

Three-Layer Approximation of Two-Layer Shallow Water Equations

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Abstract. Two-layer shallow water equations describe flows that consist of two layers of inviscid fluid of different (constant) densities flowing over bottom topography. Unlike the single-layer shallow water system, the two-layer one is only conditionally hyperbolic: the system loses its hyperbolicity because of the momentum exchange terms between the layers and as a result its solutions may develop instabilities. We study a three-layer approximation of the two-layer shallow water equations by introducing an intermediate layer of a small depth. We examine the hyperbolicity range of the three-layer model and demonstrate that while it still may lose hyperbolicity, the three-layer approximation may improve stability properties of the two-layer shallow water system.

Keywords: two-layer shallow water equations, central-upwind scheme, well-balanced scheme, conditional hyperbolicity.

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1 Introduction

Shallow water models are widely used as a mathematical framework to study water flows in rivers and coastal areas as well as to investigate a variety of phenomena in atmospheric sciences and oceanography. The basic feature of shallow water flows is that the vertical effect can be neglected compared with the horizontal one with a good approximation. This allows a considerable simplification in the mathematical formulation by replacing the vertical momentum equation by the hydrostatic pressure distribution. As a result, such

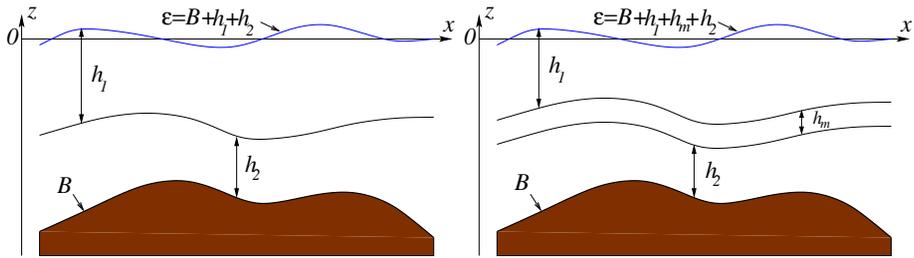


Figure 1. Two-layer (left) and three-layer (right) shallow water setup.

flows are usually described by the shallow water equations. The simplest, yet commonly used, shallow water model is the Saint-Venant system [12].

Layered shallow water equations describe the behaviour of several superposed layers of inviscid fluid of different, constant densities flowing over bottom topography, as illustrated in Fig. 1, and are derived by depth averaging the incompressible Navier–Stokes equations with the hydrostatic assumption within each layer, see [7]. For instance, the following one-dimensional (1-D) two-layer shallow water model describes a flow that consists of layers of heights h_1 (upper layer) and h_2 (lower layer) at position x at time t with the corresponding constant densities $\rho_1 < \rho_2$, velocities u_1 and u_2 , and discharges $q_1 := h_1 u_1$ and $q_2 := h_2 u_2$:

$$\begin{cases} (h_1)_t + (q_1)_x = 0, \\ (q_1)_t + \left(h_1 u_1^2 + \frac{g}{2} h_1^2 \right)_x = -gh_1 B_x - gh_1 (h_2)_x, \\ (h_2)_t + (q_2)_x = 0, \\ (q_2)_t + \left(h_2 u_2^2 + \frac{g}{2} h_2^2 \right)_x = -gh_2 B_x - gr h_2 (h_1)_x. \end{cases} \quad (1.1)$$

Here, $B(x)$ is a function describing the bottom topography, $r = \frac{\rho_1}{\rho_2}$ is the ratio of the densities, which is typically $r \sim 1$, and g is the gravitational constant.

The system (1.1) consists of four equations: the first and third of which indicate the conservation of mass, and the second and fourth equations state the momentum balance for each layer. One of the key challenges in solving the system (1.1) numerically is that it is only conditionally hyperbolic which may lead to significant instabilities. Specifically speaking, the hyperbolic region of the system (1.1) mainly depends on the difference between the velocities of the two layers: when $|u_1 - u_2|$ is large, the system is not hyperbolic and one may expect appearance of Kelvin–Helmholtz-type instabilities. Another numerical challenge is related to the presence of nonconservative momentum exchange terms on the right-hand side (RHS) of (1.1). Even though the analytic theory of nonconservative hyperbolic system has been developed (see, e.g., [11]), designing a numerical scheme whose solution is guaranteed to converge to the unique physically relevant solution is still an open problem. A theoretical framework for designing converging numerical method was presented in [26],

and a class of path-conservative schemes was developed afterwards in [8, 9, 24]. However, as it has been recently demonstrated in [2], convergence of path-conservative schemes has still not been rigorously established. In [19], an alternative approach was proposed. The system (1.1) was rewritten in a different form, which is equivalent to (1.1) for smooth solutions:

$$\begin{cases} (h_1)_t + (q_1)_x = 0, \\ (q_1)_t + \left(\frac{q_1^2}{h_1} + g\varepsilon h_1\right)_x = g\varepsilon(h_1)_x, \\ w_t + (q_2)_x = 0, \\ (q_2)_t + \left(\frac{q_2^2}{w - B} + \frac{g}{2}w^2 - \frac{g}{2}rh_1^2 - gB(rh_1 + w)\right)_x \\ = -g(rh_1 + w)B_x - gr\varepsilon(h_1)_x, \end{cases} \tag{1.2}$$

where $w := h_2 + B$ and $\varepsilon := h_1 + h_2 + B$. The main advantage of the rewritten system (1.2) over the original one (1.1) is that the coefficients of the nonconservative product terms on the RHS of (1.2) are now proportional to the water surface variable ε , which is typically small in all realistic settings provided that the reference level in the vertical direction z is selected to be 0 (see Fig. 1). This is obviously true in all of the oceanographic applications, in which the magnitude of surface waves is much smaller than the depth of the ocean. In [19], the system (1.2) was numerically solved using a well-balanced, positivity-preserving central-upwind scheme, which was originally developed for hyperbolic systems of conservation laws [16, 17, 20, 21] and then extended and applied to hyperbolic systems of balance laws arising in modelling shallow water flows, see [15, 18, 19]. For several other numerical methods for multilayer shallow water equations we refer the reader, e.g., to [1, 3, 4, 5].

