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On Order Statistics from the Gompertz–Makeham Distribution and the Lambert W Function*

Pedro Jodrá

Universidad de Zaragoza
María de Luna 3, 50018 Zaragoza, Spain
E-mail: pjodra@unizar.es

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Abstract. The aim of this paper is twofold. First, we show that the expected value of the minimum order statistic from the Gompertz–Makeham distribution can be expressed in closed form in terms of the incomplete gamma function. We also give a general formula for the moments of the minimum order statistic in terms of the generalized integro-exponential function. As a consequence, the moments of all order statistics from this probability distribution can be more easily evaluated from the moments of the minimum order statistic. Second, we show that the maximum and minimum order statistics from the Gompertz–Makeham distribution are in the domains of attraction of the Gumbel and Weibull distributions, respectively. Lambert W function plays an important role in solving these problems.

Keywords: Gompertz–Makeham distribution, order statistics, moments, domains of attraction, Lambert W function.

AMS Subject Classification: 33B30; 60E05; 33F10.

1 Introduction

The Gompertz–Makeham distribution was introduced by the British actuary William M. Makeham [16] in the second half of the 19th century as an extension of the Gompertz distribution. Since then, this probability model has been successfully used in actuarial science, biology and demography to describe mortality patterns in numerous species, including humans. Marshall and Olkin [17, Chap. 10] provides a comprehensive review of the history and theory of this probability distribution and its practical importance is strongly highlighted in Golubev [9]. Some more recent mathematical contributions can also be found in Feng et al. [7], Jodrá [12], Lagerås [14] and Teimouri and Gupta [20].

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To be more precise, let X be a continuous random variable having a Gompertz–Makeham distribution with positive real parameters α, β and λ, that is, the cumulative distribution function $F(x;\alpha,\beta,\lambda) := P(X \le x)$ is given by

$$F(x;\alpha,\beta,\lambda) = 1 - \exp\bigl\{-\lambda x - (\alpha/\beta)\bigl(e^{\beta x} - 1\bigr)\bigr\}, \quad x > 0,\ \alpha,\beta,\lambda > 0. \quad (1.1)$$

The parameters α and β are usually interpreted as the initial mortality and the mortality increase when advancing age, respectively, whereas the parameter λ represents the risk of death due to causes which do not depend on age such as accidents and acute infections, among others.

Jodrá [12, Theorem 2] expresses the inverse of the cumulative distribution function of the Gompertz–Makeham distribution in closed form, more specifically,

$$F^{-1}(u) = \frac{\alpha}{\beta\lambda} - \frac{1}{\lambda}\log(1-u) - \frac{1}{\beta}\mathrm{W}_0\left(\frac{\alpha}{\lambda}\, e^{\alpha/\lambda}(1-u)^{-\beta/\lambda}\right), \quad 0 < u < 1, \ (1.2)$$

where log denotes the natural logarithm and W_0 denotes the principal branch of the Lambert W function, which is briefly described in the following paragraph. Throughout this paper, for notational simplicity we use $F^{-1}(u)$ instead of the more cumbersome notation $F^{-1}(u;\alpha,\beta,\lambda)$.

For the sake of completeness, we recall that the Lambert W function is a multivalued complex function whose defining equation is $\mathrm{W}(z)e^{\mathrm{W}(z)} = z$, where z is a complex number. This function has attracted a great deal of attention after the seminal paper by Corless et al. [6], which summarizes a wide variety of applications in mathematics and physics together with the main properties of W. In this regard, if z is a real number such that $z \ge -1/e$ then the Lambert W function has only two real branches. The real branch taking on values in $[-1,\infty)$ (resp. $(-\infty,-1]$) is called the principal (resp. negative) branch and denoted in the literature by W_0 (resp. W_{-1}). Both real branches are depicted in Figure 1. In this paper we use only the principal branch W_0 and it is known that $\mathrm{W}_0(z)$ is increasing as z increases, $\mathrm{W}_0(-1/e) = -1$, $\mathrm{W}_0(0) = 0$, $\mathrm{W}_0(ze^z) = z$ and, in addition, the first derivative of W_0 is given by

$$\mathrm{W}_0'(z) := \frac{d\mathrm{W}_0(z)}{dz} = \frac{\mathrm{W}_0(z)}{z(1+\mathrm{W}_0(z))}, \quad \text{for } z \ne 0. \quad (1.3)$$

This paper is mainly motivated by the observation in Marshall and Olkin [17, p. 380] that the moments of the Gompertz–Makeham distribution, that is, $E[X^k] := \int_0^\infty x^k\, dF(x;\alpha,\beta,\lambda)$, $k = 1,2,\ldots$, cannot be given in closed form. Here, we show that the expected value of the minimum order statistic from the Gompertz–Makeham distribution can be expressed in closed form in terms of the incomplete gamma function. This result can be derived using Eq. (1.2) in order to highlight the importance of the Lambert W function. We also see that the kth moment of the minimum order statistic can be expressed explicitly in terms of the generalized integro-exponential function for any $k = 1,2\ldots$. Then, as a particular case, we have a closed-form expression for the moments of the Gompertz–Makeham distribution, $E[X^k]$, for any $k = 1,2,\ldots$. Additionally, using Eq. (1.2), we also determine the limit distributions of the maximum and minimum order statistics from this probability distribution.

