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Existence Results for Impulsive Systems with Initial Nonlocal Conditions

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Abstract. We study the existence of solutions for nonlinear first order impulsive systems with nonlocal initial conditions. Our approach relies in the fixed point principles of Schauder and Perov, combined with a vector approach that uses matrices that converge to zero. We prove existence and uniqueness results for these systems. Some examples are presented to illustrate the theory.
Keywords: impulsive differential system, nonlocal initial condition, vector norm, convergent to zero matrix.

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1 Introduction

Differential equations with impulses are often used when modelling a variety of phenomena in engineering, physics and life sciences. In the field of population dynamics the impulsive terms model a sudden change in the population size, for example due to stocking or harvesting, for some recent papers in this direction see for example [1,12,14]. An introduction to the theory of impulsive differential equations can be found in the books [3, 13, 23], that contain also a variety of examples.

Here we deal with a system of first order differential equations with impulsive terms subject to nonlocal initial value conditions, namely

$$\begin{cases} x'(t) = f_1\big(t, x(t), y(t)\big), \quad y'(t) = f_2\big(t, x(t), y(t)\big), \quad t \in (0,1),\ t \neq \tau, \\ \Delta x|_{t=\tau} = I_1\big(x(\tau)\big), \quad \Delta y|_{t=\tau} = I_2\big(y(\tau)\big), \quad \tau \in (0,1), \\ x(0) = \alpha_1[x], \quad y(0) = \alpha_2[y]. \end{cases} \tag{1.1}$$

Here $\Delta v|_{t=\tau}$ denotes the "jump" of the function v in $t=\tau$, that is

$$\Delta v|_{t=\tau} = v(\tau^+) - v(\tau^-),$$

where $v(\tau^-)$, $v(\tau^+)$ are the left and the right limits of v in $t=\tau$ and α_i $(i=1,2)$ are linear functionals given by Stieltjes integrals

$$\alpha_i[v] = \int_0^{t_0} v(s)\, dA_i(s), \tag{1.2}$$

where $t_0 \in (0,\tau)$ is fixed. The nonlocal conditions (1.2) are fairly general and include, as special cases, m-point and integral conditions, when

$$\alpha_i[v] = \sum_{j=1}^{m} \alpha_{ij} v(t_{ij}) \quad \text{and} \quad \alpha_i[v] = \int_0^{t_0} \alpha_i(s) v(s)\, ds$$

with $0 \le t_{ij} \le t_0$. These are widely studied objects, see for example [2, 5, 7, 8, 11, 15, 16, 18, 19, 22, 24, 25], and references therein.

Recently Nica [17] studied the system (1.1) without impulsive terms and α_1, α_2 suitable linear bounded functionals on $C[0,1]$. The methodology in [17] is to rewrite the system as an integral system of the type

$$\begin{cases} x(t) = \dfrac{1}{1-\alpha_1[1]}\alpha_1[g_1] + g_1(x,y)(t), \\ y(t) = \dfrac{1}{1-\alpha_2[1]}\alpha_2[g_2] + g_2(x,y)(t), \end{cases} \tag{1.3}$$

where $1 \neq \alpha_i[1]$ and

$$g_i(x,y)(t) := \int_0^t f_i\big(s, x(s), y(s)\big)\, ds, \quad i=1,2,$$

and to make use of some fixed point theorems combined with matrices that converge to zero and vector-valued norms.

Our idea, similar to the one utilized in [9, 10] in the context of second-order impulsive equations, is to rewrite the system (1.1) as a system of integral equations that can be seen as a perturbation of (1.3), that is

$$\begin{cases} x(t) = \dfrac{1}{1-\alpha_1[1]}\alpha_1[g_1] + g_1(x,y)(t) + G_1(x)(t), \\ y(t) = \dfrac{1}{1-\alpha_2[1]}\alpha_2[g_2] + g_2(x,y)(t) + G_2(y)(t), \end{cases}$$

where the terms G_i take into account the impulsive effect.

Here, we benefit also of a careful decomposition similar to the one proposed in [6] and later used in [17, 18], namely to rewrite the integral operator associated to the non-impulsive terms as a sum of two operators; one of Fredholm type, whose values depend only on the restrictions to the subinterval $[0,t_0]$, and another one of Volterra type depending on the restrictions to the interval

$[t_0, 1]$. This allows us to split the growth conditions on the nonlinear terms f_1 and f_2 into two parts, one for $t \in [0, t_0]$ and the other one for $[t_0, 1]$. The corresponding conditions in the existence theorems are different in the two intervals, being more relaxed in the last interval. This is the first time that this approach is used in the context of nonlocal impulsive systems.

We present two examples that illustrate the applicability of our results; this is done in the last Section.