One of the reasons the system (1.1) (and also its rewritten version (1.2)) is only conditionally hyperbolic is that it is derived using a non-mixing assumption. The loss of hyperbolicity triggers Kelvin–Helmholtz-type instabilities. Several stabilization approaches have been recently proposed. For example, in [10], interlayer friction terms were added to the two-layer shallow water system (1.1). In this paper, we study a three-layer approximation of the two-layer system (1.1), which is obtained by introducing an intermediate layer of a (small) depth h_m . The new layer is expected to contain all possible turbulence and mixing without substantially affecting the depth and horizontal velocities of the other layers. This idea has been originally presented in [6, 13] where the third layer was adaptively introduced in nonhyperbolic regions. However, the results obtained in [6, 13] seem to be inconclusive since the hyperbolicity was gained only when the width of the intermediate layer was very large. Our main objective is to study to what extent having an additional, “buffer” layer would help to regain hyperbolicity in cases it is lost. To this extent, we test a simpler (than in [6, 13]) strategy of adding a “buffer” layer at the initial time moment and then following its evolution in time. (In a more sophisticated adaptive procedure proposed in [6, 13], the width of the intermediate layer was modified dynamically to make sure that a meaningful “buffer” zone is always present between the upper and lower layers.) Although the three-layer system

may still lose hyperbolicity as it is demonstrated in Section 5 below, we show that the proposed three-layer approximation may improve stability properties of the original two-layer shallow water system. At the same time, the loss of hyperbolicity seems to be a generic feature of the studied three-layer system even though its hyperbolicity range may be much larger than the one of the corresponding two-layer system, especially when the width of the “buffer” layer is large (in the latter case, however, one cannot claim that the intermediate layer is indeed a “buffer” one). As we demonstrate in our numerical experiments reported in Section 4, even if having a “buffer” layer does not completely remove instabilities, the oscillations become much smaller. This suggests that the three-layer shallow water system may be viewed as an improved version of its two-layer counterpart.

The paper is organized as follows. In Section 2, we present the three-layer approximation of the two-layer shallow water system and discuss the strategy for choosing the intermediate layer. In Section 3, we derive the central-upwind scheme for the three-layer system. Finally, Sections 4 and 5 contain numerical examples and analysis of the hyperbolicity properties of the three-layer shallow water equations.

2 The Three-Layer Shallow Water System

In this section, we introduce a three-layer shallow water system by adding an intermediate layer with the depth h_m , velocity u_m and discharge q_m to system (1.1). The three-layer system reads (for the form of a general multilayer system we refer the reader to [3]):

$$\left\{ \begin{array}{l} (h_1)_t + (q_1)_x = 0, \\ (q_1)_t + \left(h_1 u_1^2 + \frac{g}{2} h_1^2 \right)_x = -gh_1 B_x - gh_1 (h_m + h_2)_x, \\ (h_m)_t + (q_m)_x = 0, \\ (q_m)_t + \left(h_m u_m^2 + \frac{g}{2} h_m^2 \right)_x \\ \quad = -gh_m B_x - g \frac{2r}{1+r} h_m (h_1)_x - gh_m (h_2)_x, \\ (h_2)_t + (q_2)_x = 0, \\ (q_2)_t + \left(h_2 u_2^2 + \frac{g}{2} h_2^2 \right)_x \\ \quad = -gh_2 B_x - g \frac{1+r}{2} h_2 (h_m)_x - gh_2 (h_1)_x. \end{array} \right. \quad (2.1)$$

To apply the central-upwind scheme to this system, we follow the approach in [19] and rewrite it as follows:

$$\left\{ \begin{array}{l} (h_1)_t + (q_1)_x = 0, \\ (q_1)_t + \left(\frac{q_1^2}{h_1} + g\varepsilon h_1 \right)_x = g\varepsilon (h_1)_x, \end{array} \right.$$

$$\left\{ \begin{aligned} (h_m)_t + (q_m)_x &= 0, \\ (q_m)_t + \left(\frac{q_m^2}{h_m} + g\varepsilon h_m \right)_x &= g\varepsilon(h_m)_x + g\frac{1-r}{1+r}h_m(h_1)_x, \\ w_t + (q_2)_x &= 0, \\ (q_2)_t + \left(\frac{q_2^2}{w-B} + \frac{g}{2}w^2 - \frac{g}{2}rh_1^2 - gB(rh_1+w) \right. \\ &\quad \left. - g\frac{1+r}{2}h_m(h_1+B) - g\frac{1+r}{4}h_m^2 \right)_x = -g\left(rh_1+w + \frac{1+r}{2}h_m \right) B_x \\ &\quad - g\varepsilon\left(r(h_1)_x + \frac{1+r}{2}(h_m)_x \right) - g\frac{1-r}{2}h_m(h_1)_x. \end{aligned} \right. \tag{2.2}$$

As in the two-layer system (1.2), we have introduced the new equilibrium variables $w := h_2 + B$ and the water surface $\varepsilon := h_1 + h_m + h_2 + B$. Notice that the formula for ε is now different from the one used in (1.2) since we have one extra layer to be added. At the same time, we choose the origin of coordinate system so that the reference surface level is 0 (see Fig. 1, right) as it was done in the two-layer case schematically presented in Fig. 1, left.

The new system is preferable for numerical computation thanks to the following two reasons:

(i) Since the system (2.2) is written in terms of equilibrium variables, the “lake at rest” steady-state solution takes a particularly nice form:

$$h_1 \equiv \text{const}, \quad h_m \equiv \text{const}, \quad w \equiv \text{const}, \quad u_1 \equiv u_2 \equiv u_m \equiv 0. \tag{2.3}$$

Thus, the reformulated system makes it simpler to derive a well-balanced scheme for the three-layer shallow water equations;

(ii) The coefficients in the nonconservative terms on the RHS of (2.2) are proportional to either ε or $(1-r)h_m$. Note that ε vanishes at the “lake at rest” steady states, and, what is even more important, in most oceanographic application remains very small. We also do not expect the depth of the intermediate layer to become large in any realistic scenario, which means that $(1-r)h_m$ is also expected to remain small. Smallness of ε and $(1-r)h_m$ is expected to make the computed results to be practically independent of the way the nonconservative terms are discretized as it was the case in [19].

Remark 1. Notice that if one sets $h_m = q_m \equiv 0$, the three-layer system (2.2) reduces to the two-layer system (1.2).

