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Uniformly-Convergent Numerical Methods for a System of Coupled Singularly Perturbed Convection–Diffusion Equations with Mixed Type Boundary Conditions*

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Abstract. In this paper, two hybrid difference schemes on the Shishkin mesh are constructed for solving a weakly coupled system of two singularly perturbed convection - diffusion second order ordinary differential equations subject to the mixed type boundary conditions. We prove that the method has almost second order convergence in the supremum norm independent of the diffusion parameter. Error bounds for the numerical solution and also the numerical derivative are established. Numerical results are provided to illustrate the theoretical results.

Keywords: singular perturbation problems, weakly coupled system, piecewise uniform meshes, scaled derivative, finite difference scheme, mid-point scheme, cubic spline scheme.

AMS Subject Classification: 65L10; CR G1.7.

1 Introduction

Singular perturbation problems may arise from viscous flow, edge effects in certain shell problems and the concentration or thermal layers in mass and heat transfer problems. Singularly perturbed Initial Value Problems (IVPs)/ Boundary Value Problems (BVPs) in Ordinary Differential Equations (ODEs) are characterized by the presence of a small parameter ($0 < \varepsilon \ll 1$) that multiplies the highest derivative term. Solution of such problems exhibits sharp boundary and/or interior layers when the small parameter ε is much smaller

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than 1. The numerical solution of such problems exhibits significant difficulties, particularly when the diffusion coefficient is small. Therefore, the interest in developing and analyzing efficient numerical methods for singularly perturbed problems has increased enormously (see [3,5,6,12,17] and the references therein). Most of this work has concentrated on problems involving a single differential equation. Only a few authors have developed robust parameter-uniform numerical methods for system of singularly perturbed ordinary differential equations (see [2,4,8,9,10,11,15,16,19] and references therein). While many finite difference methods have been proposed to approximate such solutions, there has been much less research into the finite difference approximations of their derivatives, even though such approximations are desirable in certain applications (flux or drag). As far as author's knowledge goes only few works are reported in the literature (see [7,13,14] and references therein) for finding approximations to scaled derivatives of the solution for problems involving singularly perturbed second order ordinary differential equations with smooth/non-smooth data.

For a singularly perturbed convection–diffusion problem, M. Stynes and H.-G. Roos [18] have established that a numerical method composed of the central difference operator in the layer region $(1-\tau, 1)$ combined with the midpoint scheme outside the layer $(0, 1-\tau]$ on the Shishkin mesh with $\tau = \min\{0.5, \frac{2\varepsilon}{\alpha}\ln N\}$ is a monotone numerical method and when $\tau < 0.5$, it satisfies a parameter-uniform error bound of the form

$$\|Y - y\| \leq \begin{cases} CN^{-1}(\varepsilon + N^{-1}), & \text{if } x_i \in [0, 1-\tau], \\ C(N^{-1}\ln N)^2, & \text{if } x_i \in (1-\tau, 1]. \end{cases}$$

In [16], the authors have analyzed a robust computational method that uses a cubic spline scheme in the fine mesh region and a classical central difference scheme in the coarse mesh region for singularly perturbed coupled system of reaction-diffusion boundary value problems. In [6], for singularly perturbed convection–diffusion problems with a continuous convection coefficient and source term for a single differential equation estimates for numerical derivatives have been derived. Here the scaled derivative is taken on the whole domain whereas Natalia Kopteva and Martin Stynes [7] have obtained approximations of derivatives with scaling in the boundary layer region and without scaling in the outer region. It may be noted that the source term and convection coefficient are smooth for the problem considered in [7]. Mythili Priyadharshini et al. [14], have determined an estimate for the scaled derivative in the boundary layer region and non-scaled derivative in the outer region for the system of singularly perturbed convection–diffusion equations with Dirichlet boundary conditions. In [13], the authors have estimated the scaled derivative for a singularly perturbed second order ordinary differential equation with a discontinuous convection coefficient using a hybrid difference scheme.

Motivated by the above works, in this paper two hybrid difference schemes are proposed to approximate the solution and its scaled first derivative of a weakly coupled system of two singularly perturbed convection–diffusion equations. Here, bounds on the errors in approximating the first derivative of the solution with a weight in the fine mesh and without a weight in the coarse mesh are also obtained.

Throughout this paper, C denotes a generic constant (sometimes subscripted) which is independent of the singular perturbation parameter ε and the dimension N of the discrete problem. Let $y : D = [a, b] \longrightarrow \mathbb{R}$. The appropriate norm for studying the convergence of numerical solution to the exact solution of a singular perturbation problem is the supremum norm $\|y\|_D = \sup_{x\in D} |y(x)|$. In case of vectors $\bar{y} = (y_1, y_2)^T$, we define

$$|\bar{y}(x)| = \big(|y_1(x)|, |y_2(x)|\big)^T, \quad \|\bar{y}\|_D = \max\big\{\|y_1\|_D, \|y_2\|_D\big\}$$

and $\bar{y} \geq \bar{0}$ provided $y_1 \geq 0$ and $y_2 \geq 0$.

We shall assume that $\varepsilon \leq CN^{-1}$ throughout the paper as is generally the case in practice for discretization of convection-dominated problem [18]. The assumption $\varepsilon \leq CN^{-1}$ makes the proof of higher order convergence significantly easier to complete. The higher order may reduce to first order in the case $1 \geq \varepsilon \geq CN^{-1}$ [1].

2 Continuous Problem

2.1 Statement of the problem

Find $y_1, y_2 \in Y \equiv C^0(\bar{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{cases} L_1\bar{y} \equiv -\varepsilon y_1'' - a_1(x)y_1' + b_{11}(x)y_1 + b_{12}(x)y_2 = f_1(x), \\ L_2\bar{y} \equiv -\varepsilon y_2'' - a_2(x)y_2' + b_{21}(x)y_1 + b_{22}(x)y_2 = f_2(x), \quad x \in \Omega \end{cases} \tag{2.1}$$

with the boundary conditions

$$\begin{cases} B_{10}y_1(0) \equiv \beta_{11}y_1(0) - \varepsilon\beta_{12}y_1'(0) = A_1, \\ B_{20}y_2(0) \equiv \beta_{21}y_2(0) - \varepsilon\beta_{22}y_2'(0) = A_2, \end{cases} \tag{2.2}$$

$$\begin{cases} B_{11}y_1(1) \equiv \gamma_{11}y_1(1) + \gamma_{12}y_1'(1) = B_1, \\ B_{21}y_2(1) \equiv \gamma_{21}y_2(1) + \gamma_{22}y_2'(1) = B_2. \end{cases} \tag{2.3}$$

Assume that $a_1(x) \geq \alpha_1 > 0$, $a_2(x) \geq \alpha_2 > 0$, $b_{12}(x) \leq 0$, $b_{21}(x) \leq 0$, $\{b_{11}(x) + b_{12}(x)\} \geq 0$, $\{b_{22}(x) + b_{21}(x)\} \geq 0$, where $\bar{y} = (y_1, y_2)^T$ and the functions $a_i(x)$, $f_i(x)$, $b_{ij}(x)$ are sufficiently smooth on $\bar{\Omega}$, $\Omega = (0, 1)$, $0 < \varepsilon \leq 1$, $\beta_{j2} > 0$, $2\beta_{j1} + \varepsilon\beta_{j2} \geq 1$, $\gamma_{j2} \geq 0$ and $\gamma_{j1} - \gamma_{j2} \geq 1$, for $i, j = 1, 2$. Let $\alpha = \min\{\alpha_1, \alpha_2\}$. The above system can be written in the matrix form as

$$\mathbf{L}\bar{y} \equiv \begin{pmatrix} L_1\bar{y} \\ L_2\bar{y} \end{pmatrix} \equiv \begin{pmatrix} -\varepsilon\frac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon\frac{d^2}{dx^2} \end{pmatrix} \bar{y} - \mathbf{A}(x)\bar{y}' + \mathbf{B}(x)\bar{y} = \bar{f}(x), \quad x \in \Omega$$

with the boundary conditions

$$\begin{pmatrix} B_{10}y_1(0) \\ B_{20}y_2(0) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \qquad \begin{pmatrix} B_{11}y_1(1) \\ B_{21}y_2(1) \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where $\mathbf{A}(x) = \begin{pmatrix} a_1(x) & 0 \\ 0 & a_2(x) \end{pmatrix}$, $\mathbf{B}(x) = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix}$ and $\bar{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$.

2.2 Analytical results

In the following, the maximum principle, stability result and derivative estimates are established for BVP (2.1)–(2.3).

Theorem 1 [Maximum Principle]. *Suppose that a function* $\bar{y}(x) = (y_1(x), y_2(x))^T$*,* $y_1, y_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ *satisfies* $B_{j0}y_j(0) \geq 0$*,* $B_{j1}y_j(1) \geq 0$*, for* $j = 1, 2$ *and* $\boldsymbol{L}\bar{y}(x) \geq \bar{0}$*,* $\forall x \in \Omega$*. Then* $\bar{y}(x) \geq \bar{0}$*,* $\forall x \in \bar{\Omega}$*.*

Proof. Define $\bar{s}(x) = (s_1(x), s_2(x))^T$ as $s_1(x) = s_2(x) = 2 - x$. Then $s_1, s_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega)$, $\bar{s}(x) > \bar{0}$, for all $x \in \bar{\Omega}$ and $\mathbf{L}\bar{s}(x) > \bar{0}, x \in \Omega$. So, we further define

$$\mu = \max\left\{\max_{x\in\bar{\Omega}}\left(-\frac{y_1}{s_1}\right), \max_{x\in\bar{\Omega}}\left(-\frac{y_2}{s_2}\right)\right\}.$$

Assume that the theorem is not true. Then $\mu > 0$ and there exists a point $x_0 \in \bar{\Omega}$, such that either $(\frac{-y_1}{s_1})(x_0) = \mu$ or $(\frac{-y_2}{s_2})(x_0) = \mu$ or both. Also $(\bar{y} + \mu\bar{s})(x) \geq \bar{0}$, $\forall x \in \bar{\Omega}$.