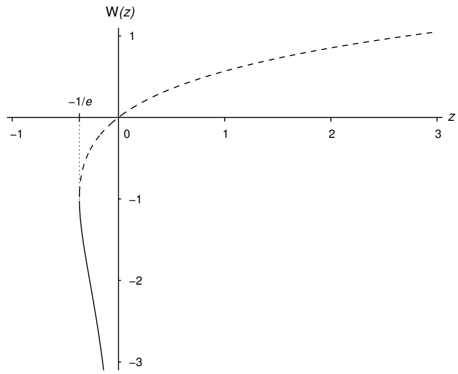

Figure 1. The two real branches of W(z): ——, $\mathrm{W}_{-1}(z)$; − − −, $\mathrm{W}_0(z)$.

The remainder of this paper is organized as follows. In Section 2, we provide a closed-form expression for an integral involving the principal branch of the Lambert W function, specifically, $\int_0^1 x^p \mathrm{W}_0(s/x^q)\,dx$, where p, q and s are positive real numbers. In Section 3, we see that the above integral arises when we compute the expected value of the minimum order statistic from the Gompertz–Makeham distribution and, as a consequence, the expected value can be expressed in closed form in terms of the incomplete gamma function. In this section, we also see that the moments of the minimum order statistic can be expressed in terms of the generalized integro-exponential function. These explicit expressions corresponding to the minimum are useful to evaluate the moments of all order statistics from the Gompertz–Makeham distribution. A numerical example illustrates the results. Finally, in Section 4, we identify the asymptotic behavior of the maximum and minimum order statistics from the Gompertz–Makeham distribution. More precisely, we establish that the domains of attraction corresponding to the maximum and minimum order statistics are the Gumbel and Weibull distributions, respectively.

2 An Integral Involving the Principal Branch of the Lambert W Function

In the following result, we provide a closed-form expression for an integral involving the principal branch of the Lambert W function. Denote by $\Gamma(a,z)$ the (upper) incomplete gamma function (cf. Abramowitz and Stegun [1, p. 260]), that is,

$$\Gamma(a,z) := \int_z^\infty t^{a-1}e^{-t}\,dt, \quad z \in \mathbb{C}\setminus\mathbb{R}^-,\ a \in \mathbb{C}. \tag{2.1}$$

Lemma 1. *For any positive real number q, we have*

$$\int_0^1 \mathrm{W}_0\big(1/x^q\big)\,dx = -q^{-1/q}\,\Gamma\Big(-\frac{1}{q}, \frac{\mathrm{W}_0(1)}{q}\Big) + \mathrm{W}_0(1) + q. \tag{2.2}$$

Proof. First, we make the change of variable $u = 1/x^q$ in the integral in Eq. (2.2) and we obtain

$$\int_0^1 \mathrm{W}_0\big(1/x^q\big)\,dx = \frac{1}{q}\int_1^\infty u^{-(q+1)/q}\mathrm{W}_0(u)\,du.$$

By setting $w = \mathrm{W}_0(u)$ on the right-hand side in the above equation, which implies $u = w\,e^w$ and also $du = (1+w)e^w\,dw$ by virtue of Eq. (1.3), we get

$$\int_1^\infty u^{-(q+1)/q}\mathrm{W}_0(u)\,du = \int_{\mathrm{W}_0(1)}^\infty (1+w)w^{-1/q}e^{-w/q}\,dw. \tag{2.3}$$

Now, on the right-hand side in Eq. (2.3) we take into account Eq. (2.1) together with the fact that $\Gamma(a,\infty) = 0$, and thus we obtain

$$\begin{aligned}&\int_{\mathrm{W}_0(1)}^\infty (1+w)w^{-1/q}e^{-w/q}\,dw\\ &\quad= q^{(q-1)/q}\left\{q\,\Gamma\left(2-\frac{1}{q},\frac{\mathrm{W}_0(1)}{q}\right) + \Gamma\left(1-\frac{1}{q},\frac{\mathrm{W}_0(1)}{q}\right)\right\}.\end{aligned} \tag{2.4}$$

It is well-known that the incomplete gamma function satisfies the recurrence formula $\Gamma(a+1,z) = a\Gamma(a,z) + z^a e^{-z}$ (see Abramowitz and Stegun [1, pp. 260–262]). According to this, Eq. (2.4) becomes

$$\begin{aligned}&\int_{\mathrm{W}_0(1)}^\infty (1+w)w^{-1/q}e^{-w/q}\,dw\\ &\quad= -q^{(q-1)/q}\,\Gamma\left(-\frac{1}{q},\frac{\mathrm{W}_0(1)}{q}\right) + q\big(\mathrm{W}_0(1)+q\big)\big(\mathrm{W}_0(1)e^{\mathrm{W}_0(1)}\big)^{-1/q}\\ &\quad= -q^{(q-1)/q}\,\Gamma\left(-\frac{1}{q},\frac{\mathrm{W}_0(1)}{q}\right) + q\big(\mathrm{W}_0(1)+q\big),\end{aligned}$$

where in the last equality we have used the fact that $\mathrm{W}_0(z)\,e^{\mathrm{W}_0(z)} = z$. This completes the proof. □

Essentially the same reasoning as in Lemma 1 leads to the result stated in Lemma 2 below, so we omit the proof here.

Lemma 2. *For any positive real numbers p, q and s, we have*

$$\begin{aligned}\int_0^1 x^p\mathrm{W}_0(s/x^q)\,dx = \frac{1}{1+p}\Bigg\{&-\left(\frac{s(1+p)}{q}\right)^{(1+p)/q}\Gamma\left(-\frac{1+p}{q},\frac{1+p}{q}\,\mathrm{W}_0(s)\right)\\ &+\mathrm{W}_0(s) + \frac{q}{(1+p)}\Bigg\}.\end{aligned}$$

In the next section, we apply Lemma 2 to obtain a closed-form expression for the expected value of the minimum order statistic from the Gompertz–Makeham distribution.