2 Preliminaries

We now give some notations, definitions and basic results which are used throughout this paper. We make use of the fixed point theorems of Perov and Schauder; in order to apply these theorems we require the notion of convergent to zero matrices (see for example [20, 21]).

Definition 1. A square matrix M with non-negative elements is said to be *convergent to zero* if

$$M^k \to 0 \quad \text{as } k \to \infty.$$

The next Lemma provides a characterization of matrices converging to zero (see [4, pp. 9, 10], [20, 21]).

Lemma 1. *Let M be a square matrix of nonnegative numbers. The following statements are equivalent:*

(i) *M is a matrix that is convergent to zero;*

(ii) *$I - M$ is nonsingular and $(I-M)^{-1} = I + M + M^2 + \cdots$ (where I stands for the unit matrix of the same order as M);*

(iii) *the eigenvalues of M are located inside the unit disc of the complex plane;*

(iv) *$I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

The following lemma is a consequence of the previous characterizations.

Lemma 2. *Let A be a matrix that is convergent to zero. Then for each matrix B of the same order whose elements are nonnegative and sufficiently small, matrix $A + B$ is also convergent to zero.*

Definition 2. By a *vector-valued metric* on a set X we mean a mapping $d : X \times X \to \mathbb{R}^n_+$ such that

(i) $d(u, v) \geq 0$ for all $u, v \in X$ and if $d(u, v) = 0$ then $u = v$;

(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$;

where, for $x, y \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, 2, \ldots, n$. We call the pair (X, d) a *generalized metric space*. For such spaces convergence and completeness are similar to those in usual metric spaces.

An operator $T : X \to X$ is said to be *contractive* with respect to a vector-valued metric d on X, if there exists a matrix M, called *Lipschitz matrix*, that is convergent to zero such that

$$d\big(T(u), T(v)\big) \leq Md(u, v), \quad \text{for every } u, v \in X.$$

The following theorem, can be found for example in [20, Theorem 10.1].

Theorem 1 [Perov]. *Let (X, d) be a complete generalized metric space and $T : X \to X$ a contractive operator with Lipschitz matrix M. Then T has a unique fixed point u^* and for each $u_0 \in X$ we have*

$$d\big(T^k(u_0), u^*\big) \leq M^k(I - M)^{-1} d\big(u_0, T(u_0)\big), \quad \textit{for every } k \in \mathbb{N}.$$

Theorem 2 [Schauder]. *Let X be a Banach space, $B \subset X$ a nonempty closed bounded convex set and $T : B \to B$ a completely continuous operator (i.e., T is continuous and $T(B)$ is relatively compact). Then T has at least one fixed point.*

We recall some earlier results of [17] valid for the non-impulsive problem

$$\begin{cases} x'(t) = f_1\big(t, x(t), y(t)\big), \quad y'(t) = f_2\big(t, x(t), y(t)\big), \quad t \in (0, 1), \\ x(0) = \alpha_1[x], \quad y(0) = \alpha_2[y]. \end{cases} \tag{2.1}$$

The approach in [17] is to rewrite the problem (2.1) as an integral system of the type (1.3). The solutions of the system (1.3) are sought as fixed points for the operator

$$T_B(x, y)(t) = \begin{pmatrix} T_{B_1}(x, y)(t) \\ T_{B_2}(x, y)(t) \end{pmatrix} := \begin{pmatrix} \frac{1}{1-\alpha_1[1]}\alpha_1[g_1] + g_1(x, y)(t) \\ \frac{1}{1-\alpha_2[1]}\alpha_2[g_2] + g_2(x, y)(t) \end{pmatrix},$$

and the operator T_B is decomposed as a sum of two operators, one of Fredholm type and another one of Volterra type, namely

$$T_B = T_F + T_V, \tag{2.2}$$

where

$$T_F(x, y)(t) = \begin{pmatrix} T_{F_1}(x, y)(t) \\ T_{F_2}(x, y)(t) \end{pmatrix}, \quad T_V(x, y)(t) = \begin{pmatrix} T_{V_1}(x, y)(t) \\ T_{V_2}(x, y)(t) \end{pmatrix},$$

with for $i = 1, 2$

$$T_{F_i}(x, y)(t) = \begin{cases} \frac{1}{1-\alpha_i[1]}\alpha_i[g_i] + \int_0^t f_i(s, x(s), y(s))\, ds, & \text{if } t < t_0, \\ \frac{1}{1-\alpha_i[1]}\alpha_i[g_i] + \int_0^{t_0} f_i(s, x(s), y(s))\, ds, & \text{if } t \geq t_0 \end{cases}$$

and

$$T_{V_i}(x, y)(t) = \begin{cases} 0, & \text{if } t < t_0, \\ \int_{t_0}^t f_i(s, x(s), y(s))\, ds, & \text{if } t \geq t_0. \end{cases}$$

