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Convergence of a Cyclic Algorithm for the Split Common Fixed Point Problem Without Continuity Assumption

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Abstract. In their recent paper (Math. Model. Anal., 17(4):457–466, 2012), Tang, Peng and Liu proposed a cyclic algorithm for solving the split common fixed point problem and established its weak convergence under some certain conditions. In this paper, we shall present a simple proof of such a result and moreover we shall remove one condition, continuity of the mapping involved, ensuring the convergence of the algorithm.

Keywords: directed mapping, split common fixed point problem, demicontractive mapping.

AMS Subject Classification: 49J53; 65K10.

1 Introduction

The *split feasibility problem* (SFP) [6] is formulated as finding

$$x \in C, \quad \text{s.t.} \quad Ax \in Q,$$

where C and Q are respectively closed convex subsets in Hilbert spaces H_1 and H_2, and $A : H_1 \to H_2$ is a bounded linear mapping. The SFP has been widely studied by many authors (see [3,8,11,13,14,15,16,17]), due to its various applications in the real word application [4,5]. An efficient algorithm for solving the SFP is Byrne's CQ algorithm: for any $x_0 \in H_1$ the CQ algorithm generates an iterative sequence as

$$x_{n+1} = P_C\big(I + \gamma A^*(P_Q - I)A\big)x_n,$$

where $0 < \gamma < 2/\|A\|^2$, and P_C denotes the projector onto C. It is known that the CQ algorithm converges weakly to a solution of the SFP if such a solution exists.

In the case whenever both C and Q consist of fixed point sets of some nonlinear mappings, the SFP is known as the two-sets split common fixed point problem (SCFP). More specifically, the two-sets SCFP requires to find

$$x \in \mathrm{Fix}(U), \quad \text{s.t.} \quad Ax \in \mathrm{Fix}(T), \tag{1.1}$$

where $\mathrm{Fix}(U)$ and $\mathrm{Fix}(T)$ stand for respectively the fixed point sets of $U : H_1 \to H_1$ and $T : H_2 \to H_2$. We note that to implement the CQ algorithm one has to calculate the metric projection at each iteration. However, it is hard to calculate the metric projection whenever the corresponding closed convex subset is fixed point set. Therefore the CQ algorithm does not work for the two-sets SCFP. Alternatively, Censor and Segal [7] introduced the following algorithm:

$$x_{n+1} = U\big(I - \gamma A^*(I - T)A\big)x_n, \tag{1.2}$$

to solve the two-sets SCFP for directed mappings. Subsequently, Moudafi [10] considered (1.1) for demicontractive mappings and proposed the following algorithm:

$$\begin{cases} u_n = x_n - \gamma A^*(I - T)Ax_n, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n U u_n. \end{cases} \tag{1.3}$$

It is known that demicontractive mappings properly include directed mappings. So in this sense, the Moudafi's algorithm (1.3) is an extension of algorithm (1.2).

Note that the two-sets SCFP is just a special case of the SCFP. More specifically, the general SCFP requires to find

$$x \in \bigcap_{i=1}^{p} \mathrm{Fix}(U_i), \quad \text{s.t.} \quad Ax \in \bigcap_{j=1}^{s} \mathrm{Fix}(T_j), \tag{1.4}$$

where $U_i : H_1 \to H_1$, $i = 1, \ldots, p$ and $T_j : H_2 \to H_2$, $j = 1, \ldots, s$ are two classes of nonlinear mappings. Recently Tang, Peng and Liu [12] considered the SCFP for demicontractive mappings and proposed a cyclic algorithm:

$$\begin{cases} u_n = x_n - \gamma A^*(I - T_{\mathrm{j}(n)})Ax_n, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n U_{\mathrm{i}(n)} u_n, \end{cases} \tag{1.5}$$

where $\mathrm{i}(n) := n \bmod p + 1$ and $\mathrm{j}(n) := n \bmod s + 1$. Clearly, the above algorithm is a further generalization of Moudafi's algorithm (1.3). Under some mild assumptions they established the weak convergence of their algorithm to a solution of the SCFP whenever such a solution exists.

We note that in [12] the continuity of the mappings U_i and T_j is one of conditions that ensures the convergence of algorithm (1.5). However the convergence of Moudafi's algorithm (1.3) does not need such a condition and more importantly many nonlinear mappings, such as directed and demicontractive mappings, are discontinuous in general [9]. In this short paper, we shall restate the weak convergence of algorithm (1.5) but we present a simple proof and moreover we can remove the continuity condition.

