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A Strongly Ill-Posed Integro-Differential Parabolic Problem with Integral Boundary Conditions*

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Abstract. Via Carleman estimates we prove uniqueness and continuous dependence results for an identification and strongly ill-posed linear integro-differential parabolic problem with the Dirichlet boundary condition, but with no initial condition. The additional information consists in a boundary linear integral condition involving the normal derivative of the temperature on the whole of the lateral boundary of the space-time domain.

Keywords: ill-posed problems, linear parabolic integro-differential equations, no initial condition, integral boundary conditions, uniqueness, continuous dependence results.

AMS Subject Classification: primary 35R30; secondary 35K20, 45K05, 45Q05.

1 Introduction

This paper is concerned with the determination of an unknown time-dependent function in a strongly ill-posed problem, where *strongly* means that no transformation can be found in order to change our problem to a well-posed one, at least, when working in classical or Sobolev function spaces of *finite order*.

We stress that the main stream concerning strongly ill-posed problems involves PDE's. For this purpose we refer the reader to the monograph [4], dealing with elliptic and parabolic problems, and the rich bibliography therein. Of course, lesser interest was devoted to recovering unknown functions in strongly ill-posed integro-differential problems. This paper is just devoted to shed some light on such problems, mainly on the questions of uniqueness and continuous dependence on the data, two fundamental topics for people working in Applied Mathematics. More exactly, we will deal here with an integro-differential linear parabolic problem, where in the integral operator the integrations involve both

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space and time and the operator is of Volterra type. Furthermore in our problem no initial condition will be supplied. It will be replaced by the requirement that the "temperature" u should assume prescribed values on $(0,T)\times\partial\Omega$ as well as an additional boundary condition involving a linear combination of the flux of u and a Volterra type integral of the temperature itself will be prescribed on $(0,T)\times\partial\Omega$.

The main task of this paper consists in recovering a function α in the source term when a mean of the temperature is known for all times $t\in[0,T]$. Moreover, we shall be able to estimate u in $C((0,T];L^2(\Omega))\cap L^2_{loc}((0,T];H^1(\Omega))$ in terms of suitable norms of our data and α in $L^2((0,T];\mathbb{R})$.

The fundamental tool to give a positive answer to our problem will be deduced by adapting to our case the celebrated (weighted) Carleman estimates for PDE's [5, 6] – of use both in Control and Inverse Problem Theory. They will ensure the *uniqueness* of the solution to our problem and also an unusual (weighted) result involving continuous dependence on the data. To improve this result to one related to usual L^2-spaces we shall add a few new assumptions on the kernels and some additional analysis.

We conclude this introduction by observing that, although there is a wide literature concerned with the problem of recovering an unknown function entering a *well-posed* parabolic problem and also a rich literature dealing with ill-posed problems for PDE's, at the best of our knowledge in these last years no paper can be found in MathScinet corresponding to the keywords *ill posed* and *identification* or *recovering* and concerned with integro-differential equations with integral boundary conditions. We quote also a few related papers concerning the identification of constants in *strongly ill-posed* parabolic problems [8, 9] or in *well-posed* ones [1, 3, 7, 10, 11, 12, 13].

The plan of the paper is the following: Section 2 reports Carleman estimates for linear second-order parabolic equations when the additional information is given on the lateral boundary. In Section 3 we express the unknown function as an affine operator involving u, so that we can transform the given integro-differential identification problem into a strongly (non-standard) ill-posed integro-differential problem for u. In Section 4 we determine sufficient conditions on our data and the integral kernels leading to a Carleman estimate of weak type, i.e. with a *fixed parameter* s_0 (cf. Theorem 1), as well as we deduce the desired *weighted* L^2-estimate for unknown function α. Finally, in Section 5 we determine sufficient conditions on the integral kernels ensuring the continuous dependence on our data of (u,α) in the non-weighted spaces $[C((0,T];L^2(\Omega))\cap L^2_{loc}((0,T];L^2(\Omega))]\times L^2((0,T];\mathbb{R})$ (cf. Theorem 2).