A key assumption utilized in the paper [17] is that the matrix

$$M := t_0 \begin{bmatrix} a_1(\frac{\|\alpha_1\|}{|1-\alpha_1[1]|}+1) & b_1(\frac{\|\alpha_1\|}{|1-\alpha_1[1]|}+1) \\ A_1(\frac{\|\alpha_2\|}{|1-\alpha_2[1]|}+1) & B_1(\frac{\|\alpha_2\|}{|1-\alpha_2[1]|}+1) \end{bmatrix}$$

is converging to zero. This is used in order to apply the theorems of Schauder and Perov for the existence of at least one and for the existence of a unique solution. The matrix M can be written as

$$M = M_N + M_V,$$

where

$$M_N := t_0 \begin{bmatrix} a_1 \frac{\|\alpha_1\|}{|1-\alpha_1[1]|} & b_1 \frac{\|\alpha_1\|}{|1-\alpha_1[1]|} \\ A_1 \frac{\|\alpha_2\|}{|1-\alpha_2[1]|} & B_1 \frac{\|\alpha_2\|}{|1-\alpha_2[1]|} \end{bmatrix} \quad \text{and} \quad M_V := t_0 \begin{bmatrix} a_1 & b_1 \\ A_1 & B_1 \end{bmatrix}.$$

Note that the matrix M_N takes into account the nonlocal conditions. The nonnegative coefficients a_i, b_i, A_i, B_i are provided by the Lipschitz conditions given by the nonlinearities, namely

$$\left|f_1(t,x,y) - f_1(t,\overline{x},\overline{y})\right| \le \begin{cases} a_1|x-\overline{x}| + b_1|y-\overline{y}|, & \text{if } t \in [0,t_0]\,, \\ a_2|x-\overline{x}| + b_2|y-\overline{y}|, & \text{if } t \in [t_0,1]\,, \end{cases}$$

$$\left|f_2(t,x,y) - f_2(t,\overline{x},\overline{y})\right| \le \begin{cases} A_1|x-\overline{x}| + B_1|y-\overline{y}|, & \text{if } t \in [0,t_0]\,, \\ A_2|x-\overline{x}| + B_2|y-\overline{y}|, & \text{if } t \in [t_0,1] \end{cases}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}$.

3 An existence result

We now consider the system

$$\begin{cases} x'(t) = f_1\big(t, x(t), y(t)\big), \quad y'(t) = f_2\big(t, x(t), y(t)\big), \quad t \in (0,1),\ t \ne \tau, \\ \Delta x|_{t=\tau} = I_1\big(x(\tau)\big), \qquad \Delta y|_{t=\tau} = I_2\big(y(\tau)\big), \quad \tau \in (0,1), \\ x(0) = \alpha_1[x], \quad y(0) = \alpha_2[y]. \end{cases} \tag{3.1}$$

Throughout the paper we assume the following:

(H1) For $i = 1,2$, $f_i : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is such that $f_i(.,x,y)$ is measurable for each $(x,y) \in \mathbb{R}^2$ and $f_i(t,.,.)$ is continuous for almost all $t \in [0,1]$, and for each $r > 0$ there exists $\phi_{i,r} \in L^1(0,1)$ such that

$$|f_i(t,u,v)| \le \phi_{i,r}(t) \ \text{ for } \ u,v \in [-r,r] \ \text{ and a.e. } t \in [0,1].$$

(H2) For $i = 1,2$, the function A_i is of bounded variation on $[0,t_0]$ with

$$\alpha_i[1] = \int_0^{t_0} 1\, dA_i(s) \ne 1.$$

(H3) For $i = 1, 2$, the function $I_i : \mathbb{R} \to \mathbb{R}$ is continuous.

We work in the Banach space $PC_\tau[0,1] \times PC_\tau[0,1]$, where

$$PC_\tau[0,1] := \{u : [0,1] \to \mathbb{R}|,\ u \text{ is continuous in } t \in [0,1]\backslash\{\tau\}, \\ \text{there exist } u(\tau^-) = u(\tau) \text{ and } \left|u(\tau^+)\right| < \infty\}.$$

The classical Ascoli–Arzelà compactness criterion cannot be applied directly to the space $PC_\tau[0,1]$, here we make use of the following extension of this criterion, see for example [13].

We recall that a set $S \subset PC_\tau[0,1]$ is said to be *quasi-equicontinuous* if for every $u \in S$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in [0,\tau]$ (or $t_1, t_2 \in (\tau, 1]$) and $|t_1 - t_2| < \delta$ implies $|u(t_1) - u(t_2)| < \varepsilon$.