Case (i): $(y_1 + \mu s_1)(x_0) = 0$, for $x_0 = 0$. It implies that $(y_1 + \mu s_1)$ attains its minimum at x_0. Therefore,

$$0 < B_{10}(y_1 + \mu s_1)(x_0) = \beta_{11}(y_1 + \mu s_1)(x_0) - \varepsilon\beta_{12}(y_1 + \mu s_1)'(x_0) \leq 0,$$

which is a contradiction.

Case (ii): $(y_1 + \mu s_1)(x_0) = 0$, for $x_0 \in \Omega$. It implies that $(y_1 + \mu s_1)$ attains its minimum at x_0. Therefore,

$$\begin{aligned} 0 < L_1\bar{y}(x) \equiv -\varepsilon(y_1 + \mu s_1)''(x) - a_1(x)(y_1 + \mu s_1)'(x) + b_{11}(x)(y_1 + \mu s_1)(x) \\ + b_{12}(x)(y_2 + \mu s_2)(x) \leq 0, \end{aligned}$$

which is a contradiction.

Case (iii): $(y_1 + \mu s_1)(x_0) = 0$, for $x_0 = 1$. It implies that $(y_1 + \mu s_1)$ attains its minimum at x_0. Therefore,

$$0 < B_{11}(y_1 + \mu s_1)(x_0) = \gamma_{11}(y_1 + \mu s_1)(x_0) + \gamma_{12}(y_1 + \mu s_1)'(x_0) \leq 0,$$

which is a contradiction.

Case (iv): $(y_2 + \mu s_2)(x_0) = 0$, for $x_0 = 0$. Similar to Case (i), it leads to a contradiction.

Case (v): $(y_2 + \mu s_2)(x_0) = 0$, for $x_0 \in \Omega$. Similar to Case (ii), it leads to a contradiction.

Case (vi): $(y_2 + \mu s_2)(x_0) = 0$, for $x_0 = 1$. Similar to Case (iii), it leads to a contradiction.

Hence $\bar{y}(x) \geq \bar{0}, \forall x \in \bar{\Omega}$. □

In the rest of the problem for continuous case the norm $\|\cdot\|$ means $\|\cdot\|_{\overline{\Omega}}$.

Theorem 2 [Stability Result]. *If* $y_1, y_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ *then*

$$\begin{aligned} |y_j(x)| \leq C\max\{|B_{10}y_1(0)|, |B_{11}y_1(1)|, |B_{20}y_2(0)|, |B_{21}y_2(1)|, \\ \|L_1\bar{y}\|_\Omega, \|L_2\bar{y}\|_\Omega\}, \quad x \in \bar{\Omega},\ j = 1, 2. \end{aligned}$$

Proof. Set

$$M = C\max\{|B_{10}y_1(0)|, |B_{11}y_1(1)|, |B_{20}y_2(0)|, |B_{21}y_2(1)|, \|L_1\bar{y}\|_\Omega, \|L_2\bar{y}\|_\Omega\}.$$

It is easy to see that $M(2\beta_{11}+\varepsilon\beta_{12}, 2\beta_{21}+\varepsilon\beta_{22})^T \pm (B_{10}y_1(0), B_{20}y_2(0))^T$ and $M(\gamma_{11}-\gamma_{12}, \gamma_{21}-\gamma_{22})^T \pm (B_{11}y_1(1), B_{21}y_2(1))^T$ are non-negative. Further

$$\begin{aligned}\mathbf{L}\big(M(2-x,2-x)^T \pm \bar{y}(x)\big) &= M\mathbf{A}(x) + M(2-x)\begin{pmatrix} b_{11}(x)+b_{12}(x) \\ b_{21}(x)+b_{22}(x)\end{pmatrix} \pm \bar{f}(x) \\ &\geq \big(M\alpha_1 \pm f_1(x) M\alpha_2 \pm f_2(x)\big) \geq \bar{0},\end{aligned}$$

by a proper choice of C. Application of Theorem 1 yields that $M(2-x, 2-x) \pm \bar{y}(x) \geq \bar{0}, x \in \bar{\Omega}$, and the desired result follows. □

Sharper bounds on the derivatives of the solution are obtained by decomposing the solution $\bar{y}$ into regular and singular components as, $\bar{y} = \bar{v}+\bar{w}$, where $\bar{v} = (v_1, v_2)^T$ and $\bar{w} = (w_1, w_2)^T$. The regular component $\bar{v}$ can be written in the form $\bar{v} = \bar{v}_0 + \varepsilon\bar{v}_1 + \varepsilon^2\bar{v}_2 + \varepsilon^3\bar{v}_3$, where $\bar{v}_0 = (v_{01}, v_{02})^T$, $\bar{v}_1 = (v_{11}, v_{12})^T$, $\bar{v}_2 = (v_{21}, v_{22})^T$, $\bar{v}_3 = (v_{31}, v_{32})^T$ are defined respectively to be the solutions of the problems

$$-\mathbf{A}(x)\bar{v}_0' + \mathbf{B}(x)\bar{v}_0 = \bar{f}(x), \quad x \in \Omega, \qquad \begin{pmatrix} B_{11}v_{01}(1) \\ B_{21}v_{02}(1)\end{pmatrix} = \begin{pmatrix} B_{11}y_1(1) \\ B_{21}y_2(1)\end{pmatrix};$$

$$-\mathbf{A}(x)\bar{v}_1' + \mathbf{B}(x)\bar{v}_1 = \begin{pmatrix} \frac{d^2}{dx^2} & 0 \\ 0 & \frac{d^2}{dx^2}\end{pmatrix}\bar{v}_0, \qquad \begin{pmatrix} B_{11}v_{11}(1) \\ B_{21}v_{12}(1)\end{pmatrix} = \bar{0};$$

$$-\mathbf{A}(x)\bar{v}_2' + \mathbf{B}(x)\bar{v}_2 = \begin{pmatrix} \frac{d^2}{dx^2} & 0 \\ 0 & \frac{d^2}{dx^2}\end{pmatrix}\bar{v}_1, \qquad \begin{pmatrix} B_{11}v_{21}(1) \\ B_{21}v_{22}(1)\end{pmatrix} = \bar{0} \quad \text{and}$$

$$\mathbf{L}\bar{v}_3 = \begin{pmatrix} \frac{d^2}{dx^2} & 0 \\ 0 & \frac{d^2}{dx^2}\end{pmatrix}\bar{v}_2, \qquad \begin{pmatrix} B_{10}v_{31}(0) \\ B_{20}v_{32}(0)\end{pmatrix} = \bar{0}, \qquad \begin{pmatrix} B_{11}v_{31}(1) \\ B_{21}v_{32}(1)\end{pmatrix} = \bar{0}.$$

Thus the regular component $\bar{v}$ is the solution of

$$\mathbf{L}\bar{v} = \bar{f}(x), \quad x \in \Omega, \tag{2.4}$$

$$\begin{cases} \begin{pmatrix} B_{10}v_1(0) \\ B_{20}v_2(0)\end{pmatrix} = \begin{pmatrix} B_{10}v_{01}(0) + \varepsilon(B_{10}v_{11}(0)) + \varepsilon^2(B_{10}v_{21}(0)) \\ B_{20}v_{02}(0) + \varepsilon(B_{20}v_{12}(0)) + \varepsilon^2(B_{20}v_{22}(0))\end{pmatrix}, \\ \begin{pmatrix} B_{11}v_1(1) \\ B_{21}v_2(1)\end{pmatrix} = \begin{pmatrix} B_{11}y_1(1) \\ B_{21}y_2(1)\end{pmatrix}. \end{cases} \tag{2.5}$$

Then the singular component $\bar{w}$ is the solution of

$$\mathbf{L}\bar{w} = \bar{0}, \tag{2.6}$$

$$\begin{pmatrix} B_{10}w_1(0) \\ B_{20}w_2(0)\end{pmatrix} = \begin{pmatrix} B_{10}y_1(0) - B_{10}v_1(0) \\ B_{20}y_2(0) - B_{20}v_2(0)\end{pmatrix}, \qquad \begin{pmatrix} B_{11}w_1(1) \\ B_{21}w_2(1)\end{pmatrix} = \bar{0}. \tag{2.7}$$

The following lemma provides the bound on the derivatives of the regular and singular components of the solution $\bar{y}$.

Lemma 1. [14] *The solution $\bar{y}$ can be decomposed into the sum $\bar{y} = \bar{v} + \bar{w}$, where, $\bar{v}$ and $\bar{w}$ are regular and singular components respectively. Further, these components and their derivatives satisfy the bounds for $j = 1, 2$*

$$\begin{aligned} &\|v_j^{(k)}\| \le C\big(1 + \varepsilon^{3-k}\big), \quad k = 0,1,2,3,4, \\ &\big|w_j^{(k)}(x)\big| \le C\varepsilon^{-k} e^{-\alpha x/\varepsilon}, \quad k = 0,1,2,3,4, \ \forall x \in \bar{\Omega}. \end{aligned}$$

3 Discrete Problem

On Ω a piecewise-uniform mesh of N mesh interval is constructed as follows. The domain $\overline{\Omega}$ is subdivided into the two subintervals $[0,\sigma] \cup [\sigma, 1]$ for some σ that satisfies $0 < \sigma \le \frac{1}{2}$. On each subinterval a uniform mesh with $N/2$ mesh-intervals is placed. The interior points of the mesh are denoted by $\bar{\Omega}_\varepsilon^N = \{x_i \mid x_i = 2i\sigma/N,\ 0 \le i \le N/2;\ x_i = x_{i-1} + 2(1-\sigma)/N,\ N/2 + 1 \le i \le N\}$ condensing at the boundary point $x_0 = 0$. The transition parameter σ is chosen to satisfies $\sigma = \min\{\frac{1}{2}, \frac{2\varepsilon}{\alpha} \ln N\}$. For our analysis we assume that $\sigma = \frac{2\varepsilon}{\alpha} \ln N$, since otherwise N^{-1} is exponentially small compared with ε. Then the mesh widths are

$$h_i = \begin{cases} H_1 = 2\sigma/N, & i = 1, \ldots, N/2 - 1, \\ H_2 = 2(1-\sigma)/N, & i = N/2, \ldots, N. \end{cases}$$

The application of second order central difference and cubic spline difference schemes on the whole domain using a Shishkin mesh may result in oscillations in the coarse mesh region. However, the use of a midpoint scheme in the whole domain results in an oscillation free scheme but with a first order convergence rate. In order to retain second order convergence and also to avoid an oscillation, we propose two hybrid difference schemes that use the central difference, cubic spline in the fine mesh region and a midpoint scheme in the coarse mesh region.