3 Central-Upwind Scheme for the Three-Layer System

In this section, we describe a second-order well-balanced, positivity preserving central-upwind scheme for the three-layer shallow water system (2.2), which we rewrite in the vector form as

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U}, B)_x = \mathbf{S}(\mathbf{U}, B) + \mathbf{N}(\mathbf{U}, B), \tag{3.1}$$

where the unknown function \mathbf{U} , flux \mathbf{F} , geometric source term \mathbf{S} and nonconservative product term \mathbf{N} are given by the following formulae:

$$\begin{aligned}
 \mathbf{U} &:= (h_1, q_1, h_m, q_m, w, q_2)^T, \\
 \mathbf{F}(\mathbf{U}, B) &:= \left(q_1, \frac{q_1^2}{h_1} + g\varepsilon h_1, q_m, \frac{q_m^2}{h_m} + g\varepsilon h_m, q_2, \frac{q_2^2}{w - B} + \frac{g}{2}w^2 - \frac{g}{2}rh_1^2 \right. \\
 &\quad \left. - gB(rh_1 + w) - g\frac{1+r}{2}h_m(h_1 + B) - g\frac{1+r}{4}h_m^2 \right)^T, \\
 \mathbf{S}(\mathbf{U}, B) &:= \left(0, 0, 0, 0, 0, -g\left(rh_1 + w + \frac{1+r}{2}h_m \right)B_x \right)^T, \\
 \mathbf{N}(\mathbf{U}, B) &:= \left(0, g\varepsilon(h_1)_x, 0, g\varepsilon(h_m)_x + g\frac{1-r}{1+r}h_m(h_1)_x, \right. \\
 &\quad \left. 0, -gr\varepsilon(h_1)_x - g\frac{1+r}{2}\left(\varepsilon(h_m)_x + \frac{1-r}{1+r}h_m(h_1)_x \right) \right)^T.
 \end{aligned} \tag{3.2}$$

We divide the computational domain into the cells $C_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, which for simplicity are assumed to be uniform so that $x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} = \Delta x$ and $x_j = j\Delta x$ for all j . We denote by

$$\bar{\mathbf{U}}_j(t) = \frac{1}{\Delta x} \int_{C_j} \mathbf{U}(x, t) dx$$

the computed cell averages of \mathbf{U} , which we assume to be available at time t . Using the notations introduced in (3.2), a semi-discretization of (3.1) can be written as the following system of ODEs:

$$\frac{d}{dt} \bar{\mathbf{U}}_j(t) = -\frac{\mathbf{H}_{j+\frac{1}{2}}(t) - \mathbf{H}_{j-\frac{1}{2}}(t)}{\Delta x} + \bar{\mathbf{S}}_j(t) + \bar{\mathbf{N}}_j(t), \tag{3.3}$$

where $\mathbf{H}_{j+\frac{1}{2}}$ are numerical fluxes, and $\bar{\mathbf{S}}_j$ and $\bar{\mathbf{N}}_j$ are discretizations of the geometric source and nonconservative product terms, respectively:

$$\bar{\mathbf{S}}_j(t) \approx \frac{1}{\Delta x} \int_{C_j} \mathbf{S}(\mathbf{U}(x, t), B(x)) dx, \quad \bar{\mathbf{N}}_j(t) \approx \frac{1}{\Delta x} \int_{C_j} \mathbf{N}(\mathbf{U}(x, t), B(x)) dx. \tag{3.4}$$

We use the central-upwind numerical fluxes derived in [17] (see also [15, 18, 19, 21]):

$$\begin{aligned}
 \mathbf{H}_{j+\frac{1}{2}} &= \frac{a_{j+\frac{1}{2}}^+ \mathbf{F}(\mathbf{U}_{j+\frac{1}{2}}^-, B_{j+\frac{1}{2}}) - a_{j+\frac{1}{2}}^- \mathbf{F}(\mathbf{U}_{j+\frac{1}{2}}^+, B_{j+\frac{1}{2}})}{a_{j+\frac{1}{2}}^+ - a_{j+\frac{1}{2}}^-} \\
 &\quad + \frac{a_{j+\frac{1}{2}}^+ a_{j+\frac{1}{2}}^-}{a_{j+\frac{1}{2}}^+ - a_{j+\frac{1}{2}}^-} [\bar{\mathbf{U}}_{j+\frac{1}{2}}^+ - \bar{\mathbf{U}}_{j+\frac{1}{2}}^-].
 \end{aligned} \tag{3.5}$$

Here, $B_{j+\frac{1}{2}} := B(x_{j+\frac{1}{2}})$ (or $B_{j+\frac{1}{2}} := \frac{1}{2}(B(x_{j+\frac{1}{2}} + 0) + B(x_{j+\frac{1}{2}} - 0))$ in the case the bottom topography function B is discontinuous at $x = x_{j+\frac{1}{2}}$), $\mathbf{U}_{j+\frac{1}{2}}^\pm$

are the right/left point values at $x = x_{j+\frac{1}{2}}$ of the conservative piecewise linear reconstruction

$$\tilde{\mathbf{U}}(x) := \bar{\mathbf{U}}_j + (\mathbf{U}_x)_j (x - x_j), \quad x \in C_j, \tag{3.6}$$

so that

$$\begin{aligned} \mathbf{U}_{j+\frac{1}{2}}^- &:= \tilde{\mathbf{U}}(x_{j+\frac{1}{2}} - 0) = \bar{\mathbf{U}}_j + \frac{\Delta x}{2} (\mathbf{U}_x)_j, \\ \mathbf{U}_{j+\frac{1}{2}}^+ &:= \tilde{\mathbf{U}}(x_{j+\frac{1}{2}} + 0) = \bar{\mathbf{U}}_{j+1} - \frac{\Delta x}{2} (\mathbf{U}_x)_j. \end{aligned} \tag{3.7}$$

The numerical derivatives $(\mathbf{U}_x)_j$ are at least first-order accurate approximations of $\mathbf{U}_x(x_j, t)$, computed using a nonlinear limiter needed to ensure a non-oscillatory nature of the reconstruction (3.6). In our numerical computations, we have used the generalized minmod limiter:

$$(\mathbf{U}_x)_j = \text{minmod} \left(\theta \frac{\bar{\mathbf{U}}_j - \bar{\mathbf{U}}_{j-1}}{\Delta x}, \frac{\bar{\mathbf{U}}_{j+1} - \bar{\mathbf{U}}_{j-1}}{2\Delta x}, \theta \frac{\bar{\mathbf{U}}_{j+1} - \bar{\mathbf{U}}_j}{\Delta x} \right), \tag{3.8}$$

where the minmod function, defined as

$$\text{minmod}(z_1, z_2, \dots) := \begin{cases} \min_j \{z_j\}, & \text{if } z_j > 0 \ \forall j, \\ \max_j \{z_j\}, & \text{if } z_j < 0 \ \forall j, \\ 0, & \text{otherwise,} \end{cases}$$

is applied in a componentwise manner. The parameter $\theta \in [1, 2]$ can be used to control the amount of numerical viscosity present in the resulting scheme (see, e.g., [22, 25] for more details). In our numerical experiments, we use $\theta = 1$, which corresponds to the most diffusive minmod reconstruction.

