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Analysis of Stabilized Finite Volume Method for Poisson Equation*

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Abstract. In this work, we systematically analyze a stabilized finite volume method for the Poisson equation. On stating the convergence of this method, optimal error estimates in different norms are obtained by establishing the adequate connections between the finite element and finite volume methods. Furthermore, some superconvergence results are established by using L^2-projection method on a coarse mesh based on some regularity assumptions for Poisson equation. Finally, some numerical experiments are presented to confirm the theoretical findings.

Keywords: Poisson equation, finite volume method, projection method, superconvergence.

AMS Subject Classification: 65M08; 65N12; 35J05.

1 Introduction

Finite volume method (FVM) as one of important numerical discretization techniques has been widely employed to solve the fluid dynamics problems [8]. The basic idea of FVM is to approximate discrete fluxes of a partial differential equation using a finite element procedure based on volumes or control volumes, so FVM is also known as covolume methods, or box methods [1], marker and cell methods [5], generalized difference methods [17]. The difficulty in the

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analysis of finite volume method arises from trial and test functions to lie in different spaces and to be associated with different meshes. This can be resolved by introducing a transfer map which allows to rewrite the finite element formulation as its classical finite-volume-like counterpart, i.e., using piecewise constant test functions. We can refer to [7,10,14,21] and the references therein for more recent developments about finite volume method.

The mixed finite element method has became a popular method for solving the partial differential equations arising in solid and fluid mechanics [4]. Its popularity is due to the fact that in some cases a vector variable is the primary variable in which one is interested. The mixed method is developed to approximate both this variable and a scalar variable simultaneously and to give accurate approximations of both variables. The mixed finite element formulations use two different approximate spaces. These two spaces must be chosen carefully to satisfy the inf–sup stability condition. There exist rich choices for these special spaces for the equations of solid and fluid mechanics (see [4,19] and the references therein).

Much attention has recently been attracted to use the P_1-P_1 pair for the fluid mechanics equations, particularly for the Stokes and Navier-Stokes equations (see [2,9,12]). Although this pair does not satisfy the inf–sup stability condition, it offers simple and practical uniform data structure and adequate accuracy. Many stabilization techniques have been proposed to stabilize the unstable pairs such as penalty method, pressure projection, velocity correction and residual stabilization methods. Among these methods, the pressure projection stabilization method [2] is a preferable choice in that it is free of stabilization parameters, does not require any calculation of high-order derivations or edge-based data structure, and can be implemented at the element level. Recent studies have been focused on stabilization of the P_1-P_1 or P_1-P_0 using this type of stabilization for the Stokes and Navier-Stokes equations, see [10,11,13,16,22] and the references therein.

There have been some attempts to use the velocity projection stabilization method for solving Poisson equation. Preliminary computational studies were given, and numerical results were reported using the low-order finite element pairs (see [23]). However, the analysis of finite volume method is still lacking for the second order equations. The analysis of the mixed finite volume method for Poisson equations is much more dedicate than for the Stokes equations since the latter equations are naturally given in mixed form. The pair $H^1(\Omega)^2 \times L^2(\Omega)$ is used for the Stokes equations while we can relax the regularity of the exact solution for Poisson equation such that this set is still valid. In this work, we provide a systematical finite volume analysis of the velocity projection stabilization method for Poisson equation. Stability results for the finite volume solution are presented. Optimal error estimates for velocity and pressure are obtained. Furthermore, some superconvergence results are established for Poisson equation by using a L^2-projection on the coarse mesh. Like other results in the family of L^2-projection methods (see [5,6,15]), the superconvergence results presented in this work are based on some regularity assumptions for Poisson problem on the quasi-uniform triangulation partitions. Different from [18], this paper considers the mixed finite volume method for Poisson equations. We not only provide the stability and optimal error estimates of numerical solution, but also present some meaningful superconvergence results by using the L^2-projections between different finite element spaces.

The rest of this paper is organized as follows. In Section 2, some results about Poisson equation and its mixed finite element scheme are recalled. Section 3 is devoted to establish the stabilized finite volume formulation for Poisson problem. Optimal error estimates are derived in Section 4. In Section 5, some superconvergence results are developed by introducing the L^2-projection methods between different finite element spaces. In Section 6, some numerical experiments are provided to verify the established theoretical findings. Finally, we end with a short conclusion in Section 7.

2 Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^2$ assumed to have a Lipschitz continuous polygonal boundary $\partial\Omega$. We consider the following Poisson equation

$$\begin{cases} -\Delta p = f & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

This equation arises from many physical and mechanical phenomena, for example, the equilibrium of an elastic membrane fixed on the boundary and subject to a load of intensity f, the equilibrium of some quantity like chemical or temperature distribution when $f = 0$.

Standard definitions are used for the Sobolev spaces $W^{m,r}(\Omega)$, with the norm $\|\cdot\|_{m,r}$ and the seminorm $|\cdot|_{m,r}$, $m, r \geq 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_m$ for $\|\cdot\|_{m,2}$. The notation $(\cdot,\cdot)$ indicates the inner product on the domain Ω.

In order to state a mixed formulation for (2.1), we define the spaces

$$V = H(div;\Omega)^2, \qquad X = H_0^1(\Omega), \qquad W = L^2(\Omega), \qquad W_1 = L^2(\Omega)^2.$$

Letting $u = -\nabla p$, the mixed formulation of equation (2.1) is to find $u \in V, p \in W$, such that

$$\begin{cases} (u,v) - (\nabla\cdot v, p) = 0 & \forall v \in V, \\ (\nabla\cdot u, w) = (f,w) & \forall w \in W. \end{cases}$$

In fact, we do not need $u \in V$, we only require $u \in W_1$. Denoting

$$a(u,v) = (u,v), \qquad d(v,p) = -(v,\nabla p).$$

Then, applying the Green formula, we have the following variational formulation for equation (2.1)

$$a(u,v) - d(v,p) + d(u,w) = (f,w) \quad \forall (u,p),(v,w) \in W_1 \times X. \tag{2.2}$$

The following lemma establishes the existence, uniqueness of the solution to Poisson equation (2.2) under some weak regularities.