Let $\delta^2 Y_j(x_i) = \frac{2}{h_i + h_{i-1}}\big(\frac{Y_j(x_{i+1}) - Y_j(x_i)}{h_i} - \frac{Y_j(x_i) - Y_j(x_{i-1})}{h_{i-1}}\big)$, $D^0 Y_j(x_i) = \frac{Y_j(x_{i+1}) - Y_j(x_{i-1})}{h_i + h_{i-1}}$, $D^+ Y_j(x_i) = \frac{Y_j(x_{i+1}) - Y_j(x_i)}{h_i}$ and $a_{j,i+1/2} \equiv a_j((x_{i+1} + x_i)/2)$; similarly for $b_{11,i+1/2}$, $b_{12,i+1/2}$, $b_{22,i+1/2}$, $b_{21,i+1/2}$ and $f_{j,i+1/2}$, for $j = 1, 2$.

Hybrid Difference Scheme - I (HDS - I) uses the central difference scheme in the fine mesh region and a midpoint difference scheme in the coarse region, that is,

$$L_1^N \bar{Y}(x_i) = \begin{cases} -\varepsilon\delta^2 Y_1(x_i) - a_1(x_i) D^0 Y_1(x_i) + b_{11}(x_i) Y_1(x_i) \\ \qquad + b_{12}(x_i) Y_2(x_i) = f_1(x_i), \quad 0 \le i < N/2, \\ -\varepsilon\delta^2 Y_1(x_i) - a_{1,i+1/2} D^+ Y_1(x_i) \\ \qquad + b_{11,i+1/2}(Y_1(x_i) + Y_1(x_{i+1}))/2 \\ \qquad + b_{12,i+1/2}(Y_2(x_i) + Y_2(x_{i+1}))/2 \\ \quad = (f_1(x_i) + f_1(x_{i+1}))/2, \quad N/2 \le i < N, \end{cases} \tag{3.1}$$

$$L_2^N \bar{Y}(x_i) = \begin{cases} -\varepsilon\delta^2 Y_2(x_i) - a_2(x_i) D^0 Y_2(x_i) + b_{21}(x_i) Y_2(x_i) \\ \qquad + b_{22}(x_i) Y_1(x_i) = f_2(x_i), \quad 0 < i \le N/2, \\ -\varepsilon\delta^2 Y_2(x_i) - a_{2,i+1/2} D^+ Y_2(x_i) \\ \qquad + b_{21,i+1/2}(Y_2(x_i) + Y_2(x_{i+1}))/2 \\ \qquad + b_{22,i+1/2}(Y_1(x_i) + Y_1(x_{i+1}))/2 \\ \quad = (f_2(x_i) + f_2(x_{i+1}))/2, \quad N/2 < i < N, \end{cases} \tag{3.2}$$

We now approximate the boundary conditions (2.2) and (2.3). The first order derivative in the left boundary conditions are approximated by the central difference operator,

$$\begin{cases} \beta_{j1} Y_j(x_0) - \varepsilon\beta_{j2} D^0 Y_j(x_0) = A_j, \\ \gamma_{j1} Y_j(x_N) + \gamma_{j2} D^- Y_j(x_N) = B_j, \quad j = 1, 2. \end{cases} \tag{3.3}$$

From (3.3), we have for $j = 1, 2$

$$Y_j(x_{-1}) = -\frac{2H_1\beta_{j1}}{\varepsilon\beta_{j2}} Y_j(x_0) + Y_j(x_1) + \frac{2H_1}{\varepsilon\beta_{j2}} A_j \tag{3.4}$$

$$Y_j(x_N) = \frac{B_j H_2 + \gamma_{j2} Y_j(x_{N-1})}{\gamma_{j1} H_2 + \gamma_{j2}}, \tag{3.5}$$

where $Y_j(x_{-1})$ are the functional values at x_{-1}. The node x_{-1} lies outside the interval $[0, 1]$ and is called a fictitious node.

The values $Y_j(x_{-1})$ may be eliminated by assuming that the difference equation (3.1) holds also for $i = 0$, that is, at the boundary point x_0. Substituting the values $Y_j(x_{-1})$ and $Y_j(x_N)$ from (3.4) and (3.5) into the equations (3.1) for $i = 0$ and $i = N - 1$, we get respectively

$$\begin{cases} B_{10}^N Y_1(x_0) \equiv \left(\frac{2\varepsilon}{H_1^2} + b_{11}(x_0) - \frac{2H_1\beta_{11}}{\varepsilon\beta_{12}} \left(-\frac{\varepsilon}{H_1^2} + \frac{a_1(x_0)}{2H_1} \right) \right) Y_1(x_0) \\ \qquad - \left(\frac{2\varepsilon}{H_1^2} \right) Y_1(x_1) + b_{12}(x_0) Y_2(x_0) \\ \quad = f_1(x_0) - \frac{2H_1 A_1}{\varepsilon\beta_{12}} \left(-\frac{\varepsilon}{H_1^2} + \frac{a_1(x_0)}{2H_1} \right), \\ B_{20}^N Y_2(x_0) \equiv \left(\frac{2\varepsilon}{H_1^2} + b_{21}(x_0) - \frac{2H_1\beta_{21}}{\varepsilon\beta_{22}} \left(-\frac{\varepsilon}{H_1^2} + \frac{a_2(x_0)}{2H_1} \right) \right) Y_2(x_0) \\ \qquad - \left(\frac{2\varepsilon}{H_1^2} \right) Y_2(x_1) + b_{22}(x_0) Y_1(x_0) \\ \quad = f_2(x_0) - \frac{2H_1 A_2}{\varepsilon\beta_{22}} \left(-\frac{\varepsilon}{H_1^2} + \frac{a_2(x_0)}{2H_1} \right) \\ B_{11}^N Y_1(x_N) \equiv \left(\frac{-\varepsilon}{H_2^2} \right) Y_1(x_{N-2}) \\ \qquad + \left(\frac{\gamma_{12}}{\gamma_{11} H_2 + \gamma_{12}} \left(\frac{-\varepsilon}{H_2^2} - \frac{a_{1,N-1/2}}{H_2} + \frac{b_{11,N-1/2}}{2} \right) \right. \end{cases} \tag{3.6}$$

and

$$\begin{cases} \qquad\qquad + \dfrac{2\varepsilon}{H_2^2} + \dfrac{a_{1,N-1/2}}{H_2} + \dfrac{b_{11,N-1/2}}{2}\Big) Y_1(x_{N-1}) \\ \qquad\qquad + \left(\dfrac{b_{12,N-1/2}}{2} + \dfrac{\gamma_{22} b_{12,N-1/2}}{2(\gamma_{21}H_2 + \gamma_{22})}\right) Y_2(x_{N-1}) \\ \qquad = f_{1,N-1/2} - \dfrac{H_2 B_1}{\gamma_{11}H_2 + \gamma_{12}}\left(-\dfrac{\varepsilon}{H_2^2} - \dfrac{a_{1,N-1/2}}{H_2} + \dfrac{b_{11,N-1/2}}{2}\right) \\ \qquad\qquad - \dfrac{B_2 H_2 b_{12,N-1/2}}{2(\gamma_{21}H_2 + \gamma_{22})}, \\ B_{21}^N Y_2(x_N) \equiv \left(\dfrac{-\varepsilon}{H_2^2}\right) Y_2(x_{N-2}) \\ \qquad\qquad + \Big(\dfrac{\gamma_{22}}{\gamma_{21}H_2 + \gamma_{22}}\left(\dfrac{-\varepsilon}{H_2^2} - \dfrac{a_{2,N-1/2}}{H_2} + \dfrac{b_{22,N-1/2}}{2}\right) \\ \qquad\qquad + \dfrac{2\varepsilon}{H_2^2} + \dfrac{a_{2,N-1/2}}{H_2} + \dfrac{b_{22,N-1/2}}{2}\Big) Y_2(x_{N-1}) \\ \qquad\qquad + \left(\dfrac{b_{21,N-1/2}}{2} + \dfrac{\gamma_{12} b_{21,N-1/2}}{2(\gamma_{11}H_2 + \gamma_{12})}\right) Y_1(x_{N-1}) \\ \qquad = f_{2,N-1/2} - \dfrac{H_2 B_2}{\gamma_{21}H_2 + \gamma_{22}}\left(-\dfrac{\varepsilon}{H_2^2} - \dfrac{a_{2,N-1/2}}{H_2} + \dfrac{b_{22,N-1/2}}{2}\right) \\ \qquad\qquad - \dfrac{B_1 H_2 b_{21,N-1/2}}{2(\gamma_{11}H_2 + \gamma_{12})}. \end{cases} \tag{3.7}$$