3 Expected Value of Order Statistics from the Gompertz – Makeham Distribution

We first introduce some notation and terminology. Denote by $\mathbb{N}$ the set of non-negative integers and by $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. For any $n \in \mathbb{N}^*$, let $X_1, \ldots, X_n$ be a random sample of size n from the Gompertz–Makeham distribution with parameters α, β and λ, and let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the corresponding order statistics obtained by arranging X_i, $i = 1, \ldots, n$, in non-decreasing order of magnitude. The rth element of this sequence, $X_{r:n}$, is called the rth order statistic and, in particular, the minimum and maximum order statistics, that is, $X_{1:n} = \min\{X_1, \ldots, X_n\}$ and $X_{n:n} = \max\{X_1, \ldots, X_n\}$, the so-called extremes, are very important in practical applications.

For any $n \in \mathbb{N}^*$ and $k \in \mathbb{N}^*$, it is known that the kth moment of $X_{r:n}$, $r = 1, \ldots, n$, can be computed as follows (cf. Balakrishnan and Rao [3, p. 7])

$$E\big[X_{r:n}^k\big] = r\binom{n}{r} \int_0^\infty x^k \big(F(x;\alpha,\beta,\lambda)\big)^{r-1}\big(1 - F(x;\alpha,\beta,\lambda)\big)^{n-r}\, dF(x;\alpha,\beta,\lambda), \tag{3.1}$$

where F is given by Eq. (1.1). For the Gompertz–Makeham distribution, even the problem of computing the expected value $E[X_{r:n}]$ by direct numerical integration of Eq. (3.1) is not efficient from a computational viewpoint, and, in particular, the problem becomes intractable when the sample size n is relatively large (see the example at the end of this section).

Our first aim in this section is to compute the expected value $E[X_{r:n}]$ in a more efficient manner, that is, avoiding the numerical integration of Eq. (3.1), and, to this end, the minimum order statistic $X_{1:n}$ plays a crucial role. More precisely, the expected value of $X_{1:n}$ can also be obtained by means of the following formula involving F^{-1} (cf. Arnold et al. [2, Chapter 5] for further details)

$$E[X_{1:n}] = n\int_0^1 (1-u)^{n-1} F^{-1}(u)\, du, \quad n \in \mathbb{N}^*, \tag{3.2}$$

where in our case F^{-1} is given by Eq. (1.2). By using Eq. (3.2) together with Lemma 2, in the following result we give an analytical expression for the expected value of the minimum order statistic $X_{1:n}$, for $n = 1, 2, \ldots$. The special case $n = 1$ corresponds to the expected value of X.

Corollary 1. Let $X_1, \ldots, X_n$ be a random sample of size n from a Gompertz–Makeham distribution with parameters α, β and λ. Then, the expected value of $X_{1:n}$ is given by

$$E[X_{1:n}] = \frac{e^{n\alpha/\beta}}{\beta}\left(\frac{n\alpha}{\beta}\right)^{n\lambda/\beta} \Gamma\left(-\frac{n\lambda}{\beta}, \frac{n\alpha}{\beta}\right), \quad n \in \mathbb{N}^*.$$

Proof. For any $n \in \mathbb{N}^*$, from Eq. (3.2) together with Eq. (1.2) we have

$$E[X_{1:n}] = \frac{\alpha}{\beta\lambda} + \frac{1}{n\lambda} - \frac{n}{\beta}\int_0^1 w^{n-1} \mathrm{W}_0\left(\frac{\alpha}{\lambda}\, e^{\alpha/\lambda} w^{-\beta/\lambda}\right) dw.$$

Now, by taking into account Lemma 2 in the above equation, we get

$$\begin{aligned} E[X_{1:n}] &= \frac{\alpha}{\beta\lambda} - \frac{1}{\beta}\,\mathrm{W}_0\left(\frac{\alpha}{\lambda}\,e^{\alpha/\lambda}\right) \\ &\quad + \frac{1}{\beta}\left(\frac{n\alpha}{\beta}\,e^{\alpha/\lambda}\right)^{n\lambda/\beta}\Gamma\left(-\frac{n\lambda}{\beta}, \frac{n\lambda}{\beta}\,\mathrm{W}_0\left(\frac{\alpha}{\lambda}\,e^{\alpha/\lambda}\right)\right) \\ &= \frac{e^{n\alpha/\beta}}{\beta}\left(\frac{n\alpha}{\beta}\right)^{n\lambda/\beta}\Gamma\left(-\frac{n\lambda}{\beta}, \frac{n\alpha}{\beta}\right), \end{aligned}$$

where in the last equality we have used the fact that $\mathrm{W}_0(ze^z) = z$. The proof is completed. □

Here, we point out the importance of the closed-form expression given in Corollary 1. On the one hand, from a theoretical point of view, the expected value of the minimum order statistic can be used to determine if two distributions with finite expected values are identical (cf. Chan [5] and also Huang [11]). On the other hand, from a numerical point of view, in order to calculate $E[X_{1:n}]$ by virtue of Corollary 1, computer algebra systems such as Maple or Mathematica, among others, can be used to directly evaluate the incomplete gamma function. Moreover, the analytical expression of $E[X_{1:n}]$ in Corollary 1 can also be used to compute $E[X_{r:n}]$, for $r = 2, \ldots, n$, avoiding the numerical integration of Eq. (3.1). To this end, for any $n \in \mathbb{N}^*$ and $k \in \mathbb{N}^*$ we can use the following relation (cf. Balakrishnan and Rao [3, p. 156])

$$E\left[X_{r:n}^k\right] = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)}\binom{n}{j}\binom{j-1}{n-r}E\left[X_{1:j}^k\right], \quad r = 2, \ldots, n. \tag{3.3}$$

By setting $k = 1$ in Eq. (3.3) and in view of Corollary 1, we get the following result.