Lemma 1. *The bilinear forms $d(\cdot,\cdot)$ satisfies the LBB condition, i.e., there exists a constant $\beta_1 > 0$, such that*

$$\inf_{w\in X}\sup_{v\in W_1}\frac{d(v,w)}{\|v\|_{W_1}\|w\|_X}\geq \beta_1.$$

Proof. See Lemma 2.2 of Ref. [18]. □

Theorem 1. *There exists a unique solution $(u,p)\in W_1\times X$ to problem* (2.2).

Proof. Combining Lemma 1 and Lax–Milgram Theorem yields the desired result immediately. □

Let $\mathcal{T}_h=\{K\}$ be a regular, quasi-uniform partition of the domain Ω into a finite number of triangulations, $h_K=\operatorname{diam}(K)$, $h=\max\{h_K\colon K\in\mathcal{T}_h\}$. We consider the following mixed finite element spaces

$$\begin{aligned} W_h &= \big\{v\in C^0(\overline{\Omega})^2\cap W_1\colon v|_K\in P_1(K)^2\ \forall K\in\mathcal{T}_h\big\},\\ X_h &= \big\{w\in C^0(\overline{\Omega})\cap X\colon w|_K\in P_1(K)\ \forall K\in\mathcal{T}_h\big\}, \end{aligned}$$

where $P_1(K)$ is the set of linear polynomials on K.

Set I_h and J_h be two interpolation operators from W_1 and X into W_h and X_h, respectively, such that for any $v\in H^1(\Omega)^2$ and $w\in H^2(\Omega)$ (see [4])

$$\|v-I_hv\|_0\leq Ch\|v\|_1,\qquad \|w-J_hw\|_i\leq C_2h^{2-i}\|w\|_2,\quad i=0,1. \tag{2.3}$$

Obviously, the lowest equal-order finite element pair does not satisfy the discrete inf–sup condition (see [19])

$$\beta\|v_h\|_0\leq \sup_{0\neq w_h\in X_h}\frac{(v_h,\nabla w_h)}{\|\nabla w_h\|_0}\quad \forall v_h\in W_h, \tag{2.4}$$

where the constant $\beta>0$ is independent of h. In order to overcome the restriction (2.4), we define a L^2-projection operator $\Pi_h: W_1\to P_0^2$ by

$$(u,v_h)=(\Pi_hu,v_h),\quad u\in W_1,\ v_h\in P_0^2, \tag{2.5}$$

where P_0^2 is the 2-D piecewise constant space associated with the triangles $\mathcal{T}_h$. And assume Π_h satisfies the following properties (see [3]):

$$\begin{aligned} \|\Pi_hu\|_0\leq C\|u\|_0\quad \forall u\in W_1;\\ \|u-\Pi_hu\|_0\leq Ch\|u\|_1\quad \forall u\in H^1(\Omega)^2. \end{aligned} \tag{2.6}$$

With the help of (2.5), we can define a bilinear form $G_h(\cdot,\cdot)$ by

$$\begin{aligned} G_h(u_h,v_h) &= (u_h-\Pi_hu_h,v_h)\\ &= (u_h-\Pi_hu_h,v_h-\Pi_hv_h)\quad \forall u_h,v_h\in W_h. \end{aligned} \tag{2.7}$$

Remark. The bilinear form $G_h(\cdot,\cdot)$ in (2.7) is symmetric, semi-positive definite form generated on each local element K.

The finite element discrete scheme for (2.1) on $W_h \times X_h$ can be defined as

$$a\big(u_h^e, v_h\big) - d\big(v_h, p_h^e\big) + d\big(u_h^e, w_h\big) + G_h\big(u_h^e, v_h\big) = (f, w_h) \quad \forall (v_h, w_h) \in W_h \times X_h. \tag{2.8}$$

Denoting the generalized bilinear form

$$B_h\big(\big(u_h^e, p_h^e\big), (v_h, w_h)\big) = a\big(u_h^e, v_h\big) - d\big(v_h, p_h^e\big) + d\big(u_h^e, w_h\big) + G_h\big(u_h^e, v_h\big). \tag{2.9}$$

It has been shown that this general bilinear form (2.9) satisfies the continuity and weak coercivity properties (see [18]):

$$B_h\big(\big(u_h^e, p_h^e\big); (v_h, w_h)\big) \le C\big(\big\|u_h^e\big\|_0 + \big\|\nabla p_h^e\big\|_0\big)\big(\|v_h\|_0 + \|\nabla w_h\|_0\big), \tag{2.10}$$

$$\sup_{0 \ne (v_h, w_h) \in W_h \times X_h} \frac{B_h((u_h^e, p_h^e); (v_h, w_h))}{\|v_h\|_0 + \|\nabla w_h\|_0} \ge \beta\big(\big\|u_h^e\big\|_0 + \big\|\nabla p_h^e\big\|_0\big), \tag{2.11}$$

where $\beta > 0$ is a constant, independent of mesh parameter h.

Thanks to (2.9), we can simplify the equation (2.8) into the following mixed finite element formulation: Find $(u_h^e, p_h^e) \in W_h \times X_h$ such that

$$B_h\big(\big(u_h^e, p_h^e\big), (v_h, w_h)\big) = (f, w_h) \quad \forall (v_h, w_h) \in W_h \times X_h. \tag{2.12}$$

Combining the classical saddle theorem and (2.10)–(2.11), we know that the system (2.12) admits a unique solution. Moreover, the following error estimates for the finite element solution (u_h^e, p_h^e) hold (see [18])

$$\big\|p - p_h^e\big\|_0 + h\big(\big\|\nabla\big(p - p_h^e\big)\big\|_0 + \big\|u - u_h^e\big\|_0\big) \le Ch^2\big(\|u\|_1 + \|p\|_2 + \|f\|_0\big). \tag{2.13}$$

3 Stabilized Finite Volume Method

This section is devoted to present the stabilized finite volume formulation for Poisson equation and establish the existence uniqueness for the approximation solution.