Lemma 1. *A set $S \subseteq PC_\tau[0,1]$ is relatively compact in $PC_\tau[0,1]$ if and only if S is bounded and quasi-equicontinuous.*

We use in $PC_\tau[0,1] \times PC_\tau[0,1]$ the vector norm

$$\left\|(x,y)\right\|_{PC_\tau[0,1]\times PC_\tau[0,1]} := \left(\|x\|, \|y\|\right),$$

where

$$\|v\| := \max\left\{|v|_{[0,t_0]}, \|v\|_{[t_0,1]}\right\}$$

and the notation $|v|_{[0,t_0]}$ stands for the sup-norm on $[0, t_0]$:

$$|v|_{[0,t_0]} = \sup\nolimits_{t\in[0,t_0]}\left|v(t)\right|,$$

while $\|v\|_{[t_0,1]}$ denotes a Bielecki-type norm on $[t_0, 1]$:

$$\|v\|_{[t_0,1]} = \sup\nolimits_{t\in[t_0,1]}\left|v(t)\right|e^{-\theta(t-t_0)}$$

for some suitable $\theta > 0$.

The norm of the functional $\alpha_i : PC_\tau[0,1] \to \mathbb{R}$, is given by

$$\|\alpha_i\| = \sup_{\|v\|=1}\left|\int_0^{t_0} v(s)\, dA_i(s)\right|.$$

Our idea is to seek a solution of the problem (3.1) as a fixed point of a perturbation of the operator (2.2), namely

$$T = T_F + T_V + G, \tag{3.2}$$

where

$$G(x,y)(t) = \begin{pmatrix} G_1(x)(t) \\ G_2(y)(t) \end{pmatrix} \quad \text{and} \quad G_i(v)(t) = \begin{cases} 0, & \text{if } t \le \tau, \\ I_i(v(\tau)), & \text{if } t > \tau. \end{cases}$$

We now show that the existence of solutions for the problem (3.1) follows from Schauder's fixed point theorem when f_1, f_2 satisfy some growth conditions of the type: there exists nonnegative coefficients a_i, b_i, c_i, A_i, B_i, C_i such that

$$\left|f_1(t,x,y)\right| \le \begin{cases} a_1|x| + b_1|y| + c_1, & \text{if } t \in [0,t_0]\,, \\ a_2|x| + b_2|y| + c_2, & \text{if } t \in [t_0,1]\,, \end{cases} \tag{3.3}$$

$$\left|f_2(t,x,y)\right| \le \begin{cases} A_1|x| + B_1|y| + C_1, & \text{if } t \in [0,t_0]\,, \\ A_2|x| + B_2|y| + C_2, & \text{if } t \in [t_0,1] \end{cases} \tag{3.4}$$

for all $x, y \in \mathbb{R}$.

We also assume that there exist $d_i, e_i \in [0,\infty)$ such that for every $v \in \mathbb{R}$ we have

$$\left|I_i(v)\right| \le d_i|v| + e_i, \quad \text{for } i = 1,2. \tag{3.5}$$

In what follows we denote by

$$M_I := \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \quad \text{and} \quad A_{\alpha_i} := \frac{1}{|1-\alpha_i[1]|}\|\alpha_i\| + 1, \quad i = 1,2.$$

The matrix M_I is essential for our arguments as it takes care of the impulsive effect. This enables us to bridge the methodology employed in [17] to the context of impulsive problems.

Theorem 3. *If the conditions* (3.3), (3.4), (3.5) *are satisfied and the matrix*

$$M_0 := M + M_I \tag{3.6}$$

converges to zero, then the problem (3.1) *has at least one solution.*

Proof. In order to apply the Schauder fixed point theorem, we look for a nonempty, bounded, closed and convex subset B of $PC_\tau[0,1] \times PC_\tau[0,1]$ so that $T(B) \subset B$. Let x, y be any elements of $PC_\tau\,[0,1]\,.$
For $t \in [0,t_0]\,,$ following the proof of Theorem 3.1 of [17], we obtain that

$$\begin{aligned} \left|T_1(x,y)\right|_{[0,t_0]} &\le \left(\frac{\|\alpha_1\|}{|1-\alpha_1[1]|} + 1\right)\left(a_1t_0|x|_{[0,t_0]} + b_1t_0|y|_{[0,t_0]}\right) + c_1t_0A_{\alpha_1} \\ &= a_1t_0A_{\alpha_1}|x|_{[0,t_0]} + b_1t_0A_{\alpha_1}|y|_{[0,t_0]} + c_1t_0A_{\alpha_1}. \end{aligned} \tag{3.7}$$