Corollary 2. Let $X_1, \ldots, X_n$ be a random sample of size n from a Gompertz–Makeham distribution with parameters α, β and λ. Then, for $r = 2, \ldots, n$, we have

$$E[X_{r:n}] = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)}\binom{n}{j}\binom{j-1}{n-r}\frac{e^{j\alpha/\beta}}{\beta}\left(\frac{j\alpha}{\beta}\right)^{j\lambda/\beta}\Gamma\left(-\frac{j\lambda}{\beta}, \frac{j\alpha}{\beta}\right).$$

Unfortunately, we have not found simple analytical expressions for the moments $E[X_{1:n}^k]$ for integers $k \geq 2$. However, we provide a general formula for the moments $E[X_{1:n}^k]$, which can be used for computing these moments for integers $k \geq 2$ in a more efficient manner than using Eq. (3.1). We first give the following preliminary result.

Lemma 3. *Let $X_1, \ldots, X_n$ be a random sample of size n from a Gompertz–Makeham distribution with parameters α, β and λ. Then, the minimum order statistic $X_{1:n}$ has a Gompertz–Makeham distribution with parameters $n\alpha$, β and $n\lambda$.*

Proof. For any $n \in \mathbb{N}^*$, recall that the cumulative distribution function of $X_{1:n}$, $F_{1:n}(x;\alpha,\beta,\lambda) := P(X_{1:n} \le x)$, can be computed as follows

$$F_{1:n}(x;\alpha,\beta,\lambda) = 1 - \big(1 - F(x;\alpha,\beta,\lambda)\big)^n, \quad x > 0.$$

Then, by using Eq. (1.1) in the above formula we obtain

$$F_{1:n}(x;\alpha,\beta,\lambda) = 1 - \exp\big\{-n\lambda x - (n\alpha/\beta)\big(e^{\beta x} - 1\big)\big\}, \quad x > 0,$$

that is, $F_{1:n}(x;\alpha,\beta,\lambda) = F(x;n\alpha,\beta,n\lambda)$ which implies the result. □

Now, we are in a position to express the moments $E[X_{1:n}^k]$ in terms of the generalized integro-exponential function. We recall that the generalized integro-exponential function can be defined by the following integral representation (cf. Milgram [18] for further details)

$$E_s^m(z) := \frac{1}{\Gamma(m+1)} \int_1^\infty (\log u)^m u^{-s} e^{-zu}\, du, \quad z > 0, \tag{3.4}$$

where $m = 0, 1, \ldots$ and s is a real number.

Proposition 1. *Let $X_1, \ldots, X_n$ be a random sample of size n from a Gompertz – Makeham distribution with parameters α, β and λ. Then, for the minimum order statistic $X_{1:n}$ we have*

$$E\big[X_{1:n}^k\big] = \frac{\Gamma(k+1) e^{n\alpha/\beta}}{\beta^k}\, E_{(\frac{n\lambda}{\beta}+1)}^{k-1}\left(\frac{n\alpha}{\beta}\right), \quad k \in \mathbb{N}^*. \tag{3.5}$$

Proof. By virtue of Lemma 3, together with Eq. (1.1), we get

$$\begin{aligned} E\big[X_{1:n}^k\big] &= \int_0^\infty x^k\, dF(x;n\alpha,\beta,n\lambda) \\ &= \int_0^\infty x^k \big(n\lambda + n\alpha e^{\beta x}\big) \exp\big\{-n\lambda x - (n\alpha/\beta)\big(e^{\beta x} - 1\big)\big\}\, dx. \end{aligned}$$

Now, the change of variable $u = e^{\beta x}$ leads to the following

$$E\big[X_{1:n}^k\big] = \frac{e^{n\alpha/\beta}}{\beta^k} \int_1^\infty \left(\frac{n\lambda}{\beta} + \frac{n\alpha}{\beta} u\right) (\log u)^k e^{-n\alpha u/\beta} u^{-(n\lambda/\beta)-1}\, du.$$

From the above equation and the definition in Eq. (3.4), for any $k \in \mathbb{N}^*$ we have

$$E\big[X_{1:n}^k\big] = \frac{\Gamma(k+1) e^{n\alpha/\beta}}{\beta^k} \left(\frac{n\lambda}{\beta}\, E_{(\frac{n\lambda}{\beta}+1)}^{k}\left(\frac{n\alpha}{\beta}\right) + \frac{n\alpha}{\beta}\, E_{\frac{n\lambda}{\beta}}^{k}\left(\frac{n\alpha}{\beta}\right)\right).$$

Finally, by taking into account the following recursion formula (cf. Milgram [18, Eq. (2.4)])

$$(1-s) E_s^m(z) = z E_{s-1}^m(z) - E_s^{m-1}(z), \quad z > 0,\ s \ne 1,\ m = 0, 1, \ldots,$$

defining $E_s^{-1}(z) := e^{-z}$, we achieve the desired result. □

It is interesting to note that if we take $k = 1$ in Proposition 1, and using the fact that $E_s^0(z) = z^{s-1}\Gamma(1-s, z)$ (cf. Milgram [18, Eq. (2.2)]), we have an alternative derivation of Corollary 1. Moreover, it is clear that if we take $n = 1$ in Proposition 1, then we have an explicit expression for the moments of the Gompertz–Makeham distribution, $E[X^k]$, for any $k = 1, 2, \ldots$. We highlight that Lenart [15] has recently given explicit expressions for the moments of the Gompertz distribution in terms of the generalized integro-exponential function. In fact, if we take $n = 1$ and $\lambda = 0$ in Eq. (3.5), then we obtain the same expression provided by Lenart for the moments of the Gompertz distribution.