Thus the scheme is given by

$$\begin{cases} L_1^N \bar{Y}(x_i) \equiv r_{1,i}^- Y_1(x_{i-1}) + r_{1,i}^c Y_1(x_i) + r_{1,i}^+ Y_1(x_{i+1}) + q_{1,i}^- Y_2(x_{i-1}) \\ \qquad\qquad + q_{1,i}^c Y_2(x_i) + q_{1,i}^+ Y_2(x_{i+1}) = F_1(x_i), \\ L_2^N \bar{Y}(x_i) \equiv r_{2,i}^- Y_2(x_{i-1}) + r_{2,i}^c Y_2(x_i) + r_{2,i}^+ Y_2(x_{i+1}) + q_{2,i}^- Y_1(x_{i-1}) \\ \qquad\qquad + q_{2,i}^c Y_1(x_i) + q_{2,i}^+ Y_1(x_{i+1}) = F_2(x_i) \end{cases} \tag{3.8}$$

with the following equations corresponding to the boundary points

$$\begin{cases} B_{10}^N Y_1(x_0) \equiv r_{1,0}^c Y_1(x_0) + r_{1,0}^+ Y_1(x_1) + q_{1,0}^c Y_2(x_0) + q_{1,0}^+ Y_2(x_1) = F_1(x_0), \\ B_{11}^N Y_1(x_N) \equiv r_{1,N-1}^- Y_1(x_{N-2}) + r_{1,N-1}^c Y_1(x_{N-1}) + q_{1,N-1}^c Y_2(x_{N-1}) \\ \qquad\qquad = F_1(x_{N-1}), \\ B_{20}^N Y_2(x_0) \equiv r_{2,0}^c Y_2(x_0) + r_{2,0}^+ Y_2(x_1) + q_{2,0}^c Y_1(x_0) + q_{2,0}^+ Y_1(x_1) = F_2(x_0), \\ B_{21}^N Y_2(x_N) \equiv r_{2,N-1}^- Y_2(x_{N-2}) + r_{2,N-1}^c Y_2(x_{N-1}) + q_{2,N-1}^c Y_1(x_{N-1}) \\ \qquad\qquad = F_2(x_{N-1}), \end{cases} \tag{3.9}$$

for $i = 0$, we have

$$r_{1,0}^c = \frac{2\varepsilon}{H_1^2} + b_{11}(x_0) - \frac{2H_1\beta_{11}}{\varepsilon\beta_{12}}\left(-\frac{\varepsilon}{H_1^2} + \frac{a_1(x_0)}{2H_1}\right), \quad r_{1,0}^+ = -\left(\frac{2\varepsilon}{H_1^2}\right),$$

$$q_{1,0}^c = b_{12}(x_0), \quad q_{1,0}^+ = 0, \quad F_1(x_0) = f_1(x_0) - \frac{2H_1A_1}{\varepsilon\beta_{12}}\left(-\frac{\varepsilon}{H_1^2} + \frac{a_1(x_0)}{2H_1}\right),$$

$$r^c_{2,0} = \frac{2\varepsilon}{H_1^2} + b_{22}(x_0) - \frac{2H_1\beta_{21}}{\varepsilon\beta_{22}}\left(-\frac{\varepsilon}{H_1^2} + \frac{a_2(x_0)}{2H_1}\right), \quad r^+_{2,0} = -\left(\frac{2\varepsilon}{H_1^2}\right),$$
$$q^c_{2,0} = b_{22}(x_0), \quad q^+_{2,0} = 0, \quad F_2(x_0) = f_2(x_0) - \frac{2H_1A_2}{\varepsilon\beta_{22}}\left(-\frac{\varepsilon}{H_1^2} + \frac{a_2(x_0)}{2H_1}\right);$$

for $i = 1, \dots, N/2-1$, we have

$$r^-_{1,i} = \frac{-2\varepsilon}{h_{i-1}(h_i+h_{i-1})} + \frac{a_1(x_i)}{h_i+h_{i-1}}, \quad r^c_{1,i} = \frac{2\varepsilon}{h_ih_{i-1}} + b_{11}(x_i),$$
$$r^+_{1,i} = \frac{-2\varepsilon}{h_i(h_i+h_{i-1})} - \frac{a_1(x_i)}{h_i+h_{i-1}}, \quad q^-_{1,i} = 0, \quad q^c_{1,i} = b_{12}(x_i), \quad q^+_{1,i} = 0,$$
$$F^-_{1,i} = 0, \quad F^c_{1,i} = 1, \quad F^+_{1,i} = 0,$$
$$r^-_{2,i} = \frac{-2\varepsilon}{h_{i-1}(h_i+h_{i-1})} + \frac{a_2(x_i)}{h_i+h_{i-1}}, \quad r^c_{2,i} = \frac{2\varepsilon}{h_ih_{i-1}} + b_{22}(x_i),$$
$$r^+_{2,i} = \frac{-2\varepsilon}{h_i(h_i+h_{i-1})} - \frac{a_2(x_i)}{h_i+h_{i-1}}, \quad q^-_{2,i} = 0, \quad q^c_{2,i} = b_{21}(x_i), \quad q^+_{2,i} = 0,$$
$$F^-_{2,i} = 0, \quad F^c_{2,i} = 1, \quad F^+_{2,i} = 0,$$

for $i = N/2, \dots, N-2$, we have

$$r^-_{1,i} = \frac{-2\varepsilon}{h_{i-1}(h_i+h_{i-1})}, \quad r^c_{1,i} = \frac{2\varepsilon}{h_ih_{i-1}} + \frac{a_{1,i+1/2}}{h_i} + \frac{b_{11,i+1/2}}{2},$$
$$r^+_{1,i} = \frac{-2\varepsilon}{h_i(h_i+h_{i-1})} - \frac{a_{1,i+1/2}}{h_i} + \frac{b_{11,i+1/2}}{2}, \quad q^+_{1,i} = \frac{b_{12,i+1/2}}{2},$$
$$q^c_{1,i} = \frac{b_{12,i+1/2}}{2}, \quad q^-_{1,i} = 0, \quad F^-_{1,i} = 0, \quad F^c_{1,i} = 1, \quad F^+_{1,i} = 0,$$
$$r^-_{2,i} = \frac{-2\varepsilon}{h_{i-1}(h_i+h_{i-1})}, \quad r^c_{2,i} = \frac{2\varepsilon}{h_ih_{i-1}} + \frac{a_{2,i+1/2}}{h_i} + \frac{b_{22,i+1/2}}{2},$$
$$r^+_{2,i} = \frac{-2\varepsilon}{h_i(h_i+h_{i-1})} - \frac{a_{2,i+1/2}}{h_i} + \frac{b_{22,i+1/2}}{2}, \quad q^+_{2,i} = \frac{b_{21,i+1/2}}{2},$$
$$q^c_{2,i} = \frac{b_{21,i+1/2}}{2}, \quad q^-_{2,i} = 0, \quad F^-_{2,i} = 0, \quad F^c_{2,i} = 1, \quad F^+_{2,i} = 0,$$

for $i = N-1$, we have

$$r^c_{1,i} = \left(\frac{\gamma_{12}}{\gamma_{11}H_2+\gamma_{12}}\left(\frac{-\varepsilon}{H_2^2} - \frac{a_{1,N-1/2}}{H_2} + \frac{b_{11,N-1/2}}{2}\right), \quad r^-_{1,i} = \frac{-\varepsilon}{H_2^2},\right.$$
$$\left. + \frac{2\varepsilon}{H_2^2} + \frac{a_{1,N-1/2}}{H_2} + \frac{b_{11,N-1/2}}{2}\right),$$
$$q^c_{1,i} = \frac{b_{12,N-1/2}}{2} + \frac{\gamma_{22}b_{12,N-1/2}}{2(\gamma_{21}H_2+\gamma_{22})},$$
$$F_1(x_{N-1}) = f_{1,N-1/2} - \frac{H_2B_1}{\gamma_{11}H_2+\gamma_{12}}\left(-\frac{\varepsilon}{H_2^2} - \frac{a_{1,N-1/2}}{H_2} + \frac{b_{11,N-1/2}}{2}\right)$$
$$- \frac{B_2H_2b_{12,N-1/2}}{2(\gamma_{21}H_2+\gamma_{22})},$$

$$r_{2,i}^c = \left(\frac{\gamma_{22}}{\gamma_{21}H_2+\gamma_{22}}\left(\frac{-\varepsilon}{H_2^2} - \frac{a_{2,N-1/2}}{H_2} + \frac{b_{22,N-1/2}}{2}\right)\right.$$
$$\left. + \frac{2\varepsilon}{H_2^2} + \frac{a_{2,N-1/2}}{H_2} + \frac{b_{22,N-1/2}}{2}\right), \quad r_{2,i}^- = \frac{-\varepsilon}{H_2^2},$$
$$q_{2,i}^c = \left(\frac{b_{21,N-1/2}}{2} + \frac{\gamma_{12}b_{21,N-1/2}}{2(\gamma_{11}H_2+\gamma_{12})}\right),$$
$$F_2(x_{N-1}) = f_{2,N-1/2} - \frac{H_2B_2}{\gamma_{21}H_2+\gamma_{22}}\left(-\frac{\varepsilon}{H_2^2} - \frac{a_{2,N-1/2}}{H_2} + \frac{b_{22,N-1/2}}{2}\right)$$
$$- \frac{B_1H_2b_{21,N-1/2}}{2(\gamma_{11}H_2+\gamma_{12})}$$

and

$$F_1(x_i) = \begin{cases} f_1(x_{i-1})F_{1,i}^- + f_1(x_i)F_{1,i}^c + f_1(x_{i+1})F_{1,i}^+, & \text{for } 1 \le i \le N/2-1, \\ f_1((x_{i+1}+x_i)/2)F_{1,i}^- + f_1((x_{i+1}+x_i)/2)F_{1,i}^c \\ \quad + f_1((x_{i+1}+x_i)/2)F_{1,i}^+, & \text{for } N/2 \le i \le N-2, \end{cases}$$

$$F_2(x_i) = \begin{cases} f_2(x_{i-1})F_{2,i}^- + f_2(x_i)F_{2,i}^c + f_2(x_{i+1})F_{2,i}^+, & \text{for } 1 \le i \le N/2-1, \\ f_2((x_{i+1}+x_i)/2)F_{2,i}^- + f_2((x_{i+1}+x_i)/2)F_{2,i}^c \\ \quad + f_2((x_{i+1}+x_i)/2)F_{2,i}^+, & \text{for } N/2 \le i \le N-2. \end{cases}$$

Remark 1. The truncation error for (3.6) is given by

$$\left|B_{10}^N(Y_1-y_1)(x_0)\right| = \left|\frac{2\varepsilon}{H_1^2} + b_{11}(x_0) + \frac{2H_1\beta_{11}}{\varepsilon\beta_{12}}\left(-\frac{\varepsilon}{H_1^2} + \frac{a_1(x_0)}{2H_1}\right))Y_1(x_0)\right.$$
$$\left. - \left(\frac{2\varepsilon}{H_1^2}\right)Y_1(x_1) + b_{12}(x_0)Y_2(x_0) - f_1(x_0) + \frac{2H_1A_1}{\varepsilon\beta_{12}}\left(-\frac{\varepsilon}{H_1^2} + \frac{a_1(x_0)}{2H_1}\right)\right|$$
$$\le C\varepsilon H_1\left|y_1^{(3)}(x_0)\right|. \tag{3.10}$$