Let $\mathcal{P}$ be the set containing all the interior nodes associated with the triangulations K, $\mathcal{N}_h$ denotes the set of all nodes $\mathcal{T}_h$. Based on the partition $\mathcal{T}_h$, we introduce the corresponding dual partition $\mathcal{T}_h^*$. Here, we choose the barycenter Q of a element $K \in \mathcal{T}_h$, and the midpoints M on the edges of K, then connect Q to M by straight line. For an arbitrary vertex $x_i \in K$, let $\widetilde{K}_i$ be the polygonal which is called a control volume. Then, we have $\overline{\Omega} = \bigcup_{x_i,\, i \in \mathcal{N}_h} \widetilde{K}_i$, the dual mesh $\mathcal{T}_h^*$ is the set of these control volumes.

The dual finite element space is defined as

$$\widetilde{X}_h = \big\{\widetilde{v} \in W : \widetilde{v} \in P_0(\widetilde{K}_i)\ \forall \widetilde{K}_i \in \mathcal{T}_h^*;\ \widetilde{v}|_{\partial \widetilde{K}_i} = 0\big\}.$$

It is clearly that the dimensions of X_h and $\widetilde{X}_h$ are the same. Furthermore, there exists an invertible linear mapping $\varGamma_h : X_h \to \widetilde{X}_h$ such that

$$\varGamma_h v_h(x) = \sum_{i=1}^{\mathcal{N}_h} v_h(x_i)\phi_i(x), \quad x \in \Omega,\ v_h \in X_h,$$

where $\phi_i(x)$ is the basis functions associated with the dual partition $\mathcal{T}_h^*$:

$$\phi_i(x) = \begin{cases} 1, & x \in \widetilde{K}_i, \\ 0, & \text{otherwise.} \end{cases}$$

The above idea of connecting the different spaces through the mapping Γ_h was introduced by Li in [17] for the elliptic problem. Furthermore, the mapping Γ_h has the following properties (see [21]).

Lemma 2. *Let $K \in \mathcal{T}_h$, if $v_h \in X_h$ and $1 \le r \le \infty$, then*

$$\int_K (v_h - \Gamma_h v_h)\, dx = 0, \quad \|v_h - \Gamma_h v_h\|_{L^r(K)^d} \le C h_K \|v_h\|_{W^{1,r}(K)^d},$$

where h_K is the diameter of the element K.

Applying Green's formula, we present the finite volume approximation to (2.1) as finding $u_h \in W_h, p_h \in X_h$, such that for $\forall (v_h, w_h) \in W_h \times X_h$

$$a(u_h, v_h) - d(v_h, p_h) + D(u_h, w_h) + G_h(u_h, v_h) = (f, \Gamma_h w_h), \tag{3.1}$$

where

$$D(u_h, w_h) = \sum_{j=1}^{\mathcal{N}_h} w_h(\mathcal{P}_j) \int_{\partial \widetilde{K}_j} u_h \cdot n\, dx, \quad u_h \in W_h,\ w_h \in X_h,$$

where n is the unit outward to $\partial \widetilde{K}_j$. Setting

$$C_h\big((u_h, p_h), (v_h, w_h)\big) = a(u_h, v_h) - d(v_h, p_h) + D(u_h, w_h) + G_h(u_h, v_h),$$

then, system (3.1) can be rewritten as

$$C_h\big((u_h, p_h), (v_h, w_h)\big) = (f, \Gamma_h w_h) \quad \forall (v_h, w_h) \in W_h \times X_h. \tag{3.2}$$

The next lemma establishes the relationship between the finite element and finite volume methods for Poisson equation.

Lemma 3. *It holds that*

$$D(u_h, w_h) = -(u_h, \nabla w_h) \quad \forall (u_h, w_h) \in W_h \times X_h.$$

Proof. From the definition of $D(\cdot,\cdot)$, and combining Lemma 2, Green's formula with the fact that $\operatorname{div} u_h$ is a constant that

$$\begin{aligned} D(u_h, w_h) &= \sum_{i=1}^{\mathcal{N}_h} \int_{\widetilde{K}_i} \operatorname{div} u_h \cdot \Gamma_h w_h\, dx = \sum_{i=1}^{\mathcal{N}_h} \int_{K \cap \widetilde{K}_i} \operatorname{div} u_h \cdot w_h\, dx \\ &= -\sum_{K \in \mathcal{T}_h} \int_K u_h \cdot \nabla w_h\, dx = -(u_h, \nabla w_h). \quad \square \end{aligned}$$

Now, we are in the position of establishing the continuity and weak coercivity for the general bilinear form $C_h((u_h, p_h), (v_h, w_h))$.

Lemma 4. *It holds that for all* $(u_h, p_h), (v_h, w_h) \in W_h \times X_h$

$$\big|C_h\big((u_h, p_h), (v_h, w_h)\big)\big| \le C\big(\|u_h\|_0 + \|\nabla p_h\|_0\big)\big(\|v_h\|_0 + \|\nabla w_h\|_0\big). \tag{3.3}$$

Moreover,

$$\beta_2\big(\|u_h\|_0 + \|\nabla p_h\|_0\big) \le \sup_{0\neq(v_h,w_h)\in(W_h,X_h)} \frac{|C_h((u_h,p_h);(v_h,w_h))|}{\|v_h\|_0 + \|\nabla w_h\|_0}, \tag{3.4}$$

where the constant $\beta_2 > 0$ *is independent of* h.

Proof. The continuity of (3.3) can easily be shown using the definition of the bilinear form $C_h((\cdot,\cdot),(\cdot,\cdot))$ and Lemma 3.