2 An Auxiliary Linear Strongly Ill-Posed Problem

Let Ω be a bounded open sets in $\mathbb{R}^n$, $\partial\Omega$ being of C^2-class. Let $A(x,D)$ be the (formal) uniformly elliptic linear operator defined by

$$\begin{aligned} A(x,D) &= \sum_{i,j=1}^{n} D_{x_i}\big[a_{i,j}(x)D_{x_j}\big] + \sum_{j=1}^{n} D_{x_j}\big[a_j(x)\cdot\big] \\ &=: A_1(x,D) + \sum_{j=1}^{n} D_{x_j}\big[a_j(x)\cdot\big], \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} &a_{i,j} \in C^1(\overline{\Omega}), \quad a_{j,i} = a_{i,j}, \quad a_j \in W^{1,\infty}(\Omega), \quad i,j = 1,\dots n, \\ &\mu_0|\xi|^n \le \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \le \mu_1|\xi|^n, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^n, \end{aligned}$$

for some positive constants μ_0 and μ_0, $\mu_0 \le \mu_1$.

We consider the following ill-posed problem, where $Q_T = (0,T) \times \Omega$ and $\Sigma_T = (0,T) \times \partial\Omega$:

$$(\mathrm{IP1}) \quad \begin{cases} u \in H^1\big((0,T);L^2(\Omega)\big) \cap L^2\big((0,T);H^2(\Omega)\big), \\ D_t u(t,x) - A(x,D)u(t,x) = q(t,x), & (t,x) \in Q_T, \\ u(t,x) = g_0(t,x), & (t,x) \in \Sigma_T, \\ D_\nu u(t,x) = g_1(t,x), & (t,x) \in \Sigma_T. \end{cases} \tag{2.2}$$

Here D_ν denotes the (outer) normal derivative, ν denoting the (outer) normal vector-field. Moreover, we assume

$$q \in L^2(Q_T), \quad g_0 \in H^1\big((0,T);L^2(\Omega)\big) \cap L^2\big((0,T);H^2(\Omega)\big).$$

Then the function $v = u - g_0$ solves the (equivalent) ill-posed problem

$$(\mathrm{IP2}) \quad \begin{cases} v \in H^1\big((0,T);L^2(\Omega)\big) \cap L^2\big((0,T);H^2(\Omega)\big), \\ D_t v(t,x) - A(x,D)v(t,x) \\ \quad = q(t,x) - D_t g_0(t,x) + A(x,D)g_0(t,x), & (t,x) \in Q_T, \\ v(t,x) = 0, & (t,x) \in \Sigma_T, \\ D_\nu v(t,x) = g_1(t,x) - D_\nu g_0(t,x), & (t,x) \in \Sigma_T. \end{cases} \tag{2.3}$$

In the sequel we shall need the Carleman estimate related to problem (IP2). For this purpose we begin by introducing the functions $\varphi_\lambda : \overline{\Omega} \to \mathbb{R}$, $\alpha_\lambda : [0,T] \times \overline{\Omega} \to \mathbb{R}$, depending on the parameter $\lambda \in [1,+\infty)$, defined by

$$\varphi_\lambda(t,x) = e^{\lambda\psi(x)}, \quad \alpha_\lambda(t,x) = \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_\infty}}{l(t)}, \quad (t,x) \in (0,T) \times \overline{\Omega},$$

where

$$l(t) = t(T-t), \tag{2.4}$$

and function $\psi \in C^4(\overline{\Omega})$ satisfies the properties (cf. [5])

$$\psi(x) > 0,\ x \in \Omega, \quad \big|\nabla\psi(x)\big| \ge \mu_2 > 0,\ x \in \overline{\Omega}, \quad D_{\nu_A}\psi(x) \le 0,\ x \in \partial\Omega \setminus \Gamma$$

for some positive constant μ_2.