Similarly, $|B_{20}^N(Y_2-y_2)(x_0)| \le C\varepsilon H_1|y_2^{(3)}(x_0)|$ and $|B_{j1}^N(Y_j-y_j)(x_N)| \le CH_2|y_j^{(2)}(x_N)|$, for $j=1,2$. Further, the truncation error bounds for the midpoint scheme and central difference scheme for $j=1,2$,

$$\left|L_j^N(\bar{Y}-\bar{y})(x_i)\right|$$
$$\le \begin{cases} \varepsilon H_1\|y_j^{(3)}\| + C_{(\|a_1\|,\|a_1'\|)}H_1^2(\|y_j^{(3)}\| + \|y_j^{(2)}\|), & i=1,\dots,N/2, \\ \varepsilon H_2^2\|y_j^{(4)}\| + \|a_1\|H_2^2\|y_j^{(3)}\|, & i=N/2+1,\dots,N-1. \end{cases} \tag{3.11}$$

[15] Hybrid Difference Scheme - II (HDS - II) uses the cubic spline difference scheme in the fine mesh region and the midpoint difference scheme in the coarse mesh region. Note that the elements in the system matrix changes for $i=0,1,\dots,N/2-1$. Thus, we have for $i=0$,

$$r_{1,0}^c = \left(\frac{3\beta_{11}}{h_0} - \frac{\beta_{12}a_1(x_0)}{h_0} + \beta_{12}b_{11}(x_0) + \frac{\beta_{12}a_1(x_1)}{2h_0} + \frac{3\varepsilon\beta_{12}}{h_0^2}\right),$$

$$r^{+}_{1,0} = \left(-\frac{\beta_{12}a_1(x_0)}{h_0} - \frac{\beta_{12}a_1(x_1)}{2h_0} + \frac{\beta_{12}b_{11}(x_1)}{2} - \frac{3\varepsilon\beta_{12}}{h_0^2}\right),$$
$$q^{c}_{1,0} = \beta_{12}b_{12}(x_0), \quad q^{+}_{1,0} = \frac{\beta_{12}b_{12}(x_1)}{2},$$
$$F_1(x_0) = \frac{3A_1}{h_0} + \beta_{12}f_1(x_0) + \frac{\beta_{12}f_1(x_1)}{2},$$
$$r^{c}_{2,0} = (\frac{3\beta_{21}}{h_0} - \frac{\beta_{22}a_2(x_0)}{h_0} + \beta_{22}b_{22}(x_0) + \frac{\beta_{22}a_2(x_1)}{2h_0} + \frac{3\varepsilon\beta_{22}}{h_0^2}),$$
$$r^{+}_{2,0} = \left(-\frac{\beta_{22}a_2(x_0)}{h_0} - \frac{\beta_{22}a_2(x_1)}{2h_0} + \frac{\beta_{22}b_{22}(x_1)}{2} - \frac{3\varepsilon\beta_{22}}{h_0^2}\right),$$
$$q^{c}_{2,0} = \beta_{12}b_{21}(x_0), \quad q^{+}_{2,0} = \frac{\beta_{22}b_{21}(x_1)}{2},$$
$$F_2(x_0) = \frac{3A_2}{h_0} + \beta_{22}f_2(x_0) + \frac{\beta_{22}f_2(x_1)}{2},$$

for $i = 1, \dots, N/2-1$,

$$r^{-}_{1,i} = -\frac{h_i^2}{2h_{i-1}(h_i+h_{i-1})}a_1(x_{i+1}) + \frac{h_i}{h_{i-1}}a_1(x_i) + \frac{(h_i+2h_{i-1})}{2(h_i+h_{i-1})}a_1(x_{i-1})$$
$$+ \frac{h_{i-1}}{2}b_{11}(x_{i-1}) - \frac{3\varepsilon}{h_{i-1}},$$
$$r^{c}_{1,i} = -\frac{-(h_i+h_{i-1})}{2h_{i-1}}a_1(x_{i+1}) - \frac{(h_i^2-h_{i-1}^2)}{h_ih_{i-1}}a_1(x_i) - \frac{(h_i+h_{i-1})}{2h_i}a_1(x_{i-1})$$
$$+ (h_i+h_{i-1})b_{11}(x_i) + \frac{3\varepsilon(h_i+h_{i-1})}{h_ih_{i-1}},$$
$$r^{+}_{1,i} = -\frac{(2h_i+h_{i-1})}{2(h_i+h_{i-1})}a_1(x_{i+1}) - \frac{h_{i-1}}{h_i}a_1(x_i) + \frac{h_{i-1}^2}{2h_i(h_i+h_{i-1})}a_1(x_{i-1})$$
$$+ \frac{h_i}{2}b_{11}(x_i) - \frac{3\varepsilon}{h_i},$$
$$q^{-}_{1,i} = \frac{h_{i-1}b_{12}(x_{i-1})}{2}, \quad q^{c}_{1,i} = (h_i+h_{i-1})b_{12}(x_i), \quad q^{+}_{1,i} = \frac{h_ib_{12}(x_{i+1})}{2},$$
$$F^{-}_{1,i} = \frac{h_{i-1}}{2}, \quad F^{c}_{1,i} = (h_i+h_{i-1}), \quad F^{+}_{1,i} = \frac{h_i}{2},$$
$$r^{-}_{2,i} = -\frac{h_i^2a_2(x_{i+1})}{2h_{i-1}(h_i+h_{i-1})} + \frac{h_i}{h_{i-1}}a_2(x_i) + \frac{(h_i+2h_{i-1})}{2(h_i+h_{i-1})}a_2(x_{i-1})$$
$$+ \frac{h_{i-1}}{2}b_{22}(x_{i-1}) - \frac{3\varepsilon}{h_{i-1}},$$
$$r^{c}_{2,i} = -\frac{-(h_i+h_{i-1})}{2h_{i-1}}a_2(x_{i+1}) - \frac{(h_i^2-h_{i-1}^2)}{h_ih_{i-1}}a_2(x_i) - \frac{(h_i+h_{i-1})}{2h_i}a_2(x_{i-1})$$
$$+ (h_i+h_{i-1})b_{22}(x_i) + \frac{3\varepsilon(h_i+h_{i-1})}{h_ih_{i-1}},$$
$$r^{+}_{2,i} = -\frac{(2h_i+h_{i-1})}{2(h_i+h_{i-1})}a_2(x_{i+1}) - \frac{h_{i-1}}{h_i}a_2(x_i) + \frac{h_{i-1}^2a_2(x_{i-1})}{2h_i(h_i+h_{i-1})}$$

$$+\frac{h_i}{2}b_{22}(x_i)-\frac{3\varepsilon}{h_i},$$

$$q^-_{2,i}=\frac{h_{i-1}}{2}b_{21}(x_{i-1}),\quad q^c_{2,i}=(h_i+h_{i-1})b_{21}(x_i),\quad q^+_{2,i}=\frac{h_i}{2}b_{21}(x_{i+1}),$$

$$F^-_{2,i}=\frac{h_{i-1}}{2},\quad F^c_{2,i}=(h_i+h_{i-1}),\quad F^+_{2,i}=\frac{h_i}{2}.$$

Remark 2. The truncation error bounds at x_0 is given by

$$\begin{aligned}
&\left|B^N_{10}(Y_1-y_1)(x_0)\right|\\
&=\Bigg|\left(\frac{3\beta_{11}}{h_0}-\frac{\beta_{12}a_1(x_0)}{h_0}+\beta_{12}b_{11}(x_0)+\frac{\beta_{12}a_1(x_1)}{2h_0}+\frac{3\varepsilon\beta_{12}}{h_0^2}\right)Y_1(x_0)\\
&\quad+\left(-\frac{\beta_{12}a_1(x_0)}{h_0}-\frac{\beta_{12}a_1(x_1)}{2h_0}+\frac{\beta_{12}b_{11}(x_1)}{2}-\frac{3\varepsilon\beta_{12}}{h_0^2}\right)Y_1(x_1)\\
&\quad+\beta_{12}b_{12}(x_0)Y_2(x_0)+\frac{\beta_{12}b_{12}(x_1)}{2}Y_2(x_1)-\frac{3A_1}{h_0}+\beta_{12}f_1(x_0)+\frac{\beta_{12}f_1(x_1)}{2}\Bigg|\\
&\le C\varepsilon H_1^2\left|y_1^{(4)}(x_0)\right|.
\end{aligned}\tag{3.12}$$

Similarly, $|B^N_{20}(Y_2-y_2)(x_0)|\le C\varepsilon H_1^2|y_1^{(4)}(x_0)|$. Further, the truncation error bounds for $i=1,\dots,N/2-1$,

$$\left|L^N_j(\bar{Y}-\bar{y})(x_i)\right|\le C\varepsilon H_1^2\left|y_j^{(4)}\right|,\quad j=1,2.\tag{3.13}$$

Note: It may be noted that the same operator symbols L^N_j, B^N_{j0} and B^N_{j1}, $j=1,2$ are used for both the schemes. In the following whatever discussion is carried out, it is true for both the schemes.

3.1 Numerical solution estimates

To guarantee the monotonicity property of the difference operator $\mathbf{L}^N$, we impose the following mild assumption on the minimum number of mesh points

$$\frac{N}{\ln N}\ge 2\max\left\{\frac{\|a_1\|}{\alpha},\frac{\|a_2\|}{\alpha}\right\}.\tag{3.14}$$

Analogous to the continuous results stated in Theorem 1 and Theorem 2 one can prove the following results.