In the following, we will prove the weak coercivity (3.4). For any fixed $p_h \in X_h$, there exists a unique $\phi \in W_1$, satisfying $-(\phi, \nabla p_h) = \|\nabla p_h\|_0^2$ and $\|\phi\|_0 \le C\|\nabla p_h\|_0$. Let $\phi_h = I_h\phi$, where I_h is the interpolation operator from W_1 to W_h, satisfies (2.3), and possesses the following property:

$$(\phi - I_h\phi, w_h) = 0 \quad \forall w_h \in W_h. \tag{3.5}$$

Combining the definition of I_h in (3.5) with the fact that ∇p_h is piecewise constant, we arrive at

$$-(\phi_h, \nabla p_h) = -(\phi, \nabla p_h) = \|\nabla p_h\|_0^2 \quad \forall p_h \in X_h. \tag{3.6}$$

By using Lemma 2, (3.5), (3.6) and Cauchy inequality yields

$$\begin{aligned} C_h\big((u_h, p_h), (\phi_h, 0)\big) &= (u_h, \phi_h) + (\phi_h, \nabla p_h) + G_h(u_h, \phi_h) \\ &\le \|u_h\|_0\|\phi_h\|_0 + (\phi, \nabla p_h) + C\|u_h\|_0\|\phi_h\|_0 \\ &\le \|u_h\|_0\|\nabla p_h\|_0 - \|\nabla p_h\|_0^2 + C\|u_h\|_0\|\nabla p_h\|_0 \\ &\le \big(1 + C^2\big)\|u_h\|_0^2 - \frac{1}{2}\|\nabla p_h\|_0^2. \end{aligned} \tag{3.7}$$

Setting $(v_h, q_h) = (u_h - \alpha\phi_h, p_h)$ with $\alpha > 0$, applying (3.7) and Lemma 3 yields

$$\begin{aligned} &C_h\big((u_h, p_h), (u_h - \alpha\phi_h, p_h)\big) \\ &\quad = C_h\big((u_h, p_h), (u_h, p_h)\big) - \alpha C_h\big((u_h, p_h), (\phi_h, 0)\big) \\ &\quad = (u_h, u_h) + G_h(u_h, u_h) - \alpha\big(1 + C^2\big)\|u_h\|_0^2 + \frac{\alpha}{2}\|\nabla p_h\|_0^2 \\ &\quad \ge \big(1 - \alpha\big(1 + C^2\big)\big)\|u_h\|_0^2 + \frac{\alpha}{2}\|\nabla p_h\|_0^2. \end{aligned}$$

Choosing $\alpha = \frac{1}{2(1+C^2)}$ we have

$$C_h\big((u_h, p_h), (v_h, w_h)\big) \ge C\big(\|u_h\|_0 + \|\nabla p_h\|_0\big)^2. \tag{3.8}$$

On the other hand,

$$\begin{aligned} \|u_h - \alpha\phi_h\|_0 + \|\nabla p_h\|_0 &\le \big(\|u_h\| + \alpha\|\phi_h\|_0 + \|\nabla p_h\|_0\big) \\ &\le C\big(\|u_h\| + \|\nabla p_h\|_0\big). \end{aligned} \tag{3.9}$$

Combining (3.8) with (3.9), we have finished the proof. □

It follows from Lemma 4 that system (3.1) has a unique solution, and satisfies

$$\|u_h\|_0 + \|\nabla p_h\|_0 \le C\|f\|_0.$$

4 Error Estimates

In this section, we present some optimal error estimates of the stabilized finite volume solutions for equation (3.1) under some regularities about f.

Theorem 2. *Let* (u, p) *and* (u_h, p_h) *be the solutions of* (2.1) *and* (3.1), *respectively. Then we have*

$$\big\|\nabla(p - p_h)\big\|_0 + \|u - u_h\|_0 \le Ch\big(\|p\|_2 + \|u\|_1 + \|f\|_0\big).$$

Proof. Subtracting (2.12) from (3.2) and using Lemma 3 gives

$$C_h\big((u_h - u_h^e, p_h - p_h^e), (v_h, w_h)\big) = (f, \Gamma_h w_h - w_h). \tag{4.1}$$

Obviously, we deduce from Lemmas 2 and 4 that

$$\sup_{(v_h,w_h)\in(W_h,X_h)} \frac{C_h((u_h - u_h^e, p_h - p_h^e), (v_h, w_h))}{\|v_h\|_0 + \|\nabla w_h\|_0} \ge \beta_2\big(\big\|u_h - u_h^e\big\|_0 + \big\|\nabla\big(p_h - p_h^e\big)\big\|_0\big),$$

$$\big|(f, \Gamma_h w_h - w_h)\big| \le \|f\|_0\|\Gamma_h w_h - w_h\|_0 \le Ch\|f\|_0\|\nabla w_h\|_0.$$

Combining above equations with (4.1), we arrive at

$$\big\|u_h - u_h^e\big\|_0 + \big\|\nabla\big(p_h - p_h^e\big)\big\|_0 \le Ch\|f\|_0,$$

which, together with (2.13), we have finished the proof. □

To obtain the estimate of the error $p - p_h$ in L^2-norm, we use the duality argument. Considering the following problem by seeking $(\phi, \varphi) \in W_1 \times X$ such that

$$(v, \phi) + d(v, \varphi) - d(\phi, w) = (p - p_h, w) \quad \forall (v, w) \in W_1 \times X. \tag{4.2}$$

When Ω is convex, the solutions of (4.2) satisfy

$$\|\phi\|_1 + \|\varphi\|_2 \le C\|p - p_h\|_0. \tag{4.3}$$

Theorem 3. *Let* (u, p) *and* (u_h, p_h) *be the solutions of* (2.1) *and* (3.1), *respectively, and* $f \in H^1(\Omega)$. *Then*

$$\|p - p_h\|_0 \le Ch^2\big(\|p\|_2 + \|u\|_1 + \|f\|_1\big).$$

Proof. Choosing $(v, w) = (u - u_h, p - p_h)$ in (4.2), letting $(v, w) = (I_h v, J_h w)$ in (2.2), $(v_h, w_h) = (I_h v, J_h w)$ in (3.2), where I_h, J_h are two interpolation operators and satisfy (2.3), we arrive at

$$\|p - p_h\|_0^2 = C_h\big((u - u_h, p - p_h), (\phi - I_h\phi, \varphi - J_h\varphi)\big) + G_h(u, I_h\phi) \\ + (f, J_h\varphi - \Gamma_h J_h\varphi) - G_h(\phi, u - u_h). \tag{4.4}$$

