines to vanish on the boundary and magnifies functions supported near the boundary for scaling purpose. This fact will become important for proving the stability aspect of the WEB-splines.

3. By forming linear combinations, the support of a WEB-spline is in general larger than that of a B-spline. However, restricting nonzero coefficient $e_{\mathbf{ij}}$ to indices with $\|\mathbf{i} - \mathbf{j}\| \preceq 1$ guarantees that the diameter of the support of WEB-spline is still $\preceq h$. In particular, WEB-splines with support well separated from the boundary are just ordinary B-splines multiplied by $\omega/\omega(x_{\mathbf{i}})$. Hence, only $\preceq h^{-1}$ WEB-splines involve linear combinations of outer B-splines.

4. The uniform boundedness of the coefficients e_{ij} prevents the WEB-spline from growing in an uncontrolled way as the grid width h tends to zero.

In next section, we give the preconditioned iterative method with Helmholtz preconditioner.

3 Preconditioned iterative method with Helmholtz preconditioner

In this section, in order to approximate the solution of nonlinear elliptic problems (1.2), u_i^*, $i = 1, 2, \ldots, n$, a computational algorithm is given based on coupling Sobolev space gradient method with Helmholtz preconditioner and WEB-spline finite element method.

To do that, let $\bar{u}_i^j$ be an approximation of u_i^*, for $j = 0, 1, 2, ...$, which is obtained by the following preconditioned Sobolev space gradient method

$$\bar{u}_i^{j+1} = \bar{u}_i^j - \frac{2}{M+m} S^{-1} \left(\aleph(\bar{u}_i^j) - g(\mathbf{x}, t_i) \right), \qquad j = 0, 1, 2, \ldots, \tag{3.1}$$

where $\bar{u}_i^0 = 0$ (or any function belonging to $H^2(\Omega) \cap H_0^1(\Omega)$). To study the spectral bounds, we introduce the derivative operator

$$\langle F'(u)h, h\rangle := \int_\Omega \left(\partial_\eta f(x, \nabla u) \nabla h \cdot \nabla h + \frac{1}{\Delta t} h^2 \right) d\mathbf{x}, \quad u, h \in H_0^1(\Omega).$$

Now, suppose that there exists some positive constant m_1 and m_2, such that

$$m_1 \int_\Omega \nabla h \cdot \nabla h \, d\mathbf{x} \le \int_\Omega \partial_\eta f(x, \nabla u) \nabla h \cdot \nabla h \, d\mathbf{x} \le m_2 \int_\Omega \nabla h \cdot \nabla h \, d\mathbf{x}.$$

So we have

$$\min\{m_1, 1\} \|h\|_H^2 \le \langle F'(u)h, h\rangle \le \max\{m_2, 1\} \|h\|_H^2 .$$

Thus the spectral bounds will be $M := \max\{m_2, 1\}$ and $m := \min\{m_1, 1\}$. Moreover, from (3.1) and [4], Helmholtz preconditioner S leads to the following upper bound of condition number for the main operator

$$\mathrm{cond}(S^{-1}\aleph) \leqslant M/m.$$

This condition number is independent of Δt and h. As a result, Helmholtz preconditioner controls the instability due to space and time discretizations.

Now, put $z_i^j = S^{-1} \left(\aleph(\bar{u}_i^j) - g(\mathbf{x}, t_i) \right)$, then we should solve the following Helmholtz problem

$$S(z_i^j) = -\Delta z_i^j + \frac{1}{\Delta t} z_i^j = \aleph(\bar{u}_i^j) - g(\mathbf{x}, t_i), \;\; \mathbf{x} \in \Omega, \quad z_i^j = 0, \;\; \mathbf{x} \in \partial\Omega. \tag{3.2}$$