Theorem 3. [14] *For any mesh function $\bar{\Psi}(x_i)$ assume that $B^N_{j0}\Psi_j(x_0)\ge 0$, $B^N_{j1}\Psi_j(x_N)\ge 0$ for $j=1,2$ and $\boldsymbol{L}^N\bar{\Psi}(x_i)\ge\bar{0}$, for all $i=1,\dots,N-1$. Then $\bar{\Psi}(x_i)\ge\bar{0}$, for all $i=0,1,\dots,N$.*

Theorem 4. [14] *If $\bar{Z}(x_i)=(Z_1(x_i),Z_2(x_i))^T$ is any mesh function then, for all $x_i\in\bar{\Omega}^N_\varepsilon$, $j=1,2$,*

$$\begin{aligned}\left|Z_j(x_i)\right|\le C\max\Big\{&\left|B^N_{10}Z_1(x_0)\right|,\left|B^N_{11}Z_1(x_N)\right|,\left|B^N_{20}Z_2(x_0)\right|,\left|B^N_{21}Z_2(x_N)\right|,\\
&\max_{1\le i\le N-1}\left|L^N_1\bar{Z}(x_i)\right|,\max_{1\le i\le N-1}\left|L^N_2\bar{Z}(x_i)\right|\Big\}.\end{aligned}$$

3.2 Error analysis

The discrete solution $\bar{Y}(x_i)$ can be decomposed into the sum $\bar{Y}(x_i) = \bar{V}(x_i) + \bar{W}(x_i)$ where $\bar{V}(x_i)$ and $\bar{W}(x_i)$ are regular and singular components respectively defined as

$$\mathbf{L}^N \bar{V}(x_i) = \bar{f}(x_i), \quad i = 1, \ldots, N-1, \tag{3.15}$$

$$\begin{pmatrix} B_{10}^N V_1(x_0) \\ B_{20}^N V_2(x_0) \end{pmatrix} = \begin{pmatrix} B_{10} v_1(0) \\ B_{20} v_2(0) \end{pmatrix}, \qquad \begin{pmatrix} B_{11}^N V_1(x_N) \\ B_{21}^N V_2(x_N) \end{pmatrix} = \begin{pmatrix} B_{11} v_1(1) \\ B_{21} v_2(1) \end{pmatrix} \tag{3.16}$$

and

$$\mathbf{L}^N \bar{W}(x_i) = \bar{0}, \quad i = 1, \ldots, N-1, \tag{3.17}$$

$$\begin{pmatrix} B_{10}^N W_1(x_0) \\ B_{20}^N W_2(x_0) \end{pmatrix} = \begin{pmatrix} B_{10} w_1(0), \\ B_{20} w_2(0) \end{pmatrix}, \qquad \begin{pmatrix} B_{11}^N W_1(x_N) \\ B_{21}^N W_2(x_N) \end{pmatrix} = \bar{0}. \tag{3.18}$$

The error in the numerical solution can be written in the form $(\bar{Y} - \bar{y})(x_i) = (\bar{V} - \bar{v})(x_i) + (\bar{W} - \bar{w})(x_i)$.

Lemma 2. *At each mesh point $x_i \in \overline{\Omega}^N$, the error of the regular component satisfies the estimate*

$$\left|(\bar{V} - \bar{v})(x_i)\right| \le \begin{pmatrix} CN^{-2}(2 - x_i) \\ CN^{-2}(2 - x_i) \end{pmatrix}.$$

Proof. Using $\varepsilon \le CN^{-1}$, (3.10), (3.12) and (3.11), (3.13) and the bounds on the derivatives of $\bar{v}$, we have for $j = 1, 2$

$$\begin{aligned} \left|B_{j0}^N (V_j - v_j)(x_0)\right| &\le C\varepsilon H_1 \left|v_j^{(3)}(x_0)\right| \le CN^{-2}, \\ \left|L_j^N (\bar{V} - \bar{v})(x_i)\right| &\le \begin{cases} CN^{-2}, & i = 1, \ldots, N/2 - 1, \\ CN^{-1}(\varepsilon + N^{-1}), & i = N/2, \ldots, N-1 \end{cases} \\ &\le CN^{-2}, \quad i = 1, \ldots, N-1, \\ \left|B_{j1}^N (V_j - v_j)(x_N)\right| &\le CH_2 \left|v_j^{(2)}(x_N)\right| \le CN^{-2}. \end{aligned}$$

Consider the barrier functions $\bar{\Psi}^{\pm}(x_i) = (\Psi_1^{\pm}(x_i), \Psi_2^{\pm}(x_i))^T$, where

$$\bar{\Psi}^{\pm}(x_i) = \begin{pmatrix} CN^{-2}(2 - x_i) \\ CN^{-2}(2 - x_i) \end{pmatrix} \pm (\bar{V} - \bar{v})(x_i).$$

Then, we have $B_{j0}^N \Psi_j^{\pm}(x_0) = 2CN^{-2}(r_{j,0}^c + r_{j,0}^+ + q_{j,0}^c + q_{j,0}^+) - CN^{-2}x_1(r_{j,0}^+ + q_{j,0}^+) \ge 0$ and $B_{j1}^N \Psi_j^{\pm}(x_N) = 2CN^{-2}(r_{j,N}^- + r_{j,N}^c + q_{j,N}^c) - CN^{-2}x_{N-1}(r_{j,N}^c + q_{j,N}^c) - CN^{-2}x_{N-2}r_{j,N}^- \ge 0$, $j = 1, 2$. For $j = 1, 2; i = 1, \ldots, N-1$ we have

$$\begin{aligned} L_j^N \bar{\Psi}^{\pm}(x_i) = CN^{-2}&(r_{j,i}^- + r_{j,i}^c + r_{j,i}^+ + q_{j,i}^- + q_{j,i}^c + q_{j,i}^+) - CN^{-2}\big([r_{j,i}^- \\ &+ q_{j,i}^-](x_{i-1}) + [r_{j,i}^c + q_{j,i}^c](x_i) + [r_{j,i}^+ + q_{j,i}^+](x_{i+1})\big) \pm CN^{-2} \\ \ge CN^{-2}&(r_{j,i}^- - r_{j,i}^+ + q_{j,i}^- - q_{j,i}^+ \pm 1) > 0. \end{aligned}$$

Applying Theorem 3 to $\bar{\Psi}^{\pm}(x_i), x_i \in \overline{\Omega}^N$, we get the required result. □

Lemma 3. *At each mesh point $x_i \in \overline{\Omega}^N$, the error of the singular component satisfies the estimate*

$$\left|(\bar{W} - \bar{w})(x_i)\right| \le \begin{pmatrix} CN^{-2}(\ln N)^3 \\ CN^{-2}(\ln N)^3 \end{pmatrix}.$$

Proof. Suppose $\sigma = \frac{2\varepsilon}{\alpha} \ln N$, so the mesh is non-uniform. We split the argument into two cases depending on the localization of the mesh point. In the first case $x_i \in \overline{\Omega}^N \cap [\sigma, 1]$, using the arguments in [1] and [14, Lemma 6], for $N/2 \le i \le N$ we have

$$\left|(\bar{W} - \bar{w})(x_i)\right| \le \begin{pmatrix} CN^{-2} \\ CN^{-2} \end{pmatrix}.$$

Now for $x_i \in \overline{\Omega}^N \cap [0, \sigma)$, using (3.11), (3.13) and the bounds on the derivatives of $\bar{w}$, we have for $j = 1, 2$

$$\left|L_j^N(\bar{W} - \bar{w})(x_i)\right| \le C\frac{H_1^2}{\varepsilon^3} \exp(-x_i\alpha/\varepsilon).$$

For all i, $0 \le i \le N/2 - 1$, we introduce the mesh functions

$$\bar{\Psi}^{\pm}(x_i) = \begin{pmatrix} CN^{-2} + C\frac{\sigma^2}{\varepsilon^3 N^2}(\sigma - x_i) \\ CN^{-2} + C\frac{\sigma^2}{\varepsilon^3 N^2}(\sigma - x_i) \end{pmatrix} \pm (\bar{W} - \bar{w})(x_i).$$

It is easy to show that $B_{j0}^N \Psi_j^{\pm}(x_0) = (CN^{-2} + C\frac{\sigma^2}{\varepsilon^3 N^2}\sigma)(r_{j,0}^c + r_{j,0}^+ + q_{j,0}^c + q_{j,0}^+) - C\frac{\sigma^2}{\varepsilon^3 N^2}x_1(r_{j,0}^+ + q_{j,0}^+) \ge 0$ and $\Psi_j^{\pm}(x_{N/2}) \ge 0$, $j = 1, 2$. For $j = 1, 2$; $i = 1, \dots, N-1$ we have

$$\begin{aligned} L_j^N \bar{\Psi}^{\pm}(x_i) &= CN^{-2}(r_{j,i}^- + r_{j,i}^c + r_{j,i}^+ + q_{j,i}^- + q_{j,i}^c + q_{j,i}^+) \\ &\quad + C\frac{\sigma^2}{\varepsilon^3 N^2}\big([r_{j,i}^- + q_{j,i}^-](\sigma - x_{i-1}) + [r_{j,i}^c + q_{j,i}^c](\sigma - x_i) \\ &\quad + [r_{j,i}^+ + q_{j,i}^+](\sigma - x_{i+1})\big) \pm C\frac{\sigma^2}{\varepsilon^3 N^2} > 0. \end{aligned}$$

Then by Theorem 3 we get $\bar{\Psi}^{\pm}(x_i) \ge \bar{0}$. Thus we get the required result. □

Theorem 5. *Let $\bar{y}(x) = (y_1(x), y_2(x))^T, x \in \bar{\Omega}$ be the solution of* (2.1)–(2.3) *and let $\bar{Y}(x_i) = (Y_1(x_i), Y_2(x_i))^T, x_i \in \bar{\Omega}^N$ be the numerical solution of problem* (3.8)–(3.9)*. Then we have*

$$\sup_{0<\varepsilon\le 1} \|Y_1 - y_1\|_{\Omega^N} \le CN^{-2}(\ln N)^3 \quad \text{and} \quad \sup_{0<\varepsilon\le 1} \|Y_2 - y_2\|_{\Omega^N} \le CN^{-2}(\ln N)^3.$$

Proof. The proof of the theorem follows immediately, if one applies the above Lemmas 2 and 3 to $\bar{Y} - \bar{y} = \bar{V} - \bar{v} + \bar{W} - \bar{w}$. □

Theorem 6. [14, 15] *Let $\bar{y}(x)$ be the solution of (2.1)–(2.3) and $\bar{Y}(x_i)$ be the corresponding numerical solution of (3.8)–(3.9). Then, for $j = 1, 2$, and $x \in \bar{\Omega}_i = [x_i, x_{i+1}]$, we have*

$$\sup_{0<\varepsilon\le 1} \big\|\varepsilon\big(D^0 Y_j(x_i) - y_j'\big)\big\|_{\overline{\Omega}_i} \le CN^{-2}(\ln N)^2, \quad 1 \le i < N/2,$$
$$\sup_{0<\varepsilon\le 1} \big\|D^+ Y_j(x_i) - y_j'\big\|_{\overline{\Omega}_i} \le CN^{-1}\ln N, \quad N/2 \le i < N.$$

Remark 3. Let $\widetilde{Y}_j, \quad j = 1, 2$, denote the piecewise linear interpolant of the finite difference solution $\{Y_j(x_i)\}_{i=0}^N$. As done in [6, p. 66], we get for $j = 1, 2$ and $x \in \bar{\Omega}_i = [x_i, x_{i+1}]$,

$$\sup_{0<\varepsilon\le 1} \big\|\varepsilon\big(\widetilde{D}^0 Y_j - y_j'\big)\big\|_{\overline{\Omega}_i} \le CN^{-2}(\ln N)^2, \quad i = 1, \dots, N/2-1 \quad \text{and}$$
$$\sup_{0<\varepsilon\le 1} \big\|\widetilde{D}^+ Y_j - Y_j'\big\|_{\overline{\Omega}_i} \le CN^{-1}\ln N, \quad i = N/2, \dots, N-1,$$

where, $\widetilde{D}^0 Y_j(x) = D^0 Y_j(x_i)$, for all $x \in [x_{i-1}, x_{i+1})$, $i = 1, \dots, N/2-1$ and $\widetilde{D}^+ Y_j(x) = D^+ Y_j(x_i)$, for all $x \in [x_i, x_{i+1})$, $i = N/2, \dots, N-1$.