The right- and left-sided local speeds $a_{j+\frac{1}{2}}^\pm$ in (3.5) should be obtained as in [16, 17, 19] using the smallest and largest eigenvalues of the system (2.1), which can be written in the following quasilinear form:

$$\begin{pmatrix} h_1 \\ q_1 \\ h_m \\ q_m \\ h_2 \\ q_2 \end{pmatrix}_t + A \begin{pmatrix} h_1 \\ q_1 \\ h_m \\ q_m \\ h_2 \\ q_2 \end{pmatrix}_x = -gB_x \begin{pmatrix} 0 \\ h_1 \\ 0 \\ h_m \\ 0 \\ h_2 \end{pmatrix}, \tag{3.9}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ gh_1 - u_1^2 & 2u_1 & gh_1 & 0 & gh_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{2r}{1+r}gh_m & 0 & gh_m - u_m^2 & 2u_m & gh_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ grh_2 & 0 & \frac{1+r}{2}gh_2 & 0 & gh_2 - u_2^2 & 2u_2 \end{pmatrix}. \tag{3.10}$$

Since no analytical expression of the eigenvalues of the matrix on the left-hand side (LHS) of (3.10) is available, we follow the approach in [19] and

overestimate the right/left-sided local speeds using upper/lower bounds on the largest/smallest eigenvalues, respectively. To this end, we write the characteristics equation in the form

$$\lambda^6 + c_1\lambda^5 + c_2\lambda^4 + c_3\lambda^3 + c_4\lambda^2 + c_5\lambda + c_6 = 0 \tag{3.11}$$

with the coefficients

$$\begin{aligned} c_1 &= -2(u_1 + u_m + u_2), \quad c_2 = \psi_1 + \psi_m + \psi_2 + 4(u_1u_m + u_1u_2 + u_mu_2), \\ c_3 &= -2(\psi_1(u_m + u_2) + \psi_m(u_1 + u_2) + \psi_2(u_1 + u_m) - 4u_1u_mu_2), \\ c_4 &= \psi_1\psi_m + \psi_1\psi_2 + \psi_m\psi_2 + 4(\psi_1u_mu_2 + \psi_mu_1u_2 + \psi_2u_1u_m) \\ &\quad - g^2 \left(\frac{2r}{1+r} h_1h_m + rh_1h_2 + \frac{1+r}{2} h_2h_m \right), \\ c_5 &= 2 \left[u_1 \left(\frac{1+r}{2} g^2h_mh_2 - \psi_m\psi_2 \right) + u_m (rg^2h_1h_2 - \psi_1\psi_2) \right. \\ &\quad \left. + u_2 \left(\frac{2r}{1+r} g^2h_1h_m - \psi_1\psi_m \right) \right], \\ c_6 &= \psi_1\psi_m\psi_2 - g^2 \left(\frac{1+r}{2} \psi_1h_mh_2 + r\psi_mh_1h_2 + \frac{2r}{1+r} \psi_2h_1h_m \right) \\ &\quad - 2rg^3h_1h_mh_2, \end{aligned} \tag{3.12}$$

where we have used the following notations:

$$\psi_1 := u_1^2 - gh_1, \quad \psi_m := u_m^2 - gh_m, \quad \psi_2 := u_2^2 - gh_2.$$

We then establish the bounds by using the Lagrange theorem (see, e.g., [23]) as it was done in [19] for the two-layer shallow water system. According to the Lagrange theorem, the largest nonnegative root is smaller than the sum of the largest and the second largest numbers in the set $\{\sqrt[j]{|c_j|} : j \in J_{\max}\}$, where J_{\max} is the set of the nonnegative coefficients of the characteristic polynomial (3.11), (3.12). Similarly, the smallest nonpositive root of this polynomial is larger than the sum of the smallest and second smallest numbers in the set $\{-\sqrt[j]{|d_j|} : j \in J_{\min}\}$, where J_{\min} is the set of the negative coefficients of the polynomial

$$\lambda^6 + d_1\lambda^5 + d_2\lambda^4 + d_3\lambda^3 + d_4\lambda^2 + d_5\lambda + d_6 = 0, \quad d_j = (-1)^j c_j \quad \forall j.$$

If we denote the obtained upper bound by $\lambda_{\max} = \lambda_{\max}(h_1, h_m, h_2, u_1, u_m, u_2)$ and the lower bound by $\lambda_{\min} = \lambda_{\min}(h_1, h_m, h_2, u_1, u_m, u_2)$, then the one-sided local speeds can be estimated by

$$\begin{aligned} a_{j+\frac{1}{2}}^+ &= \max_{\pm} \{ \lambda_{\max}((h_1)_{j+\frac{1}{2}}^{\pm}, (h_m)_{j+\frac{1}{2}}^{\pm}, (h_2)_{j+\frac{1}{2}}^{\pm}, (u_1)_{j+\frac{1}{2}}^{\pm}, (u_m)_{j+\frac{1}{2}}^{\pm}, (u_2)_{j+\frac{1}{2}}^{\pm}) \}, \\ a_{j+\frac{1}{2}}^- &= \min_{\pm} \{ \lambda_{\min}((h_1)_{j+\frac{1}{2}}^{\pm}, (h_m)_{j+\frac{1}{2}}^{\pm}, (h_2)_{j+\frac{1}{2}}^{\pm}, (u_1)_{j+\frac{1}{2}}^{\pm}, (u_m)_{j+\frac{1}{2}}^{\pm}, (u_2)_{j+\frac{1}{2}}^{\pm}) \}. \end{aligned}$$

Remark 2. Note that the quantities $\bar{U}_j, \mathbf{U}_{j+\frac{1}{2}}^{\pm}, (\mathbf{U}_x)_j$ and $a_{j+\frac{1}{2}}^{\pm}$ in (3.5)–(3.8) depend on t , but we simplify the notation by suppressing this dependence.