As a result, (3.1) is established if we obtain an approximate solution of (3.2). To this end, we apply WEB-spline finite element method. Let

$$\bar{z}_i^j(\mathbf{x}) = \sum_{k=1}^{H} c_k^{(j)} B_k(\mathbf{x}), \qquad i = 1, 2, \ldots, n, \;\; j = 0, 1, 2, \ldots,$$

be an approximate solution of z_i^j, in which $c_k^{(j)}$ and $B_k(\mathbf{x})$ for $k = 1, 2, \ldots, H$, are unknown coefficients and WEB-spline basis belongs to $H^2(\Omega) \cap H_0^1(\Omega)$, respectively. These unknown coefficients are obtained by using Ritz-Galerkin method to Helmholtz problem (3.2). This leads to the following nonsingular linear system of equations

$$AC^{(j)} = R^{(j)},$$

where

$$A = [A_{\ell q}]_{H\times H},\ C^{(j)} = \begin{bmatrix} c_1^{(j)} & c_2^{(j)} & \dots & c_H^{(j)} \end{bmatrix}^{\mathrm{T}},\ R^{(j)} = \begin{bmatrix} r_1^{(j)} & r_2^{(j)} & \dots & r_H^{(j)} \end{bmatrix}^{\mathrm{T}},$$

and

$$\begin{aligned} A_{\ell q} &= \int_{\Omega} \nabla B_\ell(\mathbf{x}) \cdot \nabla B_q(\mathbf{x}) d\mathbf{x} + \frac{1}{\Delta t}\int_{\Omega} B_\ell(\mathbf{x}) B_q(\mathbf{x}) d\mathbf{x}, \\ r_\ell^{(j)} &= \int_{\Omega} \left(\aleph(\bar{u}_i^j) - g(\mathbf{x}, t_i)\right) B_\ell(\mathbf{x}) d\mathbf{x}. \end{aligned}$$

Here T denotes the transpose of vectors.

The above considerations to construct (3.1) give the following algorithm:

Algorithm 1. Preconditioned iterative method with Helmholtz preconditioner

1: **Step 1** Set $i = 1$.
2: **Step 2** Set $j = 0$.
3: **Step 3** Solve the Helmholtz problem (3.2) and obtain $\bar{z}_i^j$ by the WEB-spline finite element method.
4: **Step 4** Obtain $\bar{u}_i^{j+1}$ by $\bar{u}_i^{j+1} = \bar{u}_i^j - \frac{2}{M+m}\bar{z}_i^j$.
5: **Step 5** For a given tolerance ϵ, if $\left\|\bar{u}_i^{j+1} - \bar{u}_i^j\right\|_{L^2(\Omega)} \le \epsilon$, then $\bar{u}_i^{j+1}$ is an acceptable approximate solution and go to step 7, else go to step 6.
6: **Step 6** Put $j = j + 1$ and go to step 3.
7: **Step 7** If $i = n$ stop, else put $i = i + 1$ and go to step 2.

In what follows, we provide an example which is given from [3].

Example 1. Consider the problem

$$\begin{aligned} & u_t - \boldsymbol{\nabla}\cdot\left(\bar{k}\left(|\nabla u|^2\right)\nabla u\right) = p(\mathbf{x}, t), && (\mathbf{x}, t) \in \Omega \times (0, 1), \\ & u(\mathbf{x}, t) = 0, && (\mathbf{x}, t) \in \partial\Omega \times (0, 1), \\ & u(\mathbf{x}, 0) = 0, && \mathbf{x} \in \Omega, \end{aligned} \tag{3.3}$$

where $\Omega = \{(x, y)\,|0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$ and

$$\bar{k}(z) = 1.02/(1 + \sqrt{1 - z/3}), \qquad 0 \leqslant z \leqslant z_0 = 2.76.$$

Using backward finite difference method, we get

$$\begin{aligned} & \aleph(u_i) := -\boldsymbol{\nabla}\cdot\left(\bar{k}\left(|\nabla u_i|^2\right)\nabla u_i\right) + \frac{1}{\Delta t}u_i(\mathbf{x}) = p(\mathbf{x}, t_i) + \frac{1}{\Delta t}u_{i-1}(\mathbf{x}), && \mathbf{x} \in \Omega, \\ & u_i(\mathbf{x}) = 0, && \mathbf{x} \in \partial\Omega. \end{aligned}$$