For $t \in [t_0,1]$ and any $\theta > 0$, we have

$$\begin{aligned} \left|T_1(x,y)(t)\right| &\le a_1t_0A_{\alpha_1}|x|_{[0,t_0]} + b_1t_0A_{\alpha_1}|y|_{[0,t_0]} + c_1t_0A_{\alpha_1} \\ &\quad + d_1\left|x(\tau)\right| + e_1 + \int_{t_0}^{t}\left(a_2\left|x(s)\right| + b_2\left|y(s)\right| + c_2\right)ds \\ &\le a_1t_0A_{\alpha_1}|x|_{[0,t_0]} + b_1t_0A_{\alpha_1}|y|_{[0,t_0]} + c_1t_0A_{\alpha_1} + (1-t_0)c_2 + e_1 \\ &\quad + d_1\left|x(\tau)\right|e^{-\theta(t-t_0)}e^{\theta(t-t_0)} + a_2\int_{t_0}^{t}\left|x(s)\right|e^{-\theta(s-t_0)}e^{\theta(s-t_0)}ds \\ &\quad + b_2\int_{t_0}^{t}\left|y(s)\right|e^{-\theta(s-t_0)}e^{\theta(s-t_0)}ds \\ &\le a_1t_0A_{\alpha_1}|x|_{[0,t_0]} + b_1t_0A_{\alpha_1}|y|_{[0,t_0]} + c_0 \\ &\quad + d_1e^{\theta(t-t_0)}\|x\|_{[t_0,1]} + \frac{a_2}{\theta}e^{\theta(t-t_0)}\|x\|_{[t_0,1]} + \frac{b_2}{\theta}e^{\theta(t-t_0)}\|y\|_{[t_0,1]}, \end{aligned}$$

where $c_0 := c_1 t_0 A_{\alpha_1} + (1-t_0)c_2 + e_1$. Dividing by $e^{\theta(t-t_0)}$ and taking the supremum, it follows that

$$\begin{aligned}\big\|T_1(x,y)\big\|_{[t_0,1]} \le{}& a_1 t_0 A_{\alpha_1} |x|_{[0,t_0]} + b_1 t_0 A_{\alpha_1} |y|_{[0,t_0]} \\ &+ \left(\frac{a_2}{\theta} + d_1\right)\|x\|_{[t_0,1]} + \frac{b_2}{\theta}\|y\|_{[t_0,1]} + c_0.\end{aligned} \tag{3.8}$$

Clearly (3.7), (3.8) give

$$\big\|T_1(x,y)\big\| \le \left(a_1 t_0 A_{\alpha_1} + d_1 + \frac{a_2}{\theta}\right)\|x\| + \left(b_1 t_0 A_{\alpha_1} + \frac{b_2}{\theta}\right)\|y\| + c_0. \tag{3.9}$$

Similarly

$$\big\|T_2(x,y)\big\| \le \left(A_1 t_0 A_{\alpha_2} + \frac{A_2}{\theta}\right)\|x\| + \left(B_1 t_0 A_{\alpha_2} + d_2 + \frac{B_2}{\theta}\right)\|y\| + C_0 \tag{3.10}$$

with $C_0 := C_1 t_0 A_{\alpha_2} + (1-t_0)C_2 + e_2$.

Now (3.9), (3.10) can be put together as

$$\begin{bmatrix}\|T_1(x,y)\| \\ \|T_2(x,y)\|\end{bmatrix} \le M_\theta \begin{bmatrix}\|x\| \\ \|y\|\end{bmatrix} + \begin{bmatrix}c_0 \\ C_0\end{bmatrix},$$

where the matrix M_θ is given by

$$M_\theta = \begin{bmatrix} A_{\alpha_1} a_1 t_0 + d_1 + \frac{a_2}{\theta} & A_{\alpha_1} b_1 t_0 + \frac{b_2}{\theta} \\ A_{\alpha_2} A_1 t_0 + \frac{A_2}{\theta} & A_{\alpha_2} B_1 t_0 + d_2 + \frac{B_2}{\theta}\end{bmatrix}. \tag{3.11}$$

Clearly the matrix M_θ can be represented as $M_\theta = M_0 + M_1$, where

$$M_1 = \begin{bmatrix}\frac{a_2}{\theta} & \frac{b_2}{\theta} \\ \frac{A_2}{\theta} & \frac{B_2}{\theta}\end{bmatrix}.$$