To ensure nonnegativity of h_1 , h_m and h_2 , we follow [18, 19] and correct the reconstructed point values of w in each cell C_j according to the following algorithm:

$$\begin{aligned} &\text{If } w_{j+\frac{1}{2}}^- < B_{j+\frac{1}{2}}, \text{ then set } w_{j+\frac{1}{2}}^- := B_{j+\frac{1}{2}}, w_{j-\frac{1}{2}}^+ := 2\bar{w}_j - B_{j+\frac{1}{2}}; \\ &\text{If } w_{j-\frac{1}{2}}^+ < B_{j-\frac{1}{2}}, \text{ then set } w_{j-\frac{1}{2}}^+ := B_{j-\frac{1}{2}}, w_{j+\frac{1}{2}}^- := 2\bar{w}_j - B_{j-\frac{1}{2}}. \end{aligned}$$

To guarantee that the resulting scheme is well-balanced in the sense that the ‘‘lake at rest’’ steady states (2.3) are preserved, we employ a well-balanced quadrature (developed in [15], see also [18, 19]) and approximate the cell averages of the geometric source term as:

$$\bar{\mathbf{S}}_j(t) = \left(0, 0, 0, 0, 0, -g \left(r(\bar{h}_1)_j + \bar{w}_j + \frac{1+r}{2} (\bar{h}_m)_j \right) \cdot \frac{B_{j+\frac{1}{2}} - B_{j-\frac{1}{2}}}{\Delta x} \right)^T.$$

Finally, we follow [19] and discretize the nonconservative product terms using the same quadrature, which results in

$$\begin{aligned} \bar{N}_j^{(2)}(t) &= g\bar{\varepsilon}_j \frac{(h_1)_{j+\frac{1}{2}}^- - (h_1)_{j-\frac{1}{2}}^+}{\Delta x}, \\ \bar{N}_j^{(4)}(t) &= g\bar{\varepsilon}_j \frac{(h_m)_{j+\frac{1}{2}}^- - (h_m)_{j-\frac{1}{2}}^+}{\Delta x} + g \frac{1-r}{1+r} (\bar{h}_m)_j \frac{(h_1)_{j+\frac{1}{2}}^- - (h_1)_{j-\frac{1}{2}}^+}{\Delta x}, \\ \bar{N}_j^{(6)}(t) &= -r\bar{N}_j^{(2)}(t) - \frac{1+r}{2} \bar{N}_j^{(4)}(t), \end{aligned}$$

where $\bar{N}_j^{(2)}$, $\bar{N}_j^{(4)}$ and $\bar{N}_j^{(6)}$ are (nonzero) components of the vector $\bar{\mathbf{N}}_j(t)$ in (3.4), that is, $\bar{\mathbf{N}}_j = (0, \bar{N}_j^{(2)}, 0, \bar{N}_j^{(4)}, 0, \bar{N}_j^{(6)})^T$.

Remark 3. A fully discrete central-upwind scheme is obtained by applying an appropriate ODE solver to (3.3). In our numerical experiments, we have used the third-order strong stability preserving Runge–Kutta (SSP-RK) method from [14].

Remark 4. As in [18, 19], one can prove that provided the ODE system (3.3) is discretized using an SSP ODE solver and the CFL condition is satisfied with the CFL number equal to 1/2, the presented central-upwind scheme will be positivity preserving in the following sense: If at a certain time level, $(\bar{h}_1)_j(t) \geq 0$, $(\bar{h}_m)_j(t) \geq 0$ and $(\bar{h}_2)_j(t) \geq 0$ for all j , then at the next time level $(\bar{h}_1)_j(t + \Delta t) \geq 0$, $(\bar{h}_m)_j(t + \Delta t) \geq 0$ and $(\bar{h}_2)_j(t + \Delta t) \geq 0$ as well. It should be observed that as in [18, 19] we set $(\bar{h}_2)_j := \bar{w}_j - \tilde{B}(x_j)$ (rather than $\bar{w}_j - B(x_j)$), where \tilde{B} is a continuous piecewise linear approximant of the bottom topography B , namely,

$$\tilde{B}(x) = B_{j-\frac{1}{2}} + (B_{j+\frac{1}{2}} - B_{j-\frac{1}{2}}) \frac{x - x_{j-\frac{1}{2}}}{\Delta x}, \quad x \in C_j.$$

Remark 5. Notice that if one sets $(\bar{h}_m)_j(0) = (\bar{q}_m)_j(0) \equiv 0$, the presented central-upwind scheme reduces to the central-upwind scheme from [19] for the two-layer system (1.2).

4 Numerical Examples

In this section, we apply the central-upwind scheme developed in Section 3 to the three-layer shallow water system (2.2). The purpose of the presented numerical experiments is to check whether (and to what extent) the three-layer approximation (2.2) is capable of curing the instabilities that are typically developed in solutions of the two-layer system (1.2) in the nonhyperbolic regime.

In both examples below, we take the gravitational constant $g = 10$ and the density ratio $r = 0.98$.

Example 1. We consider the three-layer shallow water system (2.2) subject to the initial data:

$$\begin{aligned}
 h_1(x, 0) &= \begin{cases} 1 - \frac{1}{2} \widehat{h}_m, & |x| \geq 1, \\ 1 - \frac{1}{4} \sin(2\pi x) - \frac{1}{2} \widehat{h}_m, & |x| < 1, \end{cases} \\
 h_2(x, 0) &= \begin{cases} 1 - \frac{1}{2} \widehat{h}_m, & |x| \geq 1, \\ 1 + \frac{1}{4} \sin(2\pi x) - \frac{1}{2} \widehat{h}_m, & |x| < 1, \end{cases} \\
 h_m(x, 0) &\equiv \widehat{h}_m, \quad u_1(x, 0) \equiv \widehat{u}_1, \quad u_2(x, 0) \equiv \widehat{u}_2, \quad u_m(x, 0) \equiv \frac{r\widehat{u}_1 + \widehat{u}_2}{r + 1}
 \end{aligned}$$

with $\widehat{u}_1 = 0.4$ and $\widehat{u}_1 = -0.4$. We take $B(x) \equiv -2$ and implement free boundary conditions on the computational domain $[-2, 2]$.

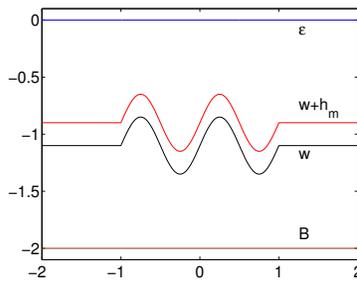


Figure 2. Example 1: The initial setting for the three-layer system with $\widehat{h}_m = 0.2$.