According to [3], we have

$$(\partial f_\eta(\mathbf{x}, \eta)\nabla h, \nabla h) = \int_{\Omega} \left(\bar{k}\left(|\nabla\eta|^2\right)|\nabla h|^2 + 2(\nabla\eta\cdot\nabla h)^2 \bar{k}'(|\nabla\eta|^2)\right) d\mathbf{x},$$

where $(\mathbf{x},\eta) \in \Omega \times \mathbb{R}^2$, $h \in H_0^1(\Omega)$ and $(u,v) := \int_\Omega uv d\mathbf{x}$. Then, since $\bar{k}$ and $\bar{k}'$ are increasing, we obtain

$$m_1 (\nabla h, \nabla h) \le (\partial f_\eta(\mathbf{x},\eta)\nabla h, \nabla h) \le m_2 (\nabla h, \nabla h)$$

where

$$\begin{aligned} m_1 &= \min_{z \geqslant 0} \bar{k}(z) = \bar{k}(0) = 0.51, \\ m_2 &= \max_{z \geqslant 0} \left\{ \bar{k}(z) + 2z\bar{k}'(z) \right\} = \bar{k}(z_0) + 2z_0\bar{k}'(z_0) = 2.81. \end{aligned}$$

So we have

$$\left. \begin{aligned} m &= \min\{0.51, 1\} = 0.51 \\ M &= \max\{2.86, 1\} = 2.86 \end{aligned} \right\} \quad \Rightarrow \quad \operatorname{cond}\left(S^{-1}\aleph\right) \le \frac{2.86}{0.51} = 5.6078.$$

In next section, we provide the error bound and convergence theorem of the proposed preconditioned iterative method with Helmholtz preconditioner.

4 Convergence and error analysis

The analysis of the proposed algorithm is mainly divided into two parts; the error of preconditioned Sobolev space gradient method and WEB-spline finite element method. In what follows, we give some theorems related to these parts.

Theorem 1. *Consider the problem (1.2) for $q(\boldsymbol{x},u_i) = \frac{1}{\Delta t}u_i$ and $g(\boldsymbol{x},u_i) = p(\boldsymbol{x},t_i) + \frac{1}{\Delta t}u_{i-1}$. Then construction of (3.1) yields the following corresponding convergence results: If Ω is C^2-diffeomorphic to a convex domain, then for any $\bar{u}_i^0 \in H^2(\Omega) \cap H_0^1(\Omega)$, we have*

$$\left\| \bar{u}_i^j - u_i^* \right\|_H \leqslant \frac{C}{\Delta t}\left(\frac{M-m}{M+m}\right)^j,$$

where $C > 0$ is positive constant.

Proof. Similar to Theorem 7.2 in [3], we have

$$\left\| \bar{u}_i^j - u_i^* \right\|_H \leqslant \frac{1}{m\varrho^{1/2}} \left\| \aleph(\bar{u}_i^0) - g(\cdot,t_i) \right\|_{L^2(\Omega)} \left(\frac{M-m}{M+m}\right)^j, \quad j = 1,2,\ldots,$$

where $\varrho > 0$ is the smallest eigenvalue of S on $H^2(\Omega) \cap H_0^1(\Omega)$. Since $\aleph(\bar{u}_i^0) = 0$, we obtain

$$\begin{aligned} \left\| \bar{u}_i^j - u_i^* \right\|_H &\leqslant \frac{1}{m\varrho^{1/2}} \left\| p(\cdot,t_i) + \frac{1}{\Delta t}u_{i-1} \right\|_{L^2(\Omega)} \left(\frac{M-m}{M+m}\right)^j \\ &\leqslant \frac{1}{m\varrho^{1/2}} \left\| p(\cdot,t_i) \right\|_{L^2(\Omega)} \left(\frac{M-m}{M+m}\right)^j + \frac{1}{m\varrho^{1/2}} \frac{1}{\Delta t} \left\| u_{i-1} \right\|_{L^2(\Omega)} \left(\frac{M-m}{M+m}\right)^j. \end{aligned} \tag{4.1}$$

Now, we find an upper bound for $\|u_{i-1}\|_{L^2(\Omega)}$. Due to this, multiply both sides of (1.2) by a function $v \in H_0^1(\Omega)$ and integrate over Ω. Then, using the boundary conditions and Green's formula, we get

$$\int_\Omega f(\mathbf{x}, \nabla u_i)\nabla v d\mathbf{x} + \frac{1}{\Delta t}\int_\Omega u_i v d\mathbf{x} = \int_\Omega p(\mathbf{x}, t_i) v d\mathbf{x} + \frac{1}{\Delta t}\int_\Omega u_{i-1} v d\mathbf{x}. \quad (4.2)$$