It follows from (2.5)–(2.7), Theorem 2 and (4.3) that

$$\begin{aligned}
&\big|G_h(u, I_h\phi)\big| \le Ch^2\|u\|_1\|\phi\|_1 \le Ch^2\|u\|_1\|p - p_h\|_0,\\
&\big|G_h(u - u_h, \phi)\big| \le Ch\|u - u_h\|_0\|\phi\|_1 \le Ch^2\|p - p_h\|_0\big(\|p\|_2 + \|u\|_1 + \|f\|_0\big),\\
&\big|C_h\big((u - u_h, p - p_h), (\phi - I_h\phi, \varphi - J_h\varphi)\big)\big|\\
&\quad \le \big(\|u - u_h\|_0 + \big\|\nabla(p - p_h)\big\|_0\big)\big(\|\phi - I_h\phi\|_0 + \big\|\nabla(\varphi - J_h\varphi)\big\|_0\big)\\
&\quad \le Ch^2\big(\|\phi\|_1 + \|\varphi\|_2\big)\big(\|p\|_2 + \|u\|_1 + \|f\|_0\big)\\
&\quad \le Ch^2\|p - p_h\|_0\big(\|p\|_2 + \|u\|_1 + \|f\|_0\big).
\end{aligned}$$

In addition, let $P_h f$ be defined by

$$P_h f|_K = \frac{1}{|K|}\int_K f(x)\,dx \quad K \in \mathcal{T}_h,$$

where $|K|$ is the area of the element K. Then, it follows from (2.5), (4.3) and Lemma 2 that

$$\begin{aligned}
\big|(f, J_h\varphi - \Gamma_h J_h\varphi)\big| &= \big|(f - P_h f, J_h\varphi - \Gamma_h J_h\varphi)\big|\\
&\le \|f - P_h f\|_0\|J_h\varphi - \Gamma_h J_h\varphi\|_0\\
&\le Ch^2\|f\|_1\|\nabla J_h\varphi\|_0 \le Ch^2\|f\|_1\|p - p_h\|_0.
\end{aligned}$$

Combining above inequalities with (4.4) yields the desired result. □

5 Superconvergence

The post-processing technique introduced by Wang [15, 20] is onto project the numerical solution to another finite element space with different mesh size. The difference in these mesh sizes can be used to achieve a superconvergence after the post-processing procedure. Let $\mathcal{T}_{\tau_i}$ $(i = 1, 2)$ be another two finite element partitions with mesh sizes τ_i, where $h \ll \tau_i$ $(i = 1, 2)$. Assume that τ_i $(i = 1, 2)$ and h have the following relation

$$\tau_i = h^{\alpha_i} \tag{5.1}$$

with $\alpha_1, \alpha_2 \in (0, 1)$, α_i play an important role in achieving the superconvergence for the stabilized finite volume solution (u_h, p_h). Let V_{τ_1} and P_{τ_2} be two finite element spaces consisting of piecewise polynomials of degree r and t, respectively, associated with the partition $\mathcal{T}_{\tau_i}$. We define two L^2 projections Q_{τ_1} and R_{τ_2} from $L^2(\Omega)$ onto the finite element spaces V_{τ_1} and P_{τ_2}, respectively. In the following , we will analyze the errors of $p - Q_{\tau_1}p_h$ and $u - R_{\tau_2}u_h$.

The following two lemmas provide the error estimates for $Q_{\tau_1}p - Q_{\tau_1}p_h$ and $R_{\tau_2}u - R_{\tau_2}u_h$, respectively.

Lemma 5. *Assume $V_{\tau_1} \in L^2(\Omega)$ and $f \in H^1(\Omega)$, then there is a constant C, independent of h and τ_1 such that*

$$\|Q_{\tau_1} p - Q_{\tau_1} p_h\|_0 \le Ch^2\big(\|u\|_1 + \|p\|_2 + \|f\|_1\big).$$

Proof. From the definition of $\|\cdot\|_0$ and Q_{τ_1}, we have

$$\|Q_{\tau_1} p - Q_{\tau_1} p_h\|_0 = \sup_{\phi\in L^2(\Omega),\, \|\phi\|_0=1} \big|(Q_{\tau_1} p - Q_{\tau_1} p_h, \phi)\big|$$
$$(Q_{\tau_1} p - Q_{\tau_1} p_h, \phi) = (p - p_h, Q_{\tau_1}\phi).$$

Then we obtain that

$$\|Q_{\tau_1} p - Q_{\tau_1} p_h\|_0 = \sup_{\phi\in L^2(\Omega),\, \|\phi\|_0=1} \big|(p - p_h, Q_{\tau_1}\phi)\big|.$$

Considering the following problem

$$(u - u_h, v) - d(v, p - p_h) + d(u - u_h, w) = (p - p_h, Q_{\tau_1}\phi). \tag{5.2}$$

Because of the convexity of domain Ω, problem (5.2) has a unique solution and satisfies

$$\|w\|_2 + \|v\|_1 \le C\|Q_{\tau_1}\phi\|_0. \tag{5.3}$$

Let $(v_h, w_h) \in W_h \times X_h$ be the usual piecewise linear interpolant of (v, w), which satisfies (2.3). From the continuous equation (2.1), we have the following finite volume variational formulation

$$(u, v_h) + (\nabla p, v_h) + (\nabla\cdot u, \Gamma_h w_h) = (f, \Gamma_h w_h). \tag{5.4}$$

The following error equation can be obtained from (3.2) and (5.4)

$$C_h\big((u - u_h, p - p_h), (v_h, w_h)\big) + G_h(u, v_h) = 0 \quad \forall (v_h, w_h) \in W_h \times X_h. \tag{5.5}$$