The initial setting for $h_m = 0.2$ is shown in Fig. 2. The simulations are run on the uniform grid with $\Delta x = 1/200$ until time $t = 0.5$.

We first compute the numerical solution of the two-layer system ($\widehat{h}_m = 0$). The obtained results are presented in Figure 3. Using the hyperbolicity condition (5.2), it is easy to check that in this case a large difference in initial velocities corresponds to the nonhyperbolic regime, which explains an oscillatory behaviour of the computed solution.

We then introduce a thin intermediate layer by taking $\widehat{h}_m = 0.05$ and compute the corresponding numerical solution of the three-layer system. As one can see in Fig. 4, the computed solution is still oscillatory, but the magnitude of oscillations is smaller than in the two-layer solution. When the initial thickness

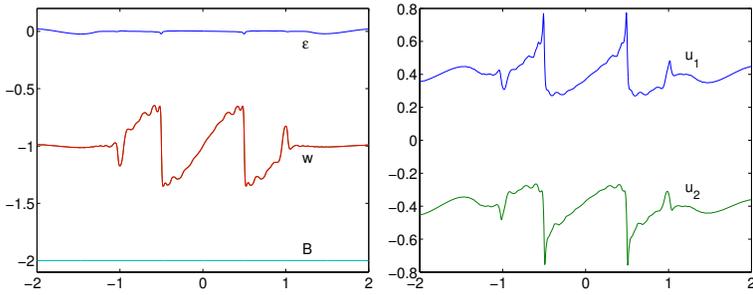


Figure 3. Example 1: Solution of the two-layer system ($\hat{h}_m = 0$).

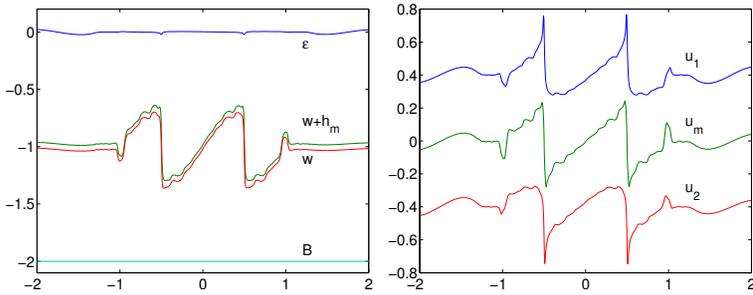


Figure 4. Example 1: Solution of the three-layer system with $\hat{h}_m = 0.05$.

of the intermediate layer is increased to $\hat{h}_m = 0.1$, the oscillations are further reduced (see Fig. 5). Finally, when we take $\hat{h}_m = 0.2$, the solution is practically non-oscillatory (see Fig. 6).

We therefore conclude that in this example, the studied three-layer approximation of the two-layer shallow water system leads to stabilization of the computed solution. However, one may argue that in the latter case of $\hat{h}_m = 0.2$ the “buffer” layer is not too thin and thus the three-layer model may not be considered as a minor modification of the original two-layer one.

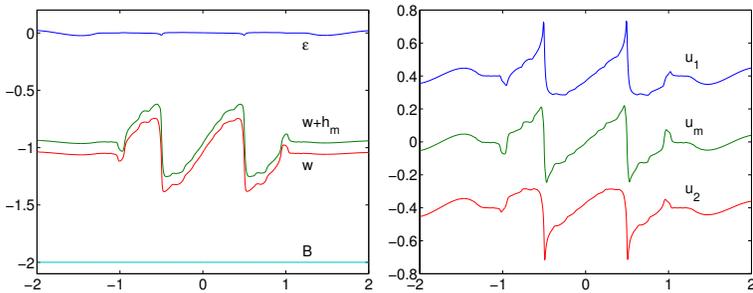


Figure 5. Example 1: Solution of the three-layer system with $\hat{h}_m = 0.1$.

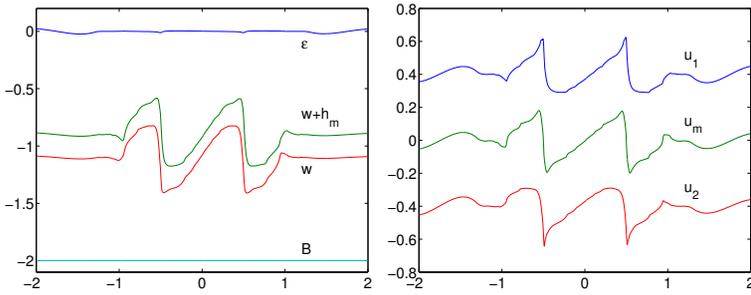


Figure 6. Example 1: Solution of the three-layer system with $\hat{h}_m = 0.2$.

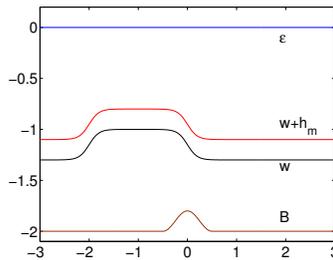


Figure 7. Example 2: The initial setting for the three-layer system with $\hat{h}_m = 0.2$.

Example 2. We now consider the three-layer shallow water system (2.2) subject to the following initial data:

$$\begin{aligned}
 h_1(x, 0) &= 1.2 + 0.15(\tanh(5x) - \tanh(5x + 10)) - 0.5\hat{h}_m, \\
 h_2(x, 0) &= -h_1(x, 0) - B(x) - \hat{h}_m, \\
 h_m(x, 0) &\equiv \hat{h}_m, \quad u_1(x, 0) \equiv \hat{u}_1, \quad u_2(x, 0) \equiv \hat{u}_2, \quad u_m(x, 0) \equiv \frac{r\hat{u}_1 + \hat{u}_2}{r + 1}
 \end{aligned}$$

with $\hat{u}_1 = 2.9$ and $\hat{u}_2 = 2.1$. We take

$$B(x) = \begin{cases} -2, & |x| \geq 0.5, \\ -1.9 + 0.1 \cos(2\pi x), & |x| < 0.5 \end{cases}$$

and implement free boundary conditions on the computational domain $[-3, 3]$. The initial setting for $h_m = 0.2$ is shown in Figure 7. Once again, we compute the numerical solution on the uniform grid with $\Delta x = 1/200$ and run the simulations until time $t = 0.5$.