Since M_0 is assumed to be convergent to zero, from Lemma 1.2 we have that M_θ also converges to zero for large enough $\theta > 0$. Next we look for two positive numbers R_1, R_2, such that if $\|x\| \le R_1$, $\|y\| \le R_2$, then $\|T_1(x,y)\| \le R_1$, $\|T_2(x,y)\| \le R_2$. To this end it is sufficient that

$$\begin{cases} \left(a_1 t_0 A_{\alpha_1} + d_1 + \frac{a_2}{\theta}\right) R_1 + \left(b_1 t_0 A_{\alpha_1} + \frac{b_2}{\theta}\right) R_2 + c_0 \le R_1, \\ \left(A_1 t_0 A_{\alpha_2} + \frac{A_2}{\theta}\right) R_1 + \left(B_1 t_0 A_{\alpha_2} + d_2 + \frac{B_2}{\theta}\right) R_2 + C_0 \le R_2 \end{cases}$$

or equivalently

$$M_\theta \begin{bmatrix}R_1 \\ R_2\end{bmatrix} + \begin{bmatrix}c_0 \\ C_0\end{bmatrix} \le \begin{bmatrix}R_1 \\ R_2\end{bmatrix},$$

and therefore

$$\begin{bmatrix}R_1 \\ R_2\end{bmatrix} \ge (I - M_\theta)^{-1} \begin{bmatrix}c_0 \\ C_0\end{bmatrix}.$$

Notice that $I - M_\theta$ is invertible and its inverse $(I - M_\theta)^{-1}$ has nonnegative elements since M_θ converges to zero. Thus, if

$$B = \{(x,y) \in PC_\tau[0,1] \times PC_\tau[0,1] \colon \|x\| \leq R_1,\ \|y\| \leq R_2\},$$

then $T(B) \subset B$. The fact that T is completely continuous follows by Lemma 1, combined with the useful decomposition (3.2), in a similar way as in the proof of the Theorem 3.3 of [10].

The result now follows from Schauder's fixed point theorem. $\square$

Remark 1. From the proof of the Theorem 3 it follows that there exists $\bar{\theta}$ such that the obtained fixed point (x,y) of the operator T satisfies the relation

$$\|x\| = \max\{|x|_{[0,t_0]}, \|x\|_{[t_0,1]}\} \leq R_1, \quad \|y\| \quad = \max\{|y|_{[0,t_0]}, \|y\|_{[t_0,1]}\} \leq R_2.$$

This implies that, in the interval $[0,t_0]$, we have that

$$|x(t)| \leq R_1, \quad |y(t)| \leq R_2$$

and in $[t_0,1]$, we have that

$$|x(t)| \leq R_1 e^{\bar{\theta}(t-t_0)}, \quad |y(t)| \leq R_2 e^{\bar{\theta}(t-t_0)}. \tag{3.12}$$

We note that a choice of $\theta > \bar{\theta}$ provides a worse estimate in (3.12).

4 An existence and uniqueness result

Here, by means of the fixed point theorem of Perov, we prove an existence and uniqueness result, provided that f_1, f_2 satisfy the Lipschitz conditions

$$\left|f_1(t,x,y) - f_1(t,\overline{x},\overline{y})\right| \leq \begin{cases} a_1|x-\overline{x}| + b_1|y-\overline{y}|, & \text{if } t \in [0,t_0]\,, \\ a_2|x-\overline{x}| + b_2|y-\overline{y}|, & \text{if } t \in [t_0,1]\,, \end{cases} \tag{4.1}$$

$$\left|f_2(t,x,y) - f_2(t,\overline{x},\overline{y})\right| \leq \begin{cases} A_1|x-\overline{x}| + B_1|y-\overline{y}|, & \text{if } t \in [0,t_0]\,, \\ A_2|x-\overline{x}| + B_2|y-\overline{y}|, & \text{if } t \in [t_0,1] \end{cases} \tag{4.2}$$

and also

$$\left|I_i(v) - I_i(\overline{v})\right| \leq d_i|v-\overline{v}| \quad \text{for } i = 1,2, \tag{4.3}$$

for all $x,y,\overline{x},\overline{y},v,\overline{v} \in \mathbb{R}$.

Theorem 4. *If the conditions* (4.1), (4.2), (4.3) *and the matrix* (3.6) *converges to zero, then the problem* (3.1) *has a unique solution.*

Proof. We have to prove that T is contractive, that is

$$\left\|T(u) - T(\overline{u})\right\|_{PC_\tau[0,1]\times PC_\tau[0,1]} \leq M_\theta \|u-\overline{u}\|_{PC_\tau[0,1]\times PC_\tau[0,1]}$$

for all $u,\overline{u} \in PC_\tau[0,1] \times PC_\tau[0,1]$ and some matrix M_θ converging to zero. To this end, let $u = (x,y), \overline{u} = (\overline{x},\overline{y})$ be any elements of $PC_\tau[0,1] \times PC_\tau[0,1]$.