Combining equations (5.2), (5.5) with Lemma 2, and the fact that $\nabla\cdot u_h$ is a constant yields

$$\begin{aligned}(Q_{\tau_1}\phi, p - p_h) &= (u - u_h, v - v_h) - d(v - v_h, p - p_h) + d(u - u_h, w - w_h)\\ &\quad + \big(\nabla\cdot(u - u_h), w_h - \Gamma_h w_h\big) - G_h(u - u_h, v_h) + G_h(u, v_h)\\ &= (u - u_h, v - v_h) - d(v - v_h, p - p_h) + d(u - u_h, w - w_h)\\ &\quad + (f, w_h - \Gamma_h w_h) - G_h(u - u_h, v_h) + G_h(u, v_h).\end{aligned} \tag{5.6}$$

Combining (2.3), (5.3) with Theorem 2 yields

$$\begin{aligned}&\big|(u - u_h, v - v_h) - d(v - v_h, p - p_h) + d(u - u_h, w - w_h)\big|\\ &\le C\|u - u_h\|_0\|v - v_h\|_0 + \|v - v_h\|_0\big\|\nabla(p - p_h)\big\|_0\\ &\quad + \|u - u_h\|_0\big\|\nabla(w - w_h)\big\|_0\\ &\le Ch^2\big(\|u\|_1 + \|p\|_2 + \|f\|_0\big)\big(\|v\|_1 + \|w\|_2\big)\\ &\le Ch^2\big(\|u\|_1 + \|p\|_2 + \|f\|_0\big)\|Q_{\tau_1}\phi\|_0.\end{aligned}$$

By (2.5)–(2.7), it suffices to show that

$$\begin{aligned}
&\big|-G_h(u-u_h,v_h)+G_h(u,v_h)\big| \\
&\quad \le \big(\|u-u_h\|_0+\|u-\Pi_h u\|_0\big)\|v_h-\Pi_h v_h\|_0 \\
&\quad \le \big(\|u-u_h\|_0+\|u-\Pi_h u\|_0\big)\big(\|v-v_h\|_0+\|v-\Pi_h v\|_0+\|\Pi_h v-\Pi_h v_h\|_0\big) \\
&\quad \le Ch^2\big(\|u\|_1+\|p\|_2+\|f\|_0\big)\|Q_{\tau_1}\phi\|_0.
\end{aligned}$$

Furthermore, from Lemma 2 and (2.6) that

$$\begin{aligned}
\big|(f,w_h-\Gamma_h w_h)\big| &= \big|(f-P_h f,w_h-\Gamma_h w_h)\big| \\
&\le \|f-P_h f\|_0\|w_h-\Gamma_h w_h\|_0 \le Ch^2\|f\|_1\|Q_{\tau_1}\phi\|_0.
\end{aligned}$$

Combining above inequalities with (5.6), we finished the proof. □

Lemma 6. *Assume $P_{\tau_2}\in H^1(\Omega)^2$ and $f\in H^1(\Omega)$, there is a constant C independent of h and τ_2 such that*

$$\|R_{\tau_2}u-R_{\tau_2}u_h\|_0 \le Ch^{2-\alpha_2}\big(\|p\|_2+\|u\|_1+\|f\|_1\big)$$

where $\alpha_2\in(0,1)$ is defined in (5.1).

Proof. From the definition of $\|\cdot\|_0$ and R_{τ_2}, we have

$$\begin{aligned}
\|R_{\tau_2}u-R_{\tau_2}u_h\|_0 &= \sup_{\phi\in L^2(\Omega)^2,\,\|\phi\|_0=1}\big|(R_{\tau_2}u-R_{\tau_2}u_h,\phi)\big|, \\
(R_{\tau_2}u-R_{\tau_2}u_h,\phi) &= (u-u_h,R_{\tau_2}\phi).
\end{aligned}$$

Then we obtain that

$$\|R_{\tau_2}u-R_{\tau_2}u_h\|_0 = \sup_{\phi\in L^2(\Omega)^2,\,\|\phi\|_0=1}\big|(u-u_h,R_{\tau_2}\phi)\big|.$$

Considering the following dual problem

$$(u-u_h,v)+d(p-p_h,v)-d(u-u_h,w)=(u-u_h,R_{\tau_2}\phi). \tag{5.7}$$

Because of the convexity of domain Ω, problem (5.7) has a unique solution and satisfies

$$\|w\|_2+\|v\|_1\le C\|R_{\tau_2}\phi\|_1.$$

Combining (5.5) with (5.7), we obtain that

$$\begin{aligned}
(u-u_h,R_{\tau_2}\phi) &= (u-u_h,v-v_h)+d(v-v_h,p-p_h)-d(u-u_h,w-w_h) \\
&\quad +\big(\nabla\cdot(u-u_h),w_h-\Gamma_h w_h\big)-G_h(u-u_h,v_h)+G_h(u,v_h) \\
&= (u-u_h,v-v_h)+d(v-v_h,p-p_h)-d(u-u_h,w-w_h) \\
&\quad +(f,w_h-\Gamma_h w_h)-G_h(u-u_h,v_h)+G_h(u,v_h).
\end{aligned} \tag{5.8}$$

Applying the tricks used in Lemma 5 and the inverse inequality, and combining (5.8), we completed the proof. □

Now, we are ready to estimate $p-Q_{\tau_1}p_h$ and $u-R_{\tau_2}u_h$.

Theorem 4. *Under the assumptions of Lemma 5, if τ_1, h and α_1 satisfy*

$$\tau_1 = \mathcal{O}\big(h^{\alpha_1}\big) \quad \textit{with } \alpha_1 = 2/(r+1),$$

then the postprocessed solution $Q_{\tau_1}p_h$ satisfies

$$\|p - Q_{\tau_1}p_h\|_0 \le Ch^2\big(\|p\|_{r+1} + \|p\|_2 + \|u\|_1 + \|f\|_1\big),$$
$$\big\|\nabla_{\tau_1}(p - Q_{\tau_1}p_h)\big\|_0 \le Ch^{\frac{2r}{1+r}}\big(\|p\|_{r+1} + \|p\|_2 + \|u\|_1 + \|f\|_1\big).$$