As in Example 1, we first compute the numerical solution of the two-layer system ($\hat{h}_m = 0$) and the situation here is even worse. The obtained results contain severe nonphysical oscillations in both the internal wave and velocities as it can be seen in Figure 8. We then compute the numerical solutions of the corresponding three-layer systems with $\hat{h}_m = 0.05, 0.1$ and 0.2 . As it can be observed in Fig. 9, the oscillations have been largely reduced even for a

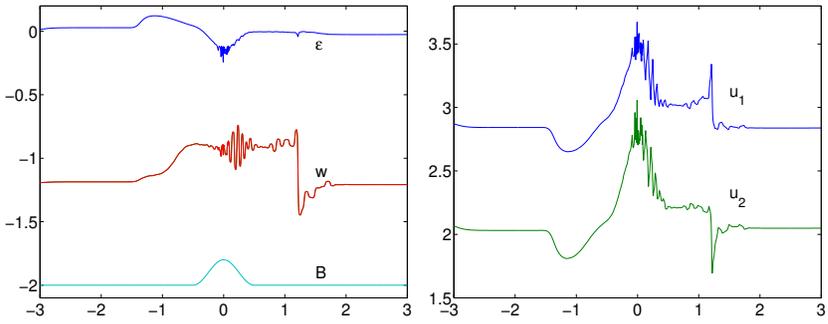


Figure 8. Example 2: Solution of the two-layer system ($\hat{h}_m = 0$).

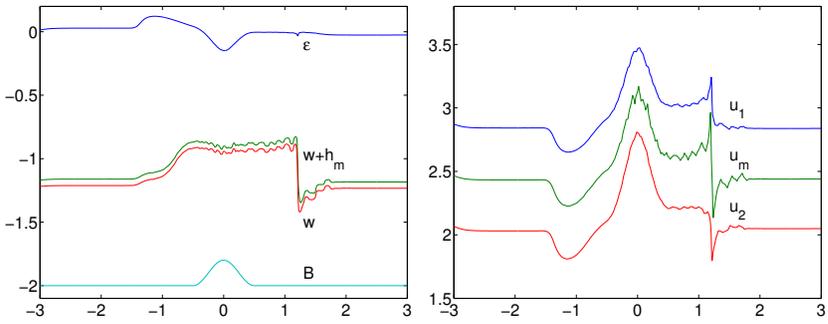


Figure 9. Example 2: Solution of the three-layer system with $\hat{h}_m = 0.05$.

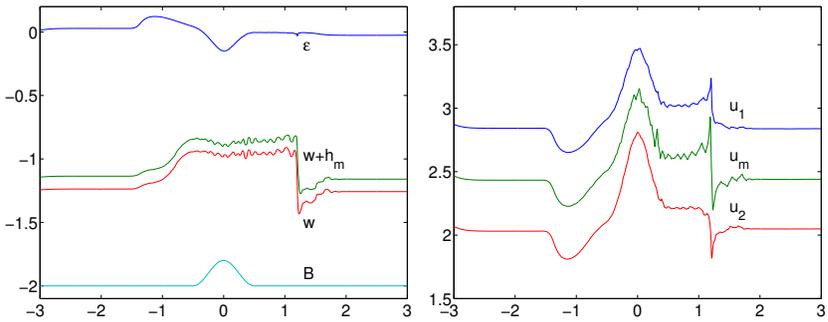


Figure 10. Example 2: Solution of the three-layer system with $\hat{h}_m = 0.1$.

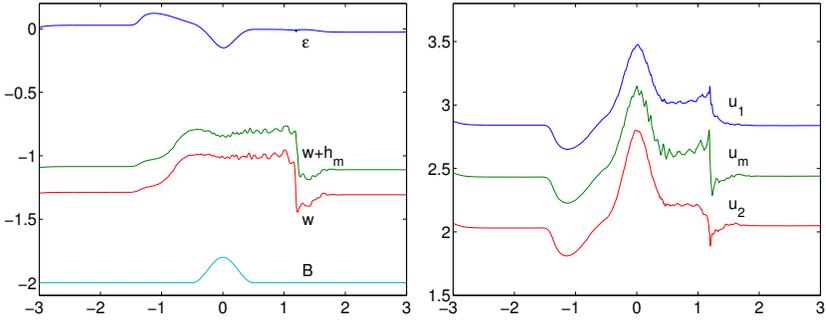


Figure 11. Example 2: Solution of the three-layer system with $\hat{h}_m = 0.2$.

small value of $\hat{h}_m = 0.05$, and as \hat{h}_m is increased (see Figures 10 and 11), the magnitude of oscillations gets smaller. Moreover, the velocity of the buffer layer (u_m) stays in the range of the upper (u_1) and lower (u_2) layer velocities. Thus, in this example, the intermediate layer helps to absorb the severe instability which is caused by the lack of hyperbolicity. We therefore conclude that the introduction of a “buffer” layer is capable of achieving its goal of improving stability properties of the two-layer shallow water system.

5 Discussion on the Hyperbolicity

As it has been mentioned above, the two-layer shallow water system (1.1) is only conditionally hyperbolic and in the nonhyperbolic regime, its solution typically develops instabilities. While in Section 4 we have demonstrated that the use of the three-layer approximation may improve stability properties of the system, in this section, we discuss and compare the hyperbolicity range of the three-layer system (2.1) and the original two-layer system (1.1).

We begin with examining the first-order approximation of the eigenvalues of the two-layer system (1.1) (see, e.g., [7, 27]), which is given by

$$\begin{aligned} \lambda_{1,2}(h_1, u_1, h_2, u_2) &\approx U_m \pm \sqrt{g(h_1 + h_2)}, \\ \lambda_{3,4}(h_1, u_1, h_2, u_2) &\approx U_c \pm \sqrt{(1-r)g \frac{h_1 h_2}{h_1 + h_2} \left(1 - \frac{(u_2 - u_1)^2}{(1-r)g(h_1 + h_2)}\right)}, \end{aligned} \tag{5.1}$$

where

$$U_m = \frac{h_1 u_1 + h_2 u_2}{h_1 + h_2}, \quad U_c = \frac{h_1 u_2 + h_2 u_1}{h_1 + h_2}.$$

From (5.1), one expects that the two-layer shallow water system (1.1) is hyperbolic as long as

$$(u_2 - u_1)^2 < (1 - r)g(h_1 + h_2), \tag{5.2}$$

and thus the hyperbolicity condition for the two-layer shallow water system (1.1) depends on the relationship between $(u_2 - u_1)^2$ and $(1 - r)g(h_1 + h_2)$. To illustrate this relation we follow [6, 13] by randomly choosing the values of h_1 ,