As a consequence of Proposition 1 together with Eq. (3.3), we give a general formula for the moments of order statistics from the Gompertz–Makeham distribution, as stated below.

Corollary 3. Let $X_1, \ldots, X_n$ be a random sample of size n from a Gompertz–Makeham distribution with parameters α, β and λ. Then, for any $k \in \mathbb{N}^*$ and for $r = 2, \ldots, n$, we have

$$E\left[X_{r:n}^k\right] = \frac{\Gamma(k+1)}{\beta^k} \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{n}{j} \binom{j-1}{n-r} e^{j\alpha/\beta} E^{k-1}_{(\frac{j\lambda}{\beta}+1)}\left(\frac{j\alpha}{\beta}\right).$$

To complete this section, we provide an example in which we compute the two first moments corresponding to order statistics from the Gompertz–Makeham distribution. Unless differently specified, all of the computations in the example below were performed with the software Mathematica 8.0, on an Intel Core2 Quad Q8200 at 2.33GHz with 4GB RAM.

Numerical Example. Promislow and Haselkorn [19] have studied the aging process for different species of flies. In particular, it is considered a sample of size $n = 990$ from a Gompertz–Makeham distribution X with parameters $\alpha = 0.00049$, $\beta = 0.071$ and $\lambda = 0.00092$. In this case, the random variable X represents the lifetime (in days) of females of *Drosophila melanogaster* (fruit fly).

With the help of the computer algebra system Mathematica and applying Corollary 1, in a preprocessing step we have first calculated and stored the expected values $E[X_{1:j}]$, for $j = 1, \ldots, 990$. Then, from Corollary 2 and taking into account the stored values $E[X_{1:j}]$, we have obtained $E[X_{r:990}]$, for $r = 2, \ldots, 990$. Table 1 below displays the results for several values of r. We remark that the expected values $E[X_{r:990}]$ could not be obtained by numerical integration of Eq. (3.1) for integers $r \geq 275$. We also note that these computational results could not be improved using the software package Maple.

Finally, in Table 1, we also show several values of $E[X_{r:990}^2]$ together with the standard deviation of $X_{r:990}$, that is, $\mathrm{Stdv}(X_{r:990}) := (E[X_{r:900}^2] - E^2[X_{r:900}])^{1/2}$. Based on Proposition 1, in a preprocessing step we have computed and stored the values $E[X_{1:j}^2]$, for $j = 1, \ldots, 990$, and, then, the values of $E[X_{r:990}^2]$ have been calculated by means of Corollary 3.

Table 1. The two first moments of $X_{r:990}$ together with the standard deviation.

r	$E[X_{r:990}]$	$E[X^2_{r:990}]$	$\mathrm{Stdv}(X_{r:990})$
1	0.7037690414	0.9732338953537	0.4779430317
100	34.671246464	1204.6301869524	1.5921231108
200	46.617592929	2174.4393369591	1.1132681618
300	53.923823454	2908.5744034664	0.8920022421
400	59.414516903	3530.6685856898	0.7640471189
500	64.018193614	4098.7930277133	0.6811130596
600	68.189517484	4650.2009359519	0.6250135998
700	72.248250231	5220.1564414941	0.5888802934
800	76.560537704	5861.8453664336	0.5739625423
900	81.951502261	6716.4130816494	0.6036215701
990	98.064118070	9621.9189792750	2.3125150810

4 Domains of Attraction of the Extreme Order Statistics

Let $X_1, \ldots, X_n$ be a random sample of size n from a Gompertz–Makeham distribution with parameters α, β and λ. Let us consider the maximum and minimum order statistics, $X_{n:n}$ and $X_{1:n}$, respectively, and denote by $F_{n:n}$ and $F_{1:n}$ the corresponding cumulative distribution functions. It is known that $F_{n:n}(x) = F^n(x)$ and $F_{1:n}(x) = 1 - (1 - F(x))^n$, $x > 0$, where F is given by Eq. (1.1). For notational simplicity, in this section we use $F(x)$ instead of $F(x; \alpha, \beta, \lambda)$ (similarly for $F_{1:n}$ and $F_{n:n}$).