Owing to Lemma 2.4, with $p = 0$, in [6], since $\varphi_\lambda(x) \ge 1$ for all $x \in \overline{\Omega}$, any solution $v \in H^1((0,T); L^2(\Omega)) \cap L^2((0,T); H^2(\Omega) \cap H^1(\Omega))$ to problem (IP2) satisfies the Carleman estimate

$$\begin{aligned}
& s^3 \int_{Q_T} l(t)^{-3}\big|v(t,x)\big|^2 \exp\big[2s\alpha_\lambda(t,x)\big]\,dt\,dx \qquad (2.5)\\
&\quad + s\int_{Q_T} l(t)^{-1}\big|\nabla v(t,x)\big|^2 \exp\big[2s\alpha_\lambda(t,x)\big]\,dt\,dx + s^{-1}e^{-\lambda\psi_m}\\
&\quad \times \int_{Q_T} l(t)\Big[\big|D_t v(t,x)\big|^2 + \sum_{i,j=1}^n \big|D_{x_i}D_{x_j}v(t,x)\big|^2\Big] \exp\big[2s\alpha_\lambda(t,x)\big]\,dt\,dx\\
&\le s^3 \int_{Q_T} l(t)^{-3}\varphi_\lambda(x)^3\big|v(t,x)\big|^2 \exp\big[2s\alpha_\lambda(t,x)\big]\,dt\,dx\\
&\quad + s\int_{Q_T} l(t)^{-1}\varphi_\lambda(x)\big|\nabla v(t,x)\big|^2 \exp\big[2s\alpha_\lambda(t,x)\big]\,dt\,dx\\
&\quad + s^{-1}\int_{Q_T} l(t)\varphi_\lambda(x)^{-1}\Big[\big|D_t v(t,x)\big|^2 + \sum_{i,j=1}^n \big|D_{x_i}D_{x_j}v(t,x)\big|^2\Big]\\
&\quad \times \exp\big[2s\alpha_\lambda(t,x)\big]\,dt\,dx\\
&\le C_1\Big\{\int_{Q_T} \big[\big|D_t g_0(t,x)\big|^2 + \big|A(x,D)g_0(t,x)\big|^2\big] \exp\big[2s\alpha_\lambda(t,x)\big]\,dt\,dx\\
&\quad + \int_{Q_T} \big|h(t,x)\big|^2 \exp\big[2s\alpha_\lambda(t,x)\big]\,dt\,dx\\
&\quad + s\int_{\Sigma_T} l(t)^{-1}\varphi_\lambda(x)\exp\big[2s\alpha_\lambda(t,x)\big]\big|D_\nu v(t,x)\big|^2\,dt\,dx\Big\}, \qquad s \ge \widehat{s}_0.
\end{aligned}$$

Here $|\psi|_\infty = \min_{x\in\overline{\Omega}} \psi(x)$ and the positive constants C_1, λ and $\widehat{s}_0$ depend on μ_0, T, $\|a_{i,j}\|_{L^\infty(\Omega)}$, $\|a_j\|_{W^{1,\infty}(\Omega)}$, $i,j = 1,\dots,n$, and Ω.

3 The Strongly Ill-Posed Identification Problem

We consider the ill-posed problem consisting in *recovering the function* $\alpha : (0,T) \to \mathbb{R}$ *in the linear integro-differential parabolic problem*

$$\text{(IP3)} \quad \begin{cases} u \in H^1\big((0,T); L^2(\Omega)\big) \cap L^2\big((0,T); H^2(\Omega)\big), & \\ D_t u(t,x) - A(x,D)u(t,x) & \\ \quad = B_1 u(t,x) + \alpha(t)f(t,x) + h(t,x), & (t,x) \in Q_T, \\ u(t,x) = 0, & (t,x) \in \Sigma_T, \\ D_\nu u(t,x) = g(t,x) + B_2 u(t,x), & (t,x) \in \Sigma_T, \end{cases} \qquad (3.1)$$