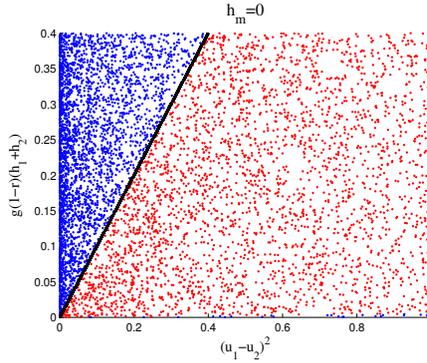


Figure 12. Hyperbolicity check for the two-layer system (1.1): The blue points represent the range of hyperbolicity.

h_2 , u_1 and u_2 and numerically checking the hyperbolicity of each data set. The results are shown in Fig. 12, where the values of $(1 - r)g(h_1 + h_2)$ are plotted against $(u_2 - u_1)^2$. In this figure, the hyperbolic region is marked using the blue points, while the nonhyperbolic one is specified using the red points. As one can see, the hyperbolic and nonhyperbolic regions are approximately separated by the straight line with the slope 1. This result is the same as the one reported in [6, 13] and it confirms the accuracy of the eigenvalue approximation used to obtain (5.2).

Next, we perform a similar hyperbolicity study of the three-layer system (2.1) using its quasilinear form (3.10) and numerically check whether all of the eigenvalues of the matrix on the LHS of (3.10) are real for the given set of data. Similarly to the two-layer case, we study the relationship between $(u_2 - u_1)^2$ and $(1 - r)gH$, where $H := h_1 + h_m + h_2$. To this end, we first randomly select the total depth $H \in [0, 2]$, parameter $\alpha \in [0.3, 0.7]$, the velocities u_1 and u_2 such that $(0 \leq |u_1 - u_2| \leq 1)$ and set $h_2 := \alpha H - h_m/2$, $h_1 = (1 - \alpha)H - h_m/2$. We then let h_m to vary from a very small value (0.01) to the larger values (0.05, 0.1 and 0.2) and up to $1.9 \min((1 - \alpha)H, \alpha H)$, which is only 5% smaller than the largest theoretically possible value of h_m . The obtained results are shown in Figures 13–15, where as before the hyperbolic regions are marked using the blue points and the nonhyperbolic ones are specified using the red points. Notice that since H cannot be less than h_m , there is a (small) horizontal white strip at the bottom of each graph in Figures 13 and 14 (larger the h_m larger the width of the white strip is).

As one can see in Figures 13 and 14, in the three-layer case (unlike the two-layer one), there is no clear dividing line between the hyperbolic and non-hyperbolic regions for small and even intermediate values of h_m . However, one can observe that the hyperbolic region increases as h_m increases. When h_m reaches $h_m = 1.9 \min((1 - \alpha)H, \alpha H)$, the hyperbolic region for the three-layer system (see Fig. 15) becomes about twice larger than the one for the corresponding two-layer system (see Figure 12). This experiment suggests that in order to significantly improve the hyperbolicity of the two-layer shallow water

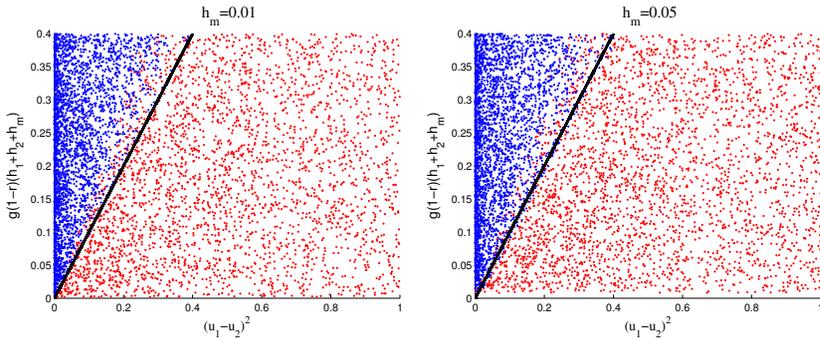


Figure 13. Hyperbolicity check for the three-layer system (2.1) with $h_m = 0.01$ (left) and $h_m = 0.05$ (right): The blue points represent the range of hyperbolicity.

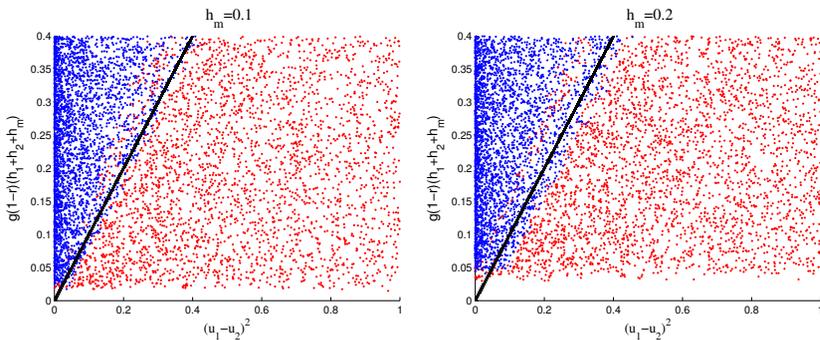


Figure 14. The same as in Fig. 13 but with $h_m = 0.1$ (left) and $h_m = 0.2$ (right).

system, we need to introduce an unrealistically deep intermediate layer (such as $1.9 \min\{(1-\alpha)H, \alpha H\}$), for which the three-layer system practically reduces to the single-layer one. Our result is consistent with the result reported in [6, 13]: It shows that to substantially gain the hyperbolicity, the intermediate layer has to be as deep as 90% of the total depth, which clearly cannot be considered as a “buffer” layer. Therefore, we arrive at the conclusion that even though adding the third layer may improve the stability properties of the two-layer shallow water system, the hyperbolicity cannot be guaranteed by a “buffer” layer approach.

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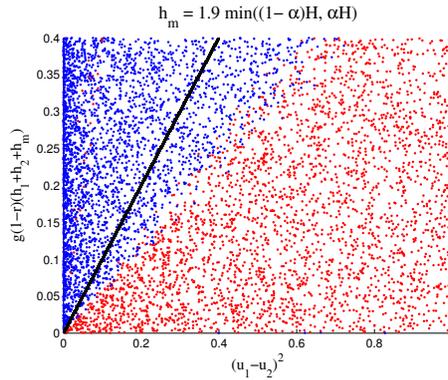


Figure 15. The same as in Figures 13 and 14 but with $h_m = 1.9 \min((1-\alpha)H, \alpha H)$.

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