4 Numerical Results

In this section, we consider the following examples to illustrate the results obtained in the paper:

Example 1.

$$\begin{aligned}
&-\varepsilon y_1''(x) - \frac{1}{3+x} y_1'(x) + 2y_1(x) - y_2(x) = (3+x)/3,\\
&-\varepsilon y_2''(x) - \frac{1}{3+x} y_2'(x) - 4y_1(x) + 5y_2(x) = (3+x)/2, \quad x \in \Omega,\\
&y_1(0) - \varepsilon y_1'(0) = 2,\ y_1(1) + y_1'(1) = 2,\ \ y_2(0) - \varepsilon y_2'(0) = 2,\ y_2(1) + y_2'(1) = 2.
\end{aligned}$$

Example 2.

$$\begin{aligned}
&-\varepsilon y_1''(x) - 3y_1'(x) + 3y_1(x) - y_2(x) = 1 + e^{-x},\\
&-\varepsilon y_2''(x) - y_2'(x) - y_1(x) + 3y_2(x) = 1 - e^{-x}, \quad x \in \Omega,\\
&3y_1(0) - \varepsilon y_1'(0){=}0,\ 2y_1(1) + y_1'(1){=}1,\ 3y_2(0) - \varepsilon y_2'(0){=}2,\ 2y_2(1) + y_2'(1){=}2.
\end{aligned}$$

Let $(Y_1^N, Y_2^N)^T$ be a numerical approximation for the exact solution $(y_1, y_2)^T$ on the mesh Ω_ε^N and N is the number of mesh points. Since the exact solutions are not available for the above test problems, for a finite set of values $\varepsilon \in R_\varepsilon = \{2^0, 2^{-1}, \dots, 2^{-25}\}$, we compute the maximum pointwise error for $j = 1, 2$, $S_{\varepsilon,j}^N = \big\|Y_j^N - \widetilde{Y}_j^{2048}\big\|_{\bar{\Omega}_\varepsilon^N}$,

$$D_{\varepsilon,j}^N = \begin{cases} \max|\varepsilon(D^0 Y_j^N - \widetilde{D}^0 Y_j^{2048})(x_i)|, & 0 < i < N/2,\\ \max|D^+ Y_j^N - \widetilde{D}^+ Y_j^{2048})(x_i)|, & N/2 \le i \le N-1, \end{cases}$$

Table 1. Values of S_1^N, r_1^N and S_2^N, r_2^N for the Example 1.

N	32	64	128	256	512
			Hybrid Difference Scheme - I		
S_1^N	9.9694e−3	3.5633e−3	1.1948e−3	3.9575e−4	1.3294e−4
r_1^N	1.4843	1.5764	1.5941	1.5738	–
S_2^N	9.3473e−3	3.3411e−3	1.1202e−3	3.7094e−4	1.2169e−4
r_2^N	1.4842	1.5766	1.5945	1.6080	–
			Hybrid Difference Scheme - II		
S_1^N	1.3008e−2	4.7605e−3	1.6042e−3	5.2783e−4	1.7402e−4
r_1^N	1.4502	1.5693	1.6037	1.6008	–
S_2^N	1.1630e−2	4.2553e−3	1.4336e−3	4.7161e−4	1.5545e−4
r_2^N	1.4505	1.5696	1.6040	1.6011	–

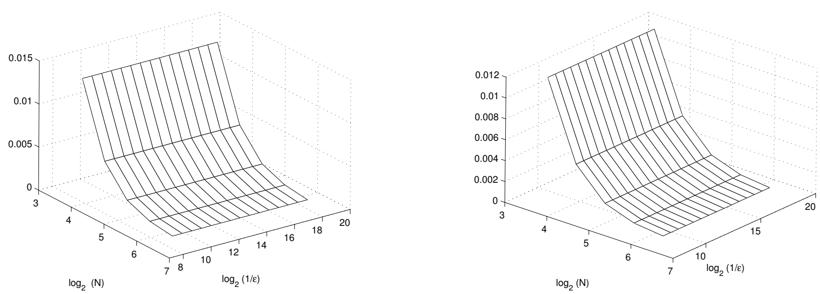

Figure 1. Surface plot of the maximum pointwise errors as a function of N and ε for the solution components Y_1 and Y_2 of the Example 1 using HDS - I.

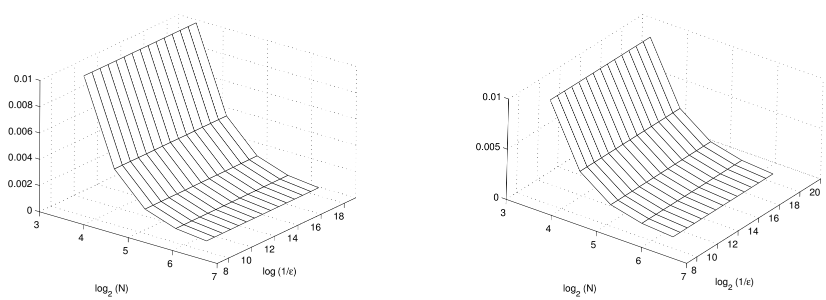

Figure 2. Surface plot of the maximum pointwise errors as a function of N and ε for the solution component Y_1 and Y_2 of the Example 1 using HDS - II.

where $\widetilde{Y}_j^{2048}$ is the piecewise linear interpolant of the mesh function Y_j^{2048} onto $[0,1]$. From these values the $\varepsilon-$uniform maximum pointwise difference

$$S_j^N = \max_{\varepsilon \in R_\varepsilon} D_{\varepsilon,j}^N, \qquad D_j^N = \max_{\varepsilon \in R_\varepsilon} D_{\varepsilon,j}^N, \quad j = 1, 2$$

Table 2. Values of D_1^N, p_1^N and D_2^N, p_2^N for the Example 1 in the fine mesh region.

N	32	64	128	256	512
Hybrid Difference Scheme - I					
D_1^N	1.3089e−2	6.1793e−3	2.4958e−3	8.8598e−4	2.7041e−4
p_1^N	1.0828	1.3079	1.4942	1.7121	–
D_2^N	1.2273e−2	5.7938e−3	2.3399e−3	8.3061e−4	2.5349e−4
p_2^N	1.0829	1.3081	1.4942	1.7122	–
Hybrid Difference Scheme - II					
D_1^N	5.4954e−3	2.7453e−3	1.1371e−3	4.0770e−4	1.2367e−4
p_1^N	1.0013	1.2716	1.4798	1.7210	–
D_2^N	4.9110e−3	2.4532e−3	1.0160e−3	3.6422e−4	1.1046e−4
p_2^N	1.0014	1.2718	1.4800	1.7213	–

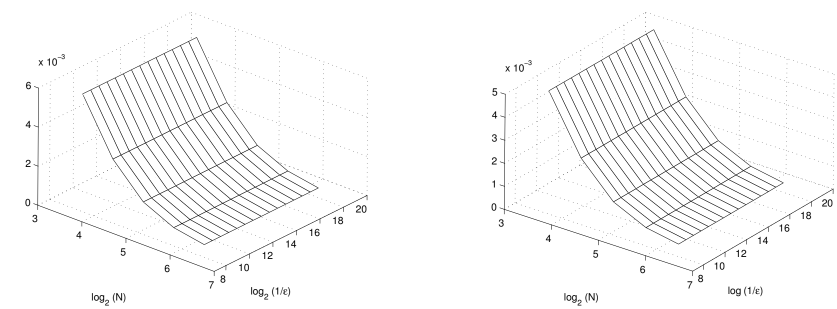

Figure 3. Surface plot of the maximum pointwise errors as a function of N and ε for the scaled discrete derivative components $\varepsilon D^0 Y_1$ and $\varepsilon D^0 Y_2$ using HDS - I for the Example 1.

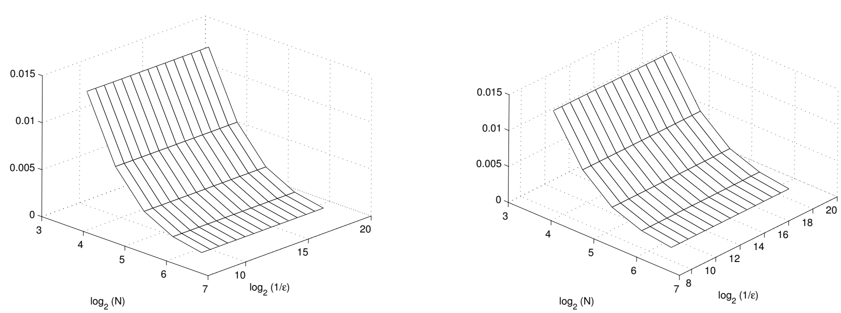

Figure 4. Surface plot of the maximum pointwise errors as a function of N and ε for the scaled discrete derivative components $\varepsilon D^0 Y_1$ and $\varepsilon D^0 Y_2$ using HDS - II for the Example 1.

are formed for each available value of N satisfying $N, 2N \in R_N$. Approximations to the $\varepsilon-$uniform order of local convergence are defined, for all $N, 4N \in$

Table 3. Values of D_1^N, p_1^N and D_2^N, p_2^N for the Example 1 in the coarse mesh region.