For the first term on the left-hand side of (4.2), we introduce the operator $A : H_0^1(\Omega) \to H_0^1(\Omega)$ in the following

$$(Au, v) = \int_\Omega f(\mathbf{x}, \nabla u)\nabla v d\mathbf{x}, \qquad u, v \in H_0^1(\Omega).$$

Thus, we have

$$(A'(u)h, h) = \int_\Omega \partial_\eta f(\mathbf{x}, \eta)\,|_{\eta=\nabla u} \nabla h \cdot \nabla h d\mathbf{x}, \quad u, h \in H_0^1(\Omega),$$
$$m_1 \|\nabla h\|^2_{L^2(\Omega)} \leqslant (A'(u)h, h) \leqslant m_2 \|\nabla h\|^2_{L^2(\Omega)},$$

which is equivalent to (see [4, Page 112, Proposition 5.2])

$$m_1 \|\nabla u - \nabla v\|^2_{L^2(\Omega)} \leqslant (A(u) - A(v), u - v) \leqslant m_2 \|\nabla u - \nabla v\|^2_{L^2(\Omega)}. \quad (4.3)$$

Moreover, substituting $v = 0$ in (4.3), it results

$$m_1 \|\nabla u\|^2_{L^2(\Omega)} \leq \int_\Omega f(\mathbf{x}, \nabla u)\nabla u d\mathbf{x} - \int_\Omega f(\mathbf{x}, 0)\nabla u d\mathbf{x} \leq m_2 \|\nabla u\|^2_{L^2(\Omega)}.$$

Then by assuming $f(\mathbf{x}, 0) = 0$, we get

$$m_1 \|\nabla u\|^2_{L^2(\Omega)} \leq \int_\Omega f(\mathbf{x}, \nabla u)\nabla u d\mathbf{x}.$$

So,

$$m_1 \|\nabla u_i\|^2_{L^2(\Omega)} \leq \int_\Omega f(\mathbf{x}, \nabla u_i)\nabla u_i d\mathbf{x}. \quad (4.4)$$

Now, substituting $v = u_i$ in (4.2) and applying (4.4) lead to

$$m_1 \|\nabla u_i\|^2_{L^2(\Omega)} + \frac{1}{\Delta t}\|u_i\|^2_{L^2(\Omega)} \leqslant \|p(\cdot, t_i)\|_{L^2(\Omega)}\|u_i\|_{L^2(\Omega)} + \frac{1}{\Delta t}\|u_{i-1}\|_{L^2(\Omega)}\|u_i\|_{L^2(\Omega)},$$

which implies that

$$\frac{1}{\Delta t}\|u_i\|^2_{L^2(\Omega)} \leqslant \|p(\cdot, t_i)\|_{L^2(\Omega)}\|u_i\|_{L^2(\Omega)} + \frac{1}{\Delta t}\|u_{i-1}\|_{L^2(\Omega)}\|u_i\|_{L^2(\Omega)},$$
$$\|u_i\|_{L^2(\Omega)} \leqslant \Delta t\|p(\cdot, t_i)\|_{L^2(\Omega)} + \|u_{i-1}\|_{L^2(\Omega)}.$$

Thus, we conclude that

$$\|u_{i-1}\|_{L^2(\Omega)} \leqslant \Delta t \sum_{k=1}^{i-1} \|p(\cdot,t_k)\|_{L^2(\Omega)} + \|u_0\|_{L^2(\Omega)}.$$

This result, help us to obtain an upper bound for (4.1), i.e.,

$$\begin{aligned} \left\|\bar{u}_i^j - u_i^*\right\|_H \leqslant & \frac{1}{m\varrho^{1/2}} \sum_{k=1}^{i} \|p(\cdot,t_k)\|_{L^2(\Omega)} \left(\frac{M-m}{M+m}\right)^j \\ & + \frac{1}{m\varrho^{1/2}} \frac{1}{\Delta t} \|u_0\|_{L^2(\Omega)} \left(\frac{M-m}{M+m}\right)^j. \end{aligned}$$