Proof. By the definition of Q_{τ_1}, we have

$$\begin{aligned}\|p - Q_{\tau_1}p_h\|_0 &\le \|p - Q_{\tau_1}p\|_0 + \|Q_{\tau_1}p - Q_{\tau_1}p_h\|_0 \\ &\le C\tau_1^{r+1}\|p\|_{r+1} + Ch^2\big(\|p\|_2 + \|u\|_1 + \|f\|_1\big) \\ &\le Ch^{\alpha_1(r+1)}\|p\|_{r+1} + h^2\big(\|p\|_2 + \|u\|_1 + \|f\|_1\big). \end{aligned} \tag{5.9}$$

The above error estimate can be optimized by choosing α_1 such that

$$\alpha_1(r+1) = 2,$$

solving the above equation gives $\alpha_1 = \frac{2}{r+1}$, we obtain the L^2-estimate of $p - Q_{\tau_1}p_h$. It is easy to see that

$$\big\|\nabla_{\tau_1}(p - Q_{\tau_1}p)\big\|_0 \le C\tau_1^r\|p\|_{r+1} = Ch^{\alpha_1 r}\|p\|_{r+1},$$

where ∇_{τ_1} is defined elementwise over the partition $\mathcal{T}_{\tau_1}$. By the inverse inequality and Lemma 5, we have

$$\begin{aligned}\big\|\nabla_{\tau_1}(p - Q_{\tau_1}p_h)\big\|_0 &\le \big\|\nabla_{\tau_1}(p - Q_{\tau_1}p)\big\|_0 + \big\|\nabla_{\tau_1}(Q_{\tau_1}p - Q_{\tau_1}p_h)\big\|_0 \\ &\le C\big(h^{\alpha_1 r}\|p\|_{r+1} + h^{2-\alpha_1}\big(\|p\|_2 + \|u\|_1 + \|f\|_1\big)\big).\end{aligned}$$

We optimize the above estimate by choosing α_1 such that

$$\alpha_1 r = 2 - \alpha_1,$$

then, we obtain the following estimate

$$\big\|\nabla_{\tau_1}(p - Q_{\tau_1}p_h)\big\|_0 \le Ch^{\frac{2r}{1+r}}\big(\|p\|_{r+1} + \|p\|_2 + \|u\|_1 + \|f\|_1\big). \tag{5.10}$$

Combining (5.9) with (5.10), we finished the proof. □

Similarly, we have the following result for velocity.

Theorem 5. *Under the assumptions of Lemma 6, if τ_2, h and α_2 satisfy*

$$\tau_2 = \mathcal{O}\big(h^{\alpha_2}\big) \quad \textit{with } \alpha_2 = 2/(t+2),$$

then the postprocessed solution $R_{\tau_2}u_h$ satisfies

$$\|u - R_{\tau_2}u_h\|_0 \le Ch^{\frac{2(1+t)}{2+t}}\big(\|p\|_2 + \|u\|_{t+1} + \|f\|_1\big).$$

Table 1. The results of standard finite volume method with P_1 element.

$1/h$	$\frac{\|p_h-p\|_0}{\|p\|_0}$	Order	$\frac{\|p_h-p\|_1}{\|p\|_1}$	Order
10	0.1105	/	0.3040	/
20	0.02891	1.9348	0.1551	0.9709
30	0.01296	1.9788	0.1038	0.9905
40	0.007310	1.9905	0.07792	0.9969
50	0.004685	1.9937	0.06237	0.9976

Proof. By the definition of R_{τ_2}, we have

$$\begin{aligned}\|u - R_{\tau_2}u_h\|_0 &\le \|u - R_{\tau_2}u\|_0 + \|R_{\tau_2}u - R_{\tau_2}u_h\|_0 \\ &\le C\big(\tau_1^{t+1}\|u\|_{t+1} + h^{2-\alpha_2}\big(\|p\|_2 + \|u\|_1 + \|f\|_1\big)\big) \\ &\le C\big(h^{\alpha_2(t+1)}\|u\|_{t+1} + h^{2-\alpha_2}\big(\|p\|_2 + \|u\|_1 + \|f\|_1\big)\big). \qquad (5.11)\end{aligned}$$

The above error estimate can be optimized by choosing α_2 such that

$$\alpha_2(t+1) = 2 - \alpha_2.$$

Solving the above equation gives $\alpha_2 = \frac{2}{t+2}$. With this chosen, combining (5.11) we completed the proof. □

Finally, we have the following error estimates by choosing the different mesh sizes relationship.

Corollary 1. Assume that $V_{\tau_1} \in L^2(\Omega)$, $P_{\tau_2} \in H^1(\Omega)^2$ and $f \in H^1(\Omega)$. If $\alpha_1 = \alpha_2 = \frac{2}{3}$, then we have

$$\begin{aligned}&\|p - Q_{\tau_1}p_h\|_0 \le Ch^2\big(\|p\|_3 + \|u\|_1 + \|f\|_1\big), \\ &\big\|\nabla_{\tau_1}(p - Q_{\tau_1}p_h)\big\|_0 \le Ch^{\frac{4}{3}}\big(\|p\|_3 + \|u\|_1 + \|f\|_1\big) \\ &\|u - R_{\tau_2}u_h\|_0 \le Ch^{\frac{4}{3}}\big(\|p\|_2 + \|u\|_2 + \|f\|_1\big).\end{aligned}$$

Besides, if $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{1}{2}$, then we arrive at

$$\begin{aligned}&\|p - Q_{\tau_1}p_h\|_0 \le Ch^2\big(\|p\|_3 + \|u\|_1 + \|f\|_1\big), \\ &\big\|\nabla_{\tau_1}(p - Q_{\tau_1}p_h)\big\|_0 \le Ch^{\frac{4}{3}}\big(\|p\|_3 + \|u\|_1 + \|f\|_1\big) \\ &\|u - R_{\tau_2}u_h\|_0 \le Ch^{\frac{3}{2}}\big(\|p\|_2 + \|u\|_3 + \|f\|_1\big).\end{aligned}$$

As a consequence, we can see that there is no improvement for the pressure in L^2-norm, but the superconvergence results for the gradient of pressure and velocity are established under some regularities of p, u and f.