N	32	64	128	256	512
			Hybrid Difference Scheme - I		
D_1^N	3.6576e−2	2.2318e−2	1.2230e−2	6.1025e−3	2.7078e−3
p_1^N	7.1269e−1	8.6778e−1	1.0030	1.1723	–
D_2^N	6.3123e−2	4.5302e−2	2.6854e−2	1.3922e−2	6.2944e−3
p_2^N	4.7859e−1	7.5444e−1	9.4777e−1	1.1452	–
			Hybrid Difference Scheme - II		
D_1^N	3.6576e−2	2.2318e−2	1.2230e−2	6.1025e−3	2.7078e−3
p_1^N	7.1269e−1	8.6778e−1	1.0030	1.1723	–
D_2^N	6.3123e−2	4.5302e−2	2.6854e−2	1.3922e−2	6.2944e−3
p_2^N	4.7859e−1	7.5444e−1	9.4777e−1	1.1452	–

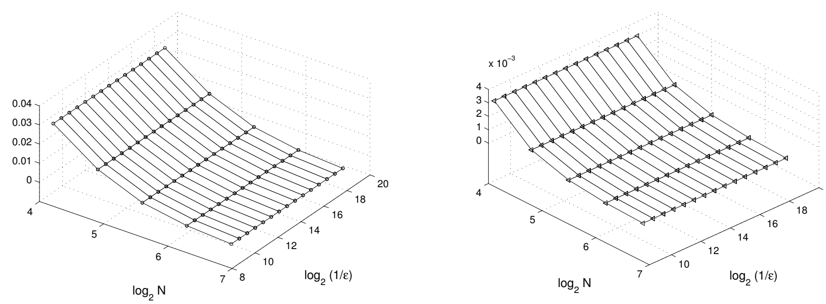

Figure 5. Surface plot of the maximum pointwise errors as a function of N and ε for the solution components Y_1 and Y_2 using HDS - I for the Example 2.

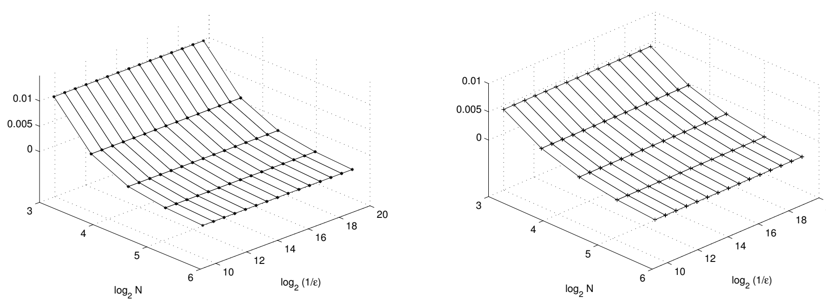

Figure 6. Surface plot of the maximum pointwise errors as a function of N and ε for the solution components Y_1 and Y_2 using HDS - II for the Example 2.

R_N, by

$$r_j^N = \log_2\left(\frac{S_j^N}{S_j^{2N}}\right), \qquad p_j^N = \log_2\left(\frac{D_j^N}{D_j^{2N}}\right), \quad j = 1, 2.$$

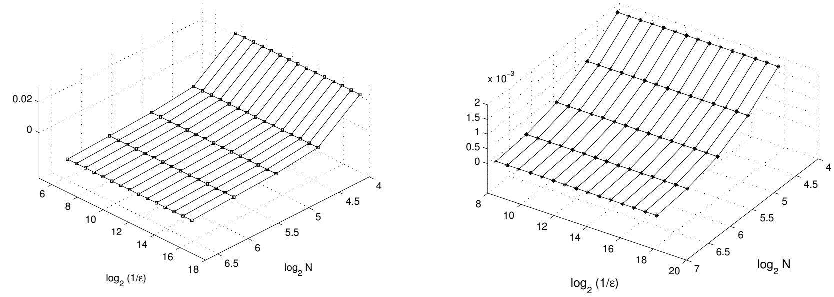

Figure 7. Surface plot of the maximum pointwise errors as a function of N and ε for the scaled discrete derivative components $\varepsilon D^0 Y_1$ and $\varepsilon D^0 Y_2$ using HDS - I for the Example 2.

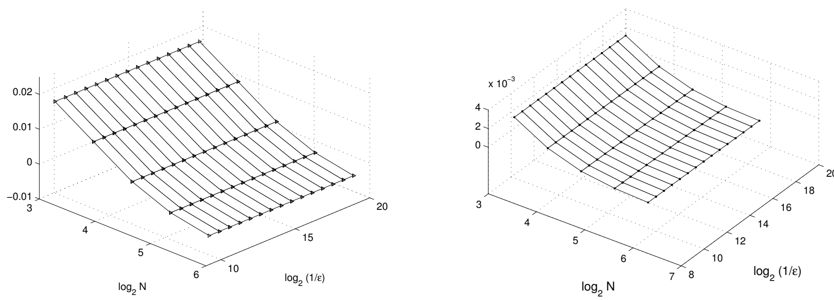

Figure 8. Surface plot of the maximum pointwise errors as a function of N and ε for the scaled discrete derivative components $\varepsilon D^0 Y_1$ and $\varepsilon D^0 Y_2$ using HDS - II for the Example 2.

Table 4. Values of S_1^N, r_1^N and S_2^N, r_2^N for the Example 2.

N	32	64	128	256	512
			Hybrid Difference Scheme - I		
S_1^N	3.1272e−2	1.5401e−2	6.1397e−3	2.1772e−3	7.2595e−4
r_1^N	1.0218	1.3268	1.4957	1.5845	–
S_2^N	3.5627e−3	1.3153e−3	4.6795e−4	1.5959e−4	5.1024e−5
r_2^N	1.4376	1.4910	1.5520	1.6451	–
			Hybrid Difference Scheme - II		
S_1^N	1.3661e−2	5.4383e−3	2.0449e−3	9.0192e−4	4.4958e−4
r_1^N	1.3288	1.4111	1.1810	1.0044	-
S_2^N	8.2171e−3	4.2905e−3	2.1935e−3	1.1092e−3	5.5774e−4
r_2^N	9.3748e−1	9.6791e−1	9.8372e−1	9.9185e−1	–

Surface plots of the maximum error for the solution as well as scaled first derivative of the above test problems are presented. In Figures 1, 2, 5, 6

Table 5. Values of D_1^N, p_1^N and D_2^N, p_2^N for the Example 2 in the fine mesh region.

N	32	64	128	256	512
Hybrid Difference Scheme - I					
D_1^N	2.7430e−2	6.5634e−3	3.4578e−3	2.1703e−3	9.5036e−4
p_1^N	2.0632	9.2459e−1	6.7196e−1	1.1913	–
D_2^N	1.9461e−3	1.0750e−3	4.7343e−4	1.8143e−4	6.3543e−5
p_2^N	8.5625e−1	1.1831	1.3837	1.5136	–
Hybrid Difference Scheme - II					
D_1^N	2.2139e−2	1.5070e−2	8.0334e−3	3.5627e−3	1.3950e−3
p_1^N	5.5491e−1	9.0760e−1	1.1730	1.3527	–
D_2^N	3.8384e−3	1.8550e−3	7.1561e−4	2.2432e−4	5.2987e−5
p_2^N	1.0491	1.3742	1.6736	2.0818	–

Table 6. Values of D_1^N, p_1^N and D_2^N, p_2^N for the Example 2 in the coarse mesh region.

N	32	64	128	256	512
Hybrid Difference Scheme - I					
D_1^N	7.4364e−3	3.8860e−3	1.9826e−3	1.0009e−3	5.0280e−4
p_1^N	9.3632e−1	9.7089e−1	9.8610e−1	9.9324e−1	–
D_2^N	2.5901e−2	1.4260e−2	7.4885e−3	3.8370e−3	1.9409e−3
p_2^N	8.6103e−1	9.2923e−1	9.6470e−1	9.8325e−1	–
Hybrid Difference Scheme - II					
D_1^N	5.0791e−3	2.4669e−3	1.2173e−3	6.0486e−4	3.0151e−4
p_1^N	1.0419	1.0190	1.0090	1.0044	–
D_2^N	4.8219e−2	2.4453e−2	1.2325e−2	6.1889e−3	3.1012e−3
p_2^N	9.7959e−1	9.8842e−1	9.9383e−1	9.9686e−1	–

and 3, 4, 7, 8 respectively we observe that as $\varepsilon \to 0$, the maximum error for the numerical approximation Y_1, Y_2 and $\varepsilon D^0 Y_1, \varepsilon D^0 Y_2$ to the exact solution y_1, y_2 and $\varepsilon y_1', \varepsilon y_2'$ respectively using HDS - I and HDS - II decreases and gets stabilized at a constant value. Tables 1 and 4 present $\varepsilon-$uniform maximum pointwise two-mesh difference and $\varepsilon-$uniform order of local convergence to the solution components Y_1 and Y_2 generated by HDS - I and HDS - II. Tables 2, 3, 5 and 6 present $\varepsilon-$uniform maximum pointwise two-mesh difference and $\varepsilon-$uniform order of local convergence to the scaled derivatives in the fine mesh region and the non scaled derivative in the coarse mesh region. From the tables, the performance of the two schemes appears to be almost the same but these two schemes are derived from different methods. It is expected that they may significantly differ for certain problems as the truncation error derived for

HDS - II is smaller than HDS - I

5 Conclusion

A weakly coupled system of two singularly perturbed convection–diffusion second order ordinary differential equations subject to mixed type boundary conditions was examined. Two hybrid difference schemes on the Shishkin mesh were constructed for solving this problem which generate $\varepsilon-$uniform convergent numerical approximations to the solution as well as to the scaled first derivative of the solution. Numerical results were presented, which are in agreement with the theoretical results. These schemes give better accuracy than the classical upwind scheme.

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