Now, if we suppose $K = \sup\limits_{0\leqslant k\leqslant i} \|p(\mathbf{x},t_k)\|_{L^2(\Omega)}$, we get

$$\begin{aligned} \left\|\bar{u}_i^j - u_i^*\right\|_{H^1(\Omega)} &\leqslant \frac{1}{m\varrho^{1/2}} iK \left(\frac{M-m}{M+m}\right)^j + \frac{1}{m\varrho^{1/2}} \frac{1}{\Delta t} \|u_0\|_{L^2(\Omega)} \left(\frac{M-m}{M+m}\right)^j \\ &= \frac{1}{m\varrho^{1/2}} \frac{t_i}{\Delta t} K \left(\frac{M-m}{M+m}\right)^j + \frac{1}{m\varrho^{1/2}} \frac{1}{\Delta t} \|u_0\|_{L^2(\Omega)} \left(\frac{M-m}{M+m}\right)^j \\ &\leqslant \frac{1}{m\varrho^{1/2}} \frac{T}{\Delta t} K \left(\frac{M-m}{M+m}\right)^j + \frac{1}{m\varrho^{1/2}} \frac{1}{\Delta t} \|u_0\|_{L^2(\Omega)} \left(\frac{M-m}{M+m}\right)^j \\ &\leqslant \left(TK + \|u_0\|_{L^2(\Omega)}\right) \frac{1}{m\varrho^{1/2}} \frac{1}{\Delta t} \left(\frac{M-m}{M+m}\right)^j \leqslant \frac{C}{\Delta t} \left(\frac{M-m}{M+m}\right)^j, \end{aligned}$$

where C is an upper bound of $\left(TK + \|u_0\|_{L^2(\Omega)}\right) \frac{1}{m\varrho^{1/2}}$. □

Theorem 2. *Let u_i^* and $\bar{u}_i^j$ be the same as defined above and $\hat{u}_i^j$ be the solution of (3.1) when WEB-spline finite element method is used to approximate the problem (3.2) in each iteration. Assume that $\partial\Omega$, g and the weight function ω are sufficiently smooth in their domains. If we suppose that there exists uniform positive constants C^1 and C^2, independent of j, defined below in the proof in (4.6), then, for $j = 1, 2, \ldots$, we have*

$$\left\|u_i^* - \hat{u}_i^j\right\|_H \leqslant \frac{C^*}{\Delta t} \left(\frac{M-m}{M+m}\right)^j + \frac{C\,h}{m\Delta t\sqrt{\Delta t}} \left(1 - \left(\frac{M-m}{M+m}\right)^j\right), \tag{4.5}$$

in which C^* and C are two positive constants.

Proof. We have

$$\left\|u_i^* - \hat{u}_i^j\right\|_H \leqslant \left\|u_i^* - \bar{u}_i^j\right\|_H + \left\|\bar{u}_i^j - \hat{u}_i^j\right\|_H.$$

For the second right-hand side, according to [3, Lemma 3.2], we have

$$\left\|\bar{u}_i^{j+1} - \hat{u}_i^{j+1}\right\|_H \leqslant \frac{M-m}{M+m} \left\|\bar{u}_i^j - \hat{u}_i^j\right\|_H + \frac{2}{M+m} \left\|z_i^j - \bar{z}_i^j\right\|_H.$$

By induction, we conclude that

$$\left\|\bar{u}_i^{j+1} - \hat{u}_i^{j+1}\right\|_H \leqslant \left(\frac{M-m}{M+m}\right)^{j+1} \left\|\bar{u}_i^0 - \hat{u}_i^0\right\|_H + \frac{2}{M+m}\sum_{k=0}^{j}\left(\frac{M-m}{M+m}\right)^k \times \left\|z_i^{j-k} - \bar{z}_i^{j-k}\right\|_H = \frac{2}{M+m}\sum_{k=0}^{j}\left(\frac{M-m}{M+m}\right)^k \left\|z_i^{j-k} - \bar{z}_i^{j-k}\right\|_H.$$

On the other hand, if $\partial\Omega$, g and the weight function ω are sufficiently smooth in their domains, then the WEB-spline finite element method offer full approximation order (see [6]), i.e.,

$$\left\|z_i^j - \bar{z}_i^j\right\|_{H^1(\Omega)} \leqslant C_j(\Omega,\omega)h\left\|z_i^j\right\|_{H^2(\Omega)}.$$