2 Preliminary and Notation

Throughout, let I denote the identity mapping, $\mathrm{Fix}(T)$ denote the set of the fixed points of an mapping T, and let $\omega_w(x_n)$ denote the set of weak cluster points of the sequence $\{x_n\}$. The notation "$\to$" stands for strong convergence and "$\rightharpoonup$" stands for weak convergence.

Let $T : H_1 \to H_1$ be a mapping with $\mathrm{Fix}(T) \neq \emptyset$. Then $I - T$ is called *demiclosed at zero* if $x_n \rightharpoonup x$, $(I-T)x_n \to 0 \Rightarrow x = Tx$. Let C be a closed convex nonempty subset and $\{x_n\}$ be a sequence in H_1. The sequence $\{x_n\}$ is called Fejér monotone with respect to C, if

$$\|x_{n+1} - c\| \leq \|x_n - c\|, \quad \forall c \in C.$$

Lemma 1 [Bauschke–Borwein [1]]. *If the sequence $\{x_n\}$ is Fejér monotone with respect to C, then $x_n \rightharpoonup x^* \in C$ if and only if $\omega_w(x_n) \subseteq C$.*

A mapping $T : H_1 \to H_1$ is called *τ-demicontractive* ($\tau < 1$) if

$$\|Tx - z\|^2 \leq \|x - z\|^2 + \tau\|x - Tx\|^2, \quad \forall x \in H_1,\ y \in \mathrm{Fix}(T)$$

or equivalently

$$\langle x - z, Tx - x\rangle \leq \frac{\tau - 1}{2}\|x - Tx\|^2, \quad \forall x \in H_1,\ y \in \mathrm{Fix}(T). \tag{2.1}$$

In particular, T is called *quasi-nonexpansive* if $\tau = 0$ and *directed if* $\tau = -1$ (cf. [2, 7, 9]).

Lemma 2. *Let $T : H_2 \to H_2$ be a τ-demicontractive mapping, $A : H_1 \to H_2$ be a linear bounded mapping and let $V_\lambda := I - \lambda A^*(I - T)A$ with $0 < \lambda < (1-\tau)/\|A\|^2$. Then*

$$\|V_\lambda x - z\|^2 \leq \|x - z\|^2 - \lambda\big(1 - \tau - \lambda\|A\|^2\big)\|Tx - x\|^2$$

for all $x \in H_1$ and all $z \in A^{-1}(\mathrm{Fix}(T)) = \{y\colon Ay \in \mathrm{Fix}(T)\}$.

Proof. Given $x \in H$ and $z \in A^{-1}(\mathrm{Fix}(T))$, we have

$$\|V_\lambda x - z\|^2 = \|x - z\|^2 - 2\lambda\big\langle x - z, A^*(I-T)Ax\big\rangle + \lambda^2\big\|A^*(I-T)Ax\big\|^2. \tag{2.2}$$

Since $Az \in \mathrm{Fix}(T)$ and

$$2\big\langle x - z, A^*(I-T)Ax\big\rangle = 2\big\langle Ax - Az, (I-T)Ax\big\rangle,$$

it follows from inequality (2.1) that

$$2\big\langle x - z, A^*(I-T)Ax\big\rangle \geq (1-\tau)\big\|(I-T)Ax\big\|^2.$$

Substituting this into (2.2) we have

$$\|V_\lambda x - z\|^2 \leq \|x - z\|^2 - \lambda(1-\tau)\big\|(I-T)Ax\big\|^2 + \lambda^2\big\|A^*(I-T)Ax\big\|^2.$$

Therefore the desired inequality follows from the fact that $\|A^*y\| \leq \|A\|\|y\|$, $\forall y \in H_2$. □

3 Weak Convergence Theorem

ASSUMPTION 1. We assume the following conditions on problem (1.4):

- The solution set to (1.4), denoted by $\mathcal{S}$, is nonempty;
- $I - U_i$, $i = 1, \dots, p$ and $I - T_j$, $j = 1, \dots, s$ are demiclosed at 0;
- U_i $(1 \le i \le p)$ is ν_i-demicontractive and T_j $(1 \le j \le s)$ is τ_i-demicontractive.

Let now $\nu := \max_{1 \le i \le p} \nu_i$ and $\tau := \max_{1 \le j \le s} \tau_j$. Clearly U_i is ν-demicontractive for all $1 \le i \le p$ and T_j is τ-demicontractive for all $1 \le j \le s$.