In order to identify the asymptotic distributions of $X_{n:n}$ and $X_{1:n}$ when the sample size n increases to ∞, it is well-known that the limits of $F_{n:n}$ and $F_{1:n}$ take only values 0 and 1, which means that the limit distributions are degenerate. To avoid degeneracy, it is necessary to look for linear transformations to find the asymptotic non-degenerate distributions. If there exist non-degenerate cumulative distribution functions H and L such that

$$\lim_{n\to\infty} F^n(a_n + b_n x) = H(x), \quad \lim_{n\to\infty} 1 - \big(1 - F(c_n + d_n x)\big)^n = L(x), \quad (4.1)$$

where a_n, $b_n > 0$, c_n and $d_n > 0$ are normalization constants depending on the sample size n, then it is said that F belongs to the maximum –resp. minimum– domain of attraction of the limit distribution H –resp. L–. Moreover, it is well-established in the statistical literature that the limit distributions H and L can only be either a Fréchet, a Gumbel or a Weibull distribution. For further details on the asymptotic theory of extremes see, for instance, Castillo at al. [4, Chap. 9], Kotz and Nadarajah [13, Chap. 1] and also Galambos [8].

In this section, we determine the limit distribution of the extremes of the Gompertz–Makeham distribution, that is, we identify H and L in Eqs. (4.1) together with the normalizing constants. With the previous notations, below we state the result and the proof is given in the remainder of this section.

Theorem 1. *The Gompertz–Makeham distribution belongs to*:

(i) *the Gumbel maximum domain of attraction,*

(ii) *the Weibull minimum domain of attraction.*

The normalization constants in Eqs. (4.1)

$$a_n = F^{-1}\Big(1-\frac{1}{n}\Big), \quad b_n = F^{-1}\Big(1-\frac{1}{ne}\Big) - F^{-1}\Big(1-\frac{1}{n}\Big) \tag{4.2}$$

$$c_n = 0, \quad d_n = F^{-1}(1/n), \tag{4.3}$$

where F^{-1} *is given by Eq.* (1.2).

The proof of Theorem 1 is based on applying Theorems 9.5 and 9.6 given in Castillo et al. [4, pp. 203–205]. Both theorems require that the inverse of the cumulative distribution function corresponding to the continuous random variable involved can be expressed analytically and, by virtue of Eq. (1.2), this is the case for the Gompertz–Makeham distribution.

Before proceeding with the proof of Theorem 1, we need some auxiliary results concerning asymptotic properties of the principal branch of the Lambert W function. Denote by

$$L_{\mathrm{W}_0}(x) := \log x - \log\log x + \frac{1}{2}\frac{\log\log x}{\log x}, \quad x \ge e,$$

$$U_{\mathrm{W}_0}(x) := \log x - \log\log x + \frac{e}{e-1}\frac{\log\log x}{\log x}, \quad x \ge e,$$

where log denotes the natural logarithm. Hoorfar and Hassani [10, Theorem 2.7] have shown that $L_{\mathrm{W}_0}(x)$ and $U_{\mathrm{W}_0}(x)$ are bounds for $\mathrm{W}_0(x)$, more specifically,

$$L_{\mathrm{W}_0}(x) \le \mathrm{W}_0(x) \le U_{\mathrm{W}_0}(x), \quad x \ge e, \tag{4.4}$$

where the equality is attained only in the case $x = e$. Now, we establish the following results.

Lemma 4. *For any* $a > 0$ *and* $b > 0$*, we have*

$$\lim_{x\to\infty}\big(\mathrm{W}_0(ax) - \mathrm{W}_0(bx)\big) = \log(a/b).$$

Proof. For any positive real numbers a and b, from Eq. (4.4) we can bound the difference $\mathrm{W}_0(ax) - \mathrm{W}_0(bx)$ as follows

$$L_{\mathrm{W}_0}(ax) - U_{\mathrm{W}_0}(bx) \le \mathrm{W}_0(ax) - \mathrm{W}_0(bx) \le U_{\mathrm{W}_0}(ax) - L_{\mathrm{W}_0}(bx), \tag{4.5}$$

which holds for $x \ge \max\{e/a, e/b\}$. Now, it can be checked that the upper bound in Eq. (4.5) can be written as below

$$U_{\mathrm{W}_0}(ax) - L_{\mathrm{W}_0}(bx) = \log\Big(\frac{a}{b}\Big) + \log\Big(\frac{\log(bx)}{\log(ax)}\Big) + \frac{e}{e-1}\frac{\log\log(ax)}{\log(ax)} - \frac{1}{2}\frac{\log\log(bx)}{\log(bx)}$$

where $x \ge \max\{e/a, e/b\}$. Then, from the above equality it can be checked that

$$\lim_{x\to\infty}\big(U_{\mathrm{W}_0}(ax) - L_{\mathrm{W}_0}(bx)\big) = \log(a/b).$$

By similar arguments, for the lower bound in Eq. (4.5) we obtain the following

$$\lim_{x\to\infty}\big(L_{\mathrm{W}_0}(ax) - U_{\mathrm{W}_0}(bx)\big) = \log(a/b).$$

As a consequence, the statement of this lemma is obtained by applying the squeeze theorem to Eq. (4.5). □

Lemma 5. *For any* $a > 0$, *we have*

$$\lim_{x\to\infty} \frac{1 + \mathrm{W}_0(ax)}{1 + \mathrm{W}_0(x)} = 1.$$

Proof. For any positive real number a, we can bound $(1+\mathrm{W}_0(ax))/(1+\mathrm{W}_0(x))$ by using Eq. (4.4) as follows

$$\frac{1 + L_{\mathrm{W}_0}(ax)}{1 + U_{\mathrm{W}_0}(x)} \le \frac{1 + \mathrm{W}_0(ax)}{1 + \mathrm{W}_0(x)} \le \frac{1 + U_{\mathrm{W}_0}(ax)}{1 + L_{\mathrm{W}_0}(x)}, \quad x \ge e/a. \tag{4.6}$$