Now, we have

$$\left\|z_i^j - \bar{z}_i^j\right\|_H = \left(\left\|z_i^j - \bar{z}_i^j\right\|_{H^1(\Omega)}^2 + \frac{1}{\Delta t}\left\|z_i^j - \bar{z}_i^j\right\|_{L^2(\Omega)}^2\right)^{1/2} \leqslant \left\|z_i^j - \bar{z}_i^j\right\|_{H^1(\Omega)} + \frac{1}{\sqrt{\Delta t}}\left\|z_i^j - \bar{z}_i^j\right\|_{L^2(\Omega)} \leqslant \left(1 + \frac{k}{\sqrt{\Delta t}}\right)\left\|z_i^j - \bar{z}_i^j\right\|_{H^1(\Omega)} \leqslant C_j\frac{h}{\sqrt{\Delta t}}\left\|z_i^j\right\|_{H^2(\Omega)},$$

where k is Poincaré constant and $C_j > 0$ is an upper bound of $(\sqrt{\Delta t} + k)C_j(\Omega,\omega)$. Therefore, we derive

$$\begin{aligned}\left\|\bar{u}_i^{j+1} - \hat{u}_i^{j+1}\right\|_H &\leqslant \frac{2h}{(M+m)\sqrt{\Delta t}}\sum_{k=0}^{j}\left(\frac{M-m}{M+m}\right)^k C_{j-k}\left\|z_i^{j-k}\right\|_{H^2(\Omega)} \\
&= \frac{2h}{(M+m)\sqrt{\Delta t}}\sum_{k=0}^{j}\left(\frac{M-m}{M+m}\right)^k C_{j-k}\left\|S^{-1}\left(\aleph(\bar{u}_i^{j-k}) - g(\cdot,t_i)\right)\right\|_{H^2(\Omega)} \\
&\leqslant \frac{2h}{(M+m)\sqrt{\Delta t}}\sum_{k=0}^{j}\left(\frac{M-m}{M+m}\right)^k {C'}_{j-k}\left\|\aleph(\bar{u}_i^{j-k}) - g(\cdot,t_i)\right\|_{L^2(\Omega)} \\
&\leqslant \frac{2h}{(M+m)\sqrt{\Delta t}}\sum_{k=0}^{j}\left(\frac{M-m}{M+m}\right)^k {C'}_{j-k}\left\|\aleph(\bar{u}_i^{j-k})\right\|_{L^2(\Omega)} \\
&+ \frac{2h}{(M+m)\sqrt{\Delta t}}\sum_{k=0}^{j}\left(\frac{M-m}{M+m}\right)^k {C'}_{j-k}\left(\|p(\cdot,t_i)\|_{L^2(\Omega)} + \frac{1}{\Delta t}\|u_{i-1}\|_{L^2(\Omega)}\right).\end{aligned}$$

If we define positive constants C^1 and C^2, independent of j such that

$$C^1 = \sup_{0\leqslant k\leqslant j} {C'}_{j-k}\left\|\aleph(\bar{u}_i^{j-k})\right\|_{L^2(\Omega)}, \quad C^2 = \sup_{0\leqslant k\leqslant j} {C'}_{j-k}, \quad j\in\mathbb{N}, \tag{4.6}$$

then we derive

$$\left\|\bar{u}_i^{j+1} - \hat{u}_i^{j+1}\right\|_H \leqslant \frac{C^1 h}{m\sqrt{\Delta t}}\left(1 - \left(\frac{M-m}{M+m}\right)^{j+1}\right)$$

$$+\frac{C^2 h}{m\sqrt{\Delta t}}\left(\sum_{k=1}^{i}\|p(\cdot,t_k)\|_{L^2(\Omega)}+\frac{1}{\Delta t}\|u_0\|_{L^2(\Omega)}\right)\left(1-\left(\frac{M-m}{M+m}\right)^{j+1}\right)$$
$$\leqslant \frac{C\,h}{m\Delta t\sqrt{\Delta t}}\left(1-\left(\frac{M-m}{M+m}\right)^{j+1}\right),$$

where $C>0$ is an upper bound of $C^1\Delta t+C^2\left(TK+\|u_0\|_{L^2(\Omega)}\right)$. So, we have

$$\left\|\bar{u}_i^{j+1}-\hat{u}_i^{j+1}\right\|_H \leqslant \frac{C\,h}{m\Delta t\sqrt{\Delta t}}\left(1-\left(\frac{M-m}{M+m}\right)^{j}\right). \tag{4.7}$$