Theorem 1. *Let Assumption* 1 *be satisfied,* $\gamma \in (0, (1-\tau)/\|A\|^2)$ *and* $\{\alpha_n\} \subseteq [\epsilon, 1 - \nu - \epsilon]$ *with* ϵ *a sufficiently small number. If* $\mathcal{S} \neq \emptyset$*, then the sequence* $\{x_n\}$*, generated by* (1.5)*, converges weakly to some* $x^* \in \mathcal{S}$.

Proof. Take $z \in \mathcal{S}$. It then follows from inequality (2.1) that

$$\begin{aligned}\|u_n - z\|^2 &= \big\|x_n - \gamma A^*(I - T_{\mathrm{j}(n)})Ax_n - z\big\|^2 \\ &\le \|x_n - z\|^2 - \gamma\big(1 - \tau - \gamma\|A\|^2\big)\big\|(I - T_{\mathrm{j}(n)})Ax_n\big\|^2\end{aligned}$$

and also that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \big\|u_n - z + \alpha_n(U_{\mathrm{i}(n)}u_n - u_n)\big\|^2 \\ &= \|u_n - z\|^2 + 2\alpha_n\langle u_n - z, U_{\mathrm{i}(n)}u_n - u_n\rangle + \alpha_n^2\big\|(I - U_{\mathrm{i}(n)})u_n\big\|^2 \\ &\le \|u_n - z\|^2 - \alpha_n(1 - \nu - \alpha_n)\big\|(I - U_{\mathrm{i}(n)})u_n\big\|^2.\end{aligned}$$

Let

$$\mu := \min(\gamma(1 - \tau - \gamma\|A\|^2), \epsilon(1 - \nu - \epsilon)) > 0.$$

Combining the last two inequalities, we get

$$\|x_{n+1} - z\|^2 \le \|x_n - z\|^2 - \mu\Big(\big\|(I - U_{\mathrm{i}(n)})u_n\big\|^2 + \big\|(I - T_{\mathrm{j}(n)})Ax_n\big\|^2\Big).$$

Clearly $\{x_n\}$ is Fejér monotone with respect to $\mathcal{S}$, and moreover

$$\sum_{n=0}^{\infty}\big\|(I - U_{\mathrm{i}(n)})u_n\big\|^2 < \infty, \qquad \sum_{n=0}^{\infty}\big\|(I - T_{\mathrm{j}(n)})Ax_n\big\|^2 < \infty. \tag{3.1}$$

In view of Lemma 1, to finish the proof it remains to show that $\omega_w(x_n) \subseteq \mathcal{S}$. To see this let $\hat{x} \in \omega_w(x_n)$ and let an index $j \in \{1, 2, \dots, s\}$ be fixed. Noticing that the pool of indexes is finite, we can find a subsequence $\{x_{m_k}\}$ of $\{x_n\}$ such that it converges weakly to $\hat{x}$ and $\mathrm{j}(m_k) = j$ for all k. Since, by weak continuity of A, Ax_{m_k} converges weakly to $A\hat{x}$ and

$$\big\|(I - T_j)Ax_{m_k}\big\| = \big\|(I - T_{\mathrm{j}(m_k)})Ax_{m_k}\big\| \to 0,$$

this together with the demiclosedness of $I - T_j$ at zero yields $A\hat{x} \in \mathrm{Fix}(T_j)$. Now let an index $i \in \{1, 2, \dots, p\}$ be fixed. Similarly we can find a subsequence

$\{x_{p_k}\}$ of $\{x_n\}$ such that it converges weakly to $\hat{x}$ and $\mathrm{i}(p_k) = i$ for all k. Noting $\|(I - T_{\mathrm{j}(n)})Ax_n\| \to 0$ thanks to (3.1) and by definition of u_n, we have

$$\|x_n - u_n\| \le \gamma \|A\| \big\|(I - T_{\mathrm{j}(n)})Ax_n\big\| \to 0,$$

and thus u_{p_k} converges weakly to $\hat{x}$. Since by (3.1) $\|(I - U_{\mathrm{i}(n)})u_n\| \to 0$, the demiclosedness of $I - U_i$ at zero yields $\hat{x} \in \mathrm{Fix}(U_i)$. Altogether $\hat{x} \in \mathcal{S}$, and therefore the proof is complete. □

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