For $t \in [0,t_0]$, following the proof of Theorem 2.1 of [17], we have

$$\begin{aligned}
&\left|T_1(x,y)-T_1(\overline{x},\overline{y})\right|_{[0,t_0]}\\
&\quad\le\left(\frac{\|\alpha_1\|}{|1-\alpha_1[1]|}+1\right)\left(a_1t_0|x-\overline{x}|_{[0,t_0]}+b_1t_0|y-\overline{y}|_{[0,t_0]}\right)\\
&\quad=A_{\alpha_1}a_1t_0|x-\overline{x}|_{[0,t_0]}+A_{\alpha_1}b_1t_0|y-\overline{y}|_{[0,t_0]}.
\end{aligned}\tag{4.4}$$

For $t \in [t_0,1]$ and any $\theta>0$, we have

$$\begin{aligned}
&\left|T_1(x,y)(t)-T_1(\overline{x},\overline{y})(t)\right|\le A_{\alpha_1}a_1t_0|x-\overline{x}|_{[0,t_0]}+A_{\alpha_1}b_1t_0|y-\overline{y}|_{[0,t_0]}\\
&\qquad+\left|I_1(x)(\tau)-I_1(\overline{x})(\tau)\right|+\int_{t_0}^{t}\left|f_1\big(s,x(s),y(s)\big)-f_1\big(s,\overline{x}(s),\overline{y}(s)\big)\right|ds\\
&\quad\le A_{\alpha_1}a_1t_0|x-\overline{x}|_{[0,t_0]}+A_{\alpha_1}b_1t_0|y-\overline{y}|_{[0,t_0]}\\
&\qquad+d_1\left|x(\tau)-\overline{x}(\tau)\right|+\int_{t_0}^{t}\left(a_2\left|x(s)-\overline{x}(s)\right|+b_2\left|y(s)-\overline{y}(s)\right|\right)ds\\
&\quad=A_{\alpha_1}a_1t_0|x-\overline{x}|_{[0,t_0]}+A_{\alpha_1}b_1t_0|y-\overline{y}|_{[0,t_0]}+d_1\left|x(\tau)-\overline{x}(\tau)\right|e^{-\theta(t-t_0)}e^{\theta(t-t_0)}\\
&\qquad+a_2\int_{t_0}^{t}\left|x(s)-\overline{x}(s)\right|\cdot e^{-\theta(s-t_0)}\cdot e^{\theta(s-t_0)}ds\\
&\qquad+b_2\int_{t_0}^{t}\left|y(s)-\overline{y}(s)\right|\cdot e^{-\theta(s-t_0)}\cdot e^{\theta(s-t_0)}ds\\
&\quad\le A_{\alpha_1}a_1t_0|x-\overline{x}|_{[0,t_0]}+A_{\alpha_1}b_1t_0|y-\overline{y}|_{[0,t_0]}\\
&\qquad+d_1e^{\theta(t-t_0)}\|x-\overline{x}\|_{[t_0,1]}+\frac{a_2}{\theta}e^{\theta(t-t_0)}\|x-\overline{x}\|_{[t_0,1]}+\frac{b_2}{\theta}e^{\theta(t-t_0)}\|y-\overline{y}\|_{[t_0,1]}.
\end{aligned}$$

Dividing by $e^{\theta(t-t_0)}$ and taking the supremum when $t\in[t_0,1]$, we obtain

$$\begin{aligned}
\left\|T_1(x,y)-T_1(\overline{x},\overline{y})\right\|_{[t_0,1]}&\le A_{\alpha_1}a_1t_0|x-\overline{x}|_{[0,t_0]}+A_{\alpha_1}b_1t_0|y-\overline{y}|_{[0,t_0]}\\
&\quad+\left(\frac{a_2}{\theta}+d_1\right)\|x-\overline{x}\|_{[t_0,1]}+\frac{b_2}{\theta}\|y-\overline{y}\|_{[t_0,1]}.
\end{aligned}\tag{4.5}$$

Now (4.4) and (4.5) imply that

$$\left\|T_1(x,y)-T_1(\overline{x},\overline{y})\right\|\le\left(A_{\alpha_1}a_1t_0+d_1+\frac{a_2}{\theta}\right)\|x-\overline{x}\|+\left(A_{\alpha_1}b_1t_0+\frac{b_2}{\theta}\right)\|y-\overline{y}\|.\tag{4.6}$$

Similarly we have

$$\left\|T_2(x,y)-T_2(\overline{x},\overline{y})\right\|\le\left(A_{\alpha_2}A_1t_0+\frac{A_2}{\theta}\right)\|x-\overline{x}\|+\left(A_{\alpha_2}B_1t_0+d_2+\frac{B_2}{\theta}\right)\|y-\overline{y}\|.\tag{4.7}$$

Using the vector norm we can put the inequalities (4.6), (4.7) in the form

$$\left\|T(u)-T(\overline{u})\right\|_{PC_\tau[0,1]\times PC_\tau[0,1]}\le M_\theta\|u-\overline{u}\|_{PC_\tau[0,1]\times PC_\tau[0,1]},$$

where M_θ is given by (3.11) and converges to zero for large enough $\theta>0$.