Consequently, by using Theorem 1 and (4.7), we get

$$\left\|u_i^*-\hat{u}_i^j\right\|_H \leqslant \left\|u_i^*-\bar{u}_i^j\right\|_H+\left\|\bar{u}_i^j-\hat{u}_i^j\right\|_H$$
$$\leqslant \frac{C^*}{\Delta t}\left(\frac{M-m}{M+m}\right)^j+\frac{C\,h}{m\Delta t\sqrt{\Delta t}}\left(1-\left(\frac{M-m}{M+m}\right)^j\right).$$

□

Remark 1. According to upper error bound (4.5), in order to derive convergence, we should use WEB-spline basis functions with $h<\Delta t\sqrt{\Delta t}$.

5 Numerical results

In order to demonstrate the effectiveness of the preconditioned iterative method, we consider the problem (3.3) and apply the proposed algorithm to solve it numerically. In this section, all the experiments were performed in Mathematica 6.0. The exact solution of the problem (3.3) is $u(x,y,t)=tP(x)(y-y^2)$ in which

$$P(x)=\begin{cases}0.2(1+(2x-1)^3), & 0\le x\le 0.5,\\ 0.2(1-(2x-1)^4), & 0.5\le x\le 1.\end{cases}$$

In the proposed Algorithm 1, we apply the quadratic WEB-spline basis with the weight function (2.2) for $h=0.5$ and $h=0.2$. In addition, the algorithm runs for $\Delta t=0.5$ and 0.2, 0.1. The convergence rate here is

$$\frac{M-m}{M+m}=\frac{2.86-0.51}{2.86+0.51}=0.6973.$$

The L^2-norm errors of the proposed method are listed in Table 1, for $h=0.5$, 0.2 and $\Delta t=0.5$, 0.2, 0.1. Also in order to compare our proposed method to Helmholtz preconditioner, we give the L^2-norm errors of the proposed method with Laplacian preconditioner in Table 2.

It is worth to point out that, when we apply Laplacian preconditioner in the proposed method, the condition number will depend on $\frac{1}{\Delta t}$. So, we cannot decrease Δt without any restrictions, but in Helmholtz preconditioner we do not have any limitations.

Table 1. L^2-norm errors using preconditioned iterative method with Helmholtz preconditioner.

h	T	Δt	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
0.2	0.5	0.5	2.12 e-2	7.44 e-3	3.09 e-3	8.80 e-4	5.51 e-4
		0.2	3.88 e-3	2.70 e-3	5.79 e-4	1.01 e-4	8.32 e-5
		0.1	1.41 e-3	9.55 e-4	3.16 e-4	9.92 e-5	5.95 e-5
	1	0.5	3.89 e-2	8.95 e-3	6.71 e-3	2.42 e-3	8.92 e-4
		0.2	9.19 e-3	3.49 e-3	8.89 e-4	3.51 e-4	6.41 e-4
		0.1	6.59 e-3	1.09 e-3	6.75 e-4	3.28 e-4	9.13 e-5
0.5	0.5	0.5	3.27 e-2	1.01 e-2	8.02 e-3	4.80 e-3	2.52 e-3
		0.2	5.56 e-3	3.97 e-3	9.59 e-4	7.35 e-4	3.45 e-4
		0.1	4.89 e-3	1.11 e-3	6.66 e-4	3.98 e-4	1.76 e-4
	1	0.5	5.42 e-2	2.89 e-2	1.08 e-2	7.89 e-3	5.41 e-3
		0.2	7.49 e-3	6.52 e-3	1.98 e-3	7.71 e-4	4.57 e-4
		0.1	6.99 e-3	4.92 e-3	8.40 e-4	5.49 e-4	3.55 e-4

Table 2. L^2-norm errors using preconditioned iterative method with Laplacian preconditioner.