It can be seen that $\lim_{x\to\infty} U_{\mathrm{W}_0}(ax) = \infty$ and also that $\lim_{x\to\infty} L_{\mathrm{W}_0}(ax) = \infty$. Then, we can compute $\lim_{x\to\infty}(1 + U_{\mathrm{W}_0}(ax))/(1 + L_{\mathrm{W}_0}(x))$ via l'Hôpital's rule. Denote by U'_{W_0} (resp. L'_{W_0}) the first derivative of U_{W_0} (resp. L_{W_0}). In addition, denote by

$$h(x) := 2\log^2 x - 2\log x - \log\log x + 1, \quad x > 0. \tag{4.7}$$

After a bit of algebra, $U'_{\mathrm{W}_0}(ax)/L'_{\mathrm{W}_0}(x)$ can be written as below

$$\frac{U'_{\mathrm{W}_0}(ax)}{L'_{\mathrm{W}_0}(x)} = \frac{2\log^2 x}{h(x)} - \frac{2\log^2 x}{\log(ax)h(x)} + \frac{2e(\log\log(ax) - 1)\log x}{(e-1)\log^2(ax)h(x)}, \quad x \ge e/a,$$

where $h(x)$ is given by Eq. (4.7). Moreover, it can be checked the following

$$\lim_{x\to\infty} \frac{2\log^2 x}{h(x)} = 1, \quad \lim_{x\to\infty} \frac{2\log^2 x}{\log(ax)h(x)} = 0,$$
$$\lim_{x\to\infty} \frac{2e(\log\log(ax) - 1)\log x}{(e-1)\log^2(ax)h(x)} = 0.$$

Thereby, we have

$$\lim_{x\to\infty} \frac{1 + U_{\mathrm{W}_0}(ax)}{1 + L_{\mathrm{W}_0}(x)} = \lim_{x\to\infty} \frac{U'_{\mathrm{W}_0}(ax)}{L'_{\mathrm{W}_0}(x)} = 1. \tag{4.8}$$

On the other hand, by similar arguments, it can also be verified that

$$\lim_{x\to\infty} \frac{1 + L_{\mathrm{W}_0}(ax)}{1 + U_{\mathrm{W}_0}(x)} = 1. \tag{4.9}$$

Finally, from Eqs. (4.8) and (4.9) the statement of this lemma is obtained by applying the squeeze theorem to Eq. (4.6). □

With the previous results, we are now in a position to prove Theorem 1.

Proof of Theorem 1. (i) By Theorem 9.5 in Castillo et al. [4, pp. 203–204], a necessary and sufficient condition to determine whether the Gompertz–Makeham distribution belongs to the maximum domain of attraction of the Gumbel distribution is the following

$$\lim_{\varepsilon\to 0}\frac{F^{-1}(1-\varepsilon)-F^{-1}(1-2\varepsilon)}{F^{-1}(1-2\varepsilon)-F^{-1}(1-4\varepsilon)}=1, \tag{4.10}$$

where F^{-1} is given by Eq. (1.2). Below, we show that Eq. (4.10) holds.

Denote by $H_k(\varepsilon) := F^{-1}(1-k\varepsilon)-F^{-1}(1-2k\varepsilon)$, $k=1,2$. Then, we need to compute $\lim_{\varepsilon\to 0} H_1(\varepsilon)/H_2(\varepsilon)$. From Eq. (1.2), we have

$$H_k(\varepsilon)=\frac{1}{\lambda}\log 2+\frac{1}{\beta}\Big(\mathrm{W}_0\Big(\frac{\alpha}{\lambda}e^{\alpha/\lambda}(2k\varepsilon)^{-\beta/\lambda}\Big)-\mathrm{W}_0\Big(\frac{\alpha}{\lambda}\,e^{\alpha/\lambda}(k\varepsilon)^{-\beta/\lambda}\Big)\Big), \tag{4.11}$$

where α, β and λ are positive real parameters. By virtue of Lemma 4, since $\lim_{\varepsilon\to 0}\varepsilon^{-\beta/\lambda}=\infty$, for $k=1,2$ we have

$$\lim_{\varepsilon\to 0}\Big(\mathrm{W}_0\Big(\frac{\alpha}{\lambda}\,e^{\alpha/\lambda}(2k\varepsilon)^{-\beta/\lambda}\Big)-\mathrm{W}_0\Big(\frac{\alpha}{\lambda}\,e^{\alpha/\lambda}(k\varepsilon)^{-\beta/\lambda}\Big)\Big)=-\frac{\beta}{\lambda}\log 2. \tag{4.12}$$