6 Numerical Experiments

In this section, we show some numerical examples to verify the established results in Sections 4 and 5. In all tests, we choose the domain Ω as the unit

Table 2. The results of mixed finite volume method with P_1^2-P_1 element.

$1/h$	$\frac{\|p_h-p\|_0}{\|p\|_0}$	Order	$\frac{\|p_h-p\|_1}{\|p\|_1}$	Order	$\frac{\|u_h-u\|_0}{\|u\|_0}$	Order
10	0.6450	/	0.3439	/	0.1752	/
20	0.1423	2.1804	0.1631	1.0762	0.08241	1.0881
30	0.06186	2.0546	0.1079	1.0190	0.05437	1.0262
40	0.03453	2.0267	0.08067	1.0110	0.04063	1.0128
50	0.02202	2.0161	0.06445	1.0060	0.03245	1.0063

Table 3. The superconvergence results with P_1^2-P_2 element with $\alpha_1 = \alpha_2 = \frac{2}{3}$.

$1/h$	$\frac{\|p_h-p\|_0}{\|p\|_0}$	Order	$\frac{\|p_h-p\|_1}{\|p\|_1}$	Order	$\frac{\|u_h-u\|_0}{\|u\|_0}$	Order
10	0.06621	/	0.2586	/	0.3882	/
20	0.01565	2.0809	0.08199	1.6572	0.1111	1.8049
30	0.007034	1.9723	0.04768	1.3369	0.06383	1.3668
40	0.003918	2.0341	0.03063	1.5383	0.04142	1.5033
50	0.002444	2.1150	0.02178	1.5281	0.02910	1.5821

Table 4. The superconvergence results with P_2^2-P_2 element with $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{1}{2}$.

$1/h$	$\frac{\|p_h-p\|_0}{\|p\|_0}$	Order	$\frac{\|p_h-p\|_1}{\|p\|_1}$	Order	$\frac{\|u_h-u\|_0}{\|u\|_0}$	Order
10	0.07612	/	0.24585	/	0.06982	/
20	0.01679	2.1807	0.08506	1.5312	0.03062	1.1892
30	0.007622	1.9478	0.05038	1.2918	0.01641	1.5384
40	0.004063	2.1868	0.03212	1.5646	0.01065	1.5028
50	0.002641	1.9304	0.02309	1.4792	0.007387	1.6395

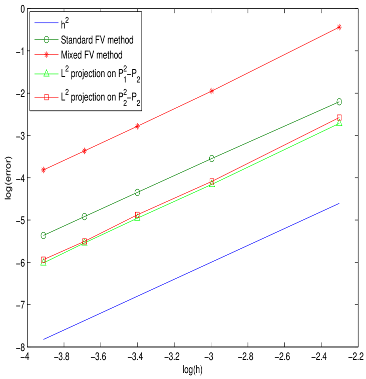
square $\Omega = [0, 1] \times [0, 1]$, the exact solution $p = \sin(2\pi x)\sin(2\pi y)$ and $f = 8\pi^2 \sin(2\pi x)\sin(2\pi y)$ is determined by (2.1).

Firstly, we provide the numerical results of standard finite volume method for (2.1) with different meshes, which are shown in Table 1. And then, we give the results obtained by mixed finite volume method (3.1) in different meshes. Compared with the results in Table 1, we can see that there are no much differences of the errors between two methods in H^1-norm, but we can obtain the better results in L^2-norm by using mixed method. Furthermore, the results in Table 2 verify the established theoretical findings of Theorems 2–3.

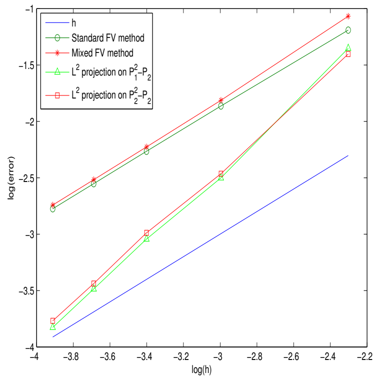
Finally, in order to achieve superconvergence for the numerical solution, the local L^2 projections are used. The key of this method is to project one finite element space onto another one based on the high order polynomials of the coarse mesh. The solution of $(Q_{\tau_1} p_h, R_{\tau_2} u_h)$ can be computed as follows: Find $(Q_{\tau_1} p_h, R_{\tau_2} u_h) \in (V_{\tau_1}, P_{\tau_2})$ for all $(w, v) \in (V_{\tau_1}, P_{\tau_2})$ such that

$$(Q_{\tau_1} p_h, w) = (p_h, w), \qquad (R_{\tau_2} u_h, v) = (u_h, v),$$

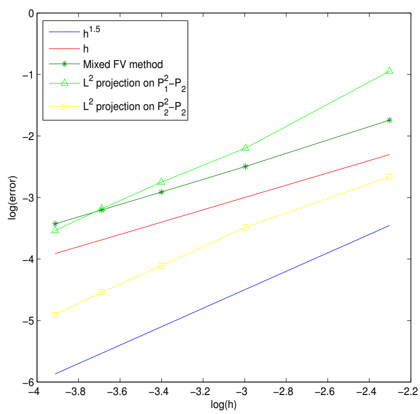
where (p_h, u_h) is the solution obtained by (3.1). Tables 3–4 list the mesh sizes among different partitions and the convergence rates of numerical solutions.



(a)



(b)



(c)

Figure 1. Errors of various variables of different methods for Poisson equations. (a) L^2-error for pressure, (b) H^1-error for the pressure, (c)L^2-error for velocity.

From those Tables, we can see that the relative errors of pressure obtained by L^2 projection method are better than these in Table 2, and the convergence orders confirm the results of Corollary 1, see also Figure 1 for details.

7 Conclusions

In this paper we have presented a theoretical analysis of stabilized finite volume method for Poisson equation. By introducing a new mixed variational formulation, we obtained the optimal error estimates for the numerical solutions based on the linear polynomials over the triangulation partitions. Furthermore, with the help of the L^2-projection method, we presented some superconvergence results for the numerical solutions under some regularity assumptions for Poisson equation. Some numerical experiments are presented at last.

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