h	T	Δt	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
0.2	0.5	0.5	8.02 e-3	5.69 e-3	4.04 e-3	2.87 e-3	2.04 e-3
		0.2	4.31 e-3	2.25 e-3	1.29 e-3	2.22 e-3	2.09 e-3
		0.1	3.81 e-3	1.69 e-3	1.36 e-3	1.55 e-3	1.34 e-3
	1	0.5	4.19 e-2	2.97 e-2	2.11 e-2	1.51 e-2	1.20 e-2
		0.2	2.01 e-2	1.76 e-2	1.43 e-2	1.15 e-2	9.98 e-3
		0.1	2.01 e-2	1.46 e-2	1.15 e-2	9.34 e-3	6.12 e-3
0.5	0.5	0.5	2.47 e-2	1.75 e-2	1.24 e-2	8.80 e-3	6.31 e-3
		0.2	9.04 e-3	5.93 e-3	5.00 e-3	4.05 e-3	3.15 e-3
		0.1	7.45 e-3	6.58 e-3	4.19 e-3	3.51 e-3	2.11 e-3
	1	0.5	5.31 e-2	3.77 e-2	2.68 e-2	1.90 e-2	1.55 e-2
		0.2	3.68 e-2	2.91 e-2	2.11 e-2	1.13 e-2	1.09 e-2
		0.1	3.59 e-2	2.31 e-2	1.91 e-2	1.09 e-2	1.03 e-2

6 Conclusions

In this article, we investigated how to solve nonlinear parabolic problems of type (1.1) by first discretizing the time variable and then applying the preconditioned Sobolev space gradient method with WEB-spline finite element method. As it is seen in the paper, for the discretized in time parabolic problems the term arising from discretization of the time derivative on the new time-level should also be included. This means that the term $\frac{1}{\Delta t}u$ should be added in the definition of Laplacian operator. Hence, in this paper we apply a Helmholtz preconditioner instead of Laplacian preconditioner and this choice of preconditioner leads to a condition number independent of Δt. This is because the Laplacian preconditioner is essentially worthless in this case, due to the fact that if Δt is small then the condition number is huge and the convergence factor is ≈ 1. At the end, the effectiveness of the proposed method has been illustrated in the numerical section.

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Appendix: Condition number

Based on [4, Page 110], the condition number of the operator $\aleph$ of an elliptic problem is defined as

$$\mathrm{cond}(\aleph) = \Lambda(\aleph)/\lambda(\aleph),$$

where

$$\Lambda(\aleph) = \sup_{u \neq v \in D(\aleph)} \frac{(\aleph(v) - \aleph(u), v - u)}{\|v - u\|^2_{L^2(\Omega)}},$$

$$\lambda(\aleph) = \inf_{u \neq v \in D(\aleph)} \frac{(\aleph(v) - \aleph(u), v - u)}{\|v - u\|^2_{L^2(\Omega)}},$$

in which $D(\aleph) = H^2(\Omega) \cap H^1_0(\Omega)$ is the domain of the operator $\aleph$. According to the given assumptions (1.1), it is easy to check that

$$(\aleph(v) - \aleph(u), v - u) = \int_\Omega (f(\mathbf{x}, \nabla v) - f(\mathbf{x}, \nabla u)) \cdot (\nabla v - \nabla u) d\mathbf{x} + \frac{1}{\Delta t} \|v - u\|^2_{L^2(\Omega)}$$

$$= \int_\Omega \frac{\partial f}{\partial \eta} (\mathbf{x}, \nabla u + \theta \nabla (v - u)) (\nabla v - \nabla u) \cdot (\nabla v - \nabla u) d\mathbf{x} + \frac{1}{\Delta t} \|v - u\|^2_{L^2(\Omega)},$$

where $0 < \theta < 1$. Hence, we have

$$\mu_1 \frac{\int_\Omega |\nabla (v - u)|^2 d\mathbf{x}}{\|v - u\|^2_{L^2(\Omega)}} \leqslant \frac{(\aleph(v) - \aleph(u), v - u)}{\|v - u\|^2_{L^2(\Omega)}} \leqslant \mu_2 \frac{\int_\Omega |\nabla (v - u)|^2 d\mathbf{x}}{\|v - u\|^2_{L^2(\Omega)}} + \frac{1}{\Delta t}.$$

So, using Sobolev inequality with constant $\zeta > 0$, we conclude that

$$\sup_{u \neq v \in D(\aleph)} \frac{\int_\Omega |\nabla (v - u)|^2 d\mathbf{x}}{\|v - u\|^2_{L^2(\Omega)}} = \infty, \quad \inf_{u \neq v \in D(\aleph)} \frac{\int_\Omega |\nabla (v - u)|^2 d\mathbf{x}}{\|v - u\|^2_{L^2(\Omega)}} = \zeta,$$

which means $\mathrm{cond}(\aleph) = \infty$.