Therefore, from Eqs. (4.11) and (4.12) we get $\lim_{\varepsilon\to 0} H_k(\varepsilon)=0$ for $k=1,2$. We can now apply l'Hôpital's rule to obtain $\lim_{\varepsilon\to 0} H_1(\varepsilon)/H_2(\varepsilon)$. Denote by H_i' the first derivative of H_i, $i=1,2$. Then, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{H_1'(\varepsilon)}{H_2'(\varepsilon)}&=\lim_{\varepsilon\to 0}\frac{\mathrm{W}_0'((\alpha/\lambda)\,e^{\alpha/\lambda}(2\varepsilon)^{-\beta/\lambda})-\mathrm{W}_0'((\alpha/\lambda)\,e^{\alpha/\lambda}(\varepsilon)^{-\beta/\lambda})}{\mathrm{W}_0'((\alpha/\lambda)\,e^{\alpha/\lambda}(4\varepsilon)^{-\beta/\lambda})-\mathrm{W}_0'((\alpha/\lambda)\,e^{\alpha/\lambda}(2\varepsilon)^{-\beta/\lambda})}\\&=\lim_{\varepsilon\to 0}\frac{\mathrm{W}_0((\alpha/\lambda)\,e^{\alpha/\lambda}(2\varepsilon)^{-\beta/\lambda})-\mathrm{W}_0((\alpha/\lambda)\,e^{\alpha/\lambda}(\varepsilon)^{-\beta/\lambda})}{\mathrm{W}_0((\alpha/\lambda)\,e^{\alpha/\lambda}(4\varepsilon)^{-\beta/\lambda})-\mathrm{W}_0((\alpha/\lambda)\,e^{\alpha/\lambda}(2\varepsilon)^{-\beta/\lambda})}\\&\quad\times\lim_{\varepsilon\to 0}\frac{1+\mathrm{W}_0((\alpha/\lambda)\,e^{\alpha/\lambda}(4\varepsilon)^{-\beta/\lambda})}{1+\mathrm{W}_0((\alpha/\lambda)\,e^{\alpha/\lambda}(\varepsilon)^{-\beta/\lambda})},\end{aligned} \tag{4.13}$$

where W_0' denotes the first derivative of W_0 with respect to ε and the second equality in Eq. (4.13) is obtained by taking into account Eq. (1.3). Finally, we apply Lemmas 4 and 5 to calculate the limit in Eq. (4.13) and then we get $\lim_{\varepsilon\to 0} H_1'(\varepsilon)/H_2'(\varepsilon)=1$, which implies that Eq. (4.10) holds. The normalizing constants a_n and b_n given in Eq. (4.2) are chosen by virtue of Castillo et al. [4, Theorem 9.5]. The proof of part (i) is completed.

(ii) By Theorem 9.6 in Castillo et al. [4, pp. 204–205], a necessary and sufficient condition for part (ii) is that there exists a constant $c>0$ such that

$$\lim_{\varepsilon\to 0}\frac{F^{-1}(\varepsilon)-F^{-1}(2\varepsilon)}{F^{-1}(2\varepsilon)-F^{-1}(4\varepsilon)}=2^{-c}, \tag{4.14}$$

where F^{-1} is given by Eq. (1.2). We claim that $c=1$ and, if this is the case, then the Gompertz–Makeham distribution belongs to the minimum domain of attraction of the Weibull distribution. To see this, we denote by $L_k(\varepsilon) :=$

$F^{-1}(k\varepsilon) - F^{-1}(2k\varepsilon)$, $k = 1, 2$. We need to compute $\lim_{\varepsilon\to 0} L_1(\varepsilon)/L_2(\varepsilon)$. From Eq. (1.2), we have

$$L_k(\varepsilon) = \frac{1}{\lambda}\log\left(\frac{1-2k\varepsilon}{1-k\varepsilon}\right) + \frac{1}{\beta}\left(\mathrm{W}_0\left(\frac{\alpha}{\lambda}\, e^{\alpha/\lambda}(1-2k\varepsilon)^{-\beta/\lambda}\right) - \mathrm{W}_0\left(\frac{\alpha}{\lambda}\, e^{\alpha/\lambda}(1-k\varepsilon)^{-\beta/\lambda}\right)\right),$$

where α, β and λ are positive real parameters. Now, by taking into account that $\mathrm{W}_0(xe^x) = x$ it is easy to check that $\lim_{\varepsilon\to 0} L_k(\varepsilon) = 0$ for $k = 1, 2$. Then, we can apply l'Hôpital's rule to compute $\lim_{\varepsilon\to 0} L_1(\varepsilon)/L_2(\varepsilon)$, so that, denoting by L_i' the first derivative of L_i, $i = 1, 2$, we have to compute

$$\lim_{\varepsilon\to 0}\frac{L_1'(\varepsilon)}{L_2'(\varepsilon)} = \lim_{\varepsilon\to 0}\frac{(F^{-1})'(\varepsilon) - (F^{-1})'(2\varepsilon)}{(F^{-1})'(2\varepsilon) - (F^{-1})'(4\varepsilon)},$$

where $(F^{-1})'$ denotes the first derivative of F^{-1} with respect to ε. For $k = 1, 2, 4$, it can be checked the following

$$\left(F^{-1}\right)'(k\varepsilon) = \frac{k}{\lambda(1-k\varepsilon)(1+\mathrm{W}_0((\alpha/\lambda)\, e^{\alpha/\lambda}(1-k\varepsilon)^{-\beta/\lambda}))}. \tag{4.15}$$

In view of Eq. (4.15) together with the fact that $\mathrm{W}_0(xe^x) = x$, we get

$$\lim_{\varepsilon\to 0}\left(F^{-1}\right)'(k\varepsilon) = \frac{k}{\alpha+\lambda}, \quad k = 1, 2, 4. \tag{4.16}$$

Now, by using Eq. (4.16) we obtain

$$\lim_{\varepsilon\to 0}\frac{(F^{-1})'(\varepsilon) - (F^{-1})'(2\varepsilon)}{(F^{-1})'(2\varepsilon) - (F^{-1})'(4\varepsilon)} = \frac{1}{2},$$

which implies that $c = 1$ in Eq. (4.14), as it was claimed. Finally, the normalizing constants c_n and d_n in Eq. (4.3) are chosen according to Castillo et al. [4, Theorem 9.6]. This concludes the proof of part (ii). The proof of Theorem 1 is completed. □

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