The result follows now from Perov's fixed point theorem. $\square$

5 Numerical examples

In what follows, we give some numerical examples to illustrate our theory.

Example 1. Consider the initial value problem

$$\begin{cases} x' = \dfrac{1}{4} x \sin\left(\dfrac{y}{x}\right) + \dfrac{1}{3} y \sin\left(\dfrac{x}{y}\right) + h_1(t) \equiv f_1(t,x,y), \\ y' = \dfrac{1}{3} x \sin\left(\dfrac{y}{x}\right) + \dfrac{1}{6} y \sin\left(\dfrac{x}{y}\right) + h_2(t) \equiv f_2(t,x,y), \\ \Delta x|_{t=\frac{3}{4}} = \dfrac{1}{3} \sin\left(x\left(\dfrac{3}{4}\right)\right), \quad \Delta y|_{t=\frac{3}{4}} = \dfrac{1}{4} \cos\left(y\left(\dfrac{3}{4}\right)\right), \\ x(0) = \dfrac{1}{2} \displaystyle\int_0^{\frac{1}{2}} x(s)\,ds, \quad y(0) = \dfrac{1}{2} \displaystyle\int_0^{\frac{1}{2}} y\,(s)\;ds, \end{cases} \qquad t \in [0,1], \quad (5.1)$$

where $h_1, h_2 \in C[0,1]$. We have that $\alpha_1\,[1] = \alpha_2\,[1] = \|\alpha_1\| = \|\alpha_2\| = \frac{1}{4}$. Since

$$|f_1\,(t,x,y)| \le \frac{1}{4}\,|x| + \frac{1}{3}\,|y| + |h_1(t)|\,, \quad |f_2\,(t,x,y)| \le \frac{1}{3}\,|x| + \frac{1}{6}\,|y| + |h_2(t)|$$

and $d_1 = \frac{1}{3},\ d_2 = \frac{1}{4}$, we obtain

$$M_0 = \frac{1}{36}\begin{pmatrix} 18 & 8 \\ 8 & 13 \end{pmatrix},$$

which is convergent to zero because its eigenvalues (rounded to the third decimal place) are $\lambda_1 = 0.198 < 1$, $\lambda_2 = 0.663 < 1$. From Theorem 3, the problem (5.1) has at least one solution.

Example 2. We present a modified version of Example 2.2 in [22] that takes into account systems and impulsive effects. Consider the initial value problem

$$\begin{cases} x' = \dfrac{1}{2} y\left[1 + e^{-\frac{4}{5}(x-1)}\right]^{-1} \equiv f_1(x,y), \\ y' = \dfrac{1}{10} x\left[1 + e^{-\frac{2}{5}(y-1)}\right]^{-1} \equiv f_2(x,y), \\ \Delta x|_{t=\frac{3}{4}} = \dfrac{1}{3} \cos\left(x\left(\dfrac{3}{4}\right)\right), \quad \Delta y|_{t=\frac{3}{4}} = \dfrac{1}{5} \sin\left(y\left(\dfrac{3}{4}\right)\right), \\ x(0) = \dfrac{1}{2} \displaystyle\int_0^{\frac{1}{2}} x(s)\,ds, \quad y(0) = \dfrac{1}{2} \displaystyle\int_0^{\frac{1}{2}} y(s)\,ds. \end{cases} \qquad t \in [0,1], \quad (5.2)$$

Here we have that $\alpha_1\,[1] = \alpha_2\,[1] = \|\alpha_1\| = \|\alpha_2\| = \frac{1}{4}$. Furthermore we have

$$\sup_{x,y\in\mathbb{R}} \left|\frac{\partial f_1\,(x,y)}{\partial x}\right| \le \frac{1}{10} = a_1, \qquad \sup_{x,y\in\mathbb{R}} \left|\frac{\partial f_1\,(x,y)}{\partial y}\right| \le \frac{1}{2} = b_1,$$

$$\sup_{x,y\in\mathbb{R}} \left|\frac{\partial f_2\,(x,y)}{\partial x}\right| \le \frac{1}{10} = A_1, \qquad \sup_{x,y\in\mathbb{R}} \left|\frac{\partial f_2\,(x,y)}{\partial y}\right| \le \frac{1}{10} = B_1$$

and $d_1 = \frac{1}{3}$, $d_2 = \frac{1}{5}$. Therefore the matrix $M_0 = \frac{1}{15}\begin{pmatrix} 6 & 5 \\ 1 & 4 \end{pmatrix}$ converges to zero since its eigenvalues are $\lambda_1 = 0.17 < 1$, $\lambda_2 = 0.5 < 1$. From Theorem 4, the problem (5.2) has a unique solution.

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