ν denoting the (outer) normal to $\partial\Omega$, under the following *additional condition*, standing for an (integral) spatial mean of u:

$$\int_\Omega u(t,x)\,dx = \beta(t), \quad t \in (0,T). \tag{3.2}$$

Here $\beta : (0,T) \to \mathbb{R}$ and B_1 and B_2 are the linear integral operators defined by

$$B_1 w(t,x) = \int_{Q_t} k_1(t,x,r,y) w(r,y)\,dr\,dy, \quad (t,x) \in Q_T,$$
$$B_2 w(t,x) = \int_{Q_t} k_2(t,x,r,y) w(r,y)\,dr\,dy, \quad (t,x) \in \Sigma_T,$$

where the kernels $k_j : E \to \mathbb{R}$, with $j = 1,2$ and $E_1 = \{(t,x,r,y) \in Q_T \times Q_T\colon r < t\}$ and $E_2 = \{(t,x,r,y) \in \Sigma_T \times Q_T\colon r < t\}$, are measurable and, for the time being, separably integrable over Q_T, if $j = 1$, and over Σ_T, if $j = 2$, with respect to (t,x) and over Q_T with respect to (r,y).

We note that, up to a translation, a problem with a non-vanishing u on Σ_T, can be reduced to one with a vanishing u on Σ_T.

We need now to compute the conormal derivative $D_{\nu_A} u$ in terms of the normal derivative $D_\nu u$. For this task we recall that the conormal vector-field ν_{A_1} is defined by

$$(\nu_A)_i = \sum_{j=1}^{n} a_{i,j}(x)\nu_i(x), \quad x \in \partial\Omega,\ i = 1,\dots,n.$$

Then we note that, if $\tau^{(j)}(x)$, $j = 1,\dots,d-1$, denote the set of $d-1$ unit vectors that are mutually orthogonal and tangent to $\partial\Omega$ at x, we have

$$\nu_A(x) = \sum_{j=1}^{d-1} c_j(x)\tau^{(j)}(x) + \big[\nu_{A_1}(x)\cdot n(x)\big]\nu(x), \quad x \in \partial\Omega.$$

Since

$$\overline{\nu}_A(x) := \nu_A(x)\cdot n(x) = \sum_{j=1}^{d} a_{i,j}(x) n_i(x) n_j(x) \geq \mu_0 > 0, \quad \text{for all } x \in \partial\Omega$$

and all the tangential derivative of u vanish according to the first boundary condition in (3.1), we easily deduce the relation

$$D_{\nu_A} u(x) = \overline{\nu}_A(x) D_\nu u(x), \quad x \in \partial\Omega.$$

Apply now the functional $Jw = \int_\Omega w(x)\,dx$ to both sides of the differential equation in (IP2) and consider the following formulae (cf. (2.2)):

$$\begin{aligned}\int_\Omega Au(t,x)\,dx &= \int_{\partial\Omega} D_{\nu_A} u(t,x)\,d\sigma(x) = \int_{\partial\Omega} \overline{\nu}_A(x) D_\nu u(t,x)\,d\sigma(x)\\ &= \int_{\partial\Omega} \overline{\nu}_A(x) g(t,x)\,d\sigma(x) + \int_{\partial\Omega} \overline{\nu}_A(x) B_2 u(t,x)\,d\sigma(x),\end{aligned}$$

where σ denotes the Lebesgue surface measure.

Consider now the identities

$$\begin{aligned}\int_{\Omega} B_1 u(t,x)\,dx &= \int_{\Omega} dx \int\limits_{(0,t)\times\Omega} k_1(t,x,r,y)u(r,y)\,dr\,dy\\ &= \int_{(0,t)\times\Omega} \widetilde{k}_1(t,r,y)u(r,y)\,dr\,dy,\\ \int_{\partial\Omega} \overline{\nu}_A(x)B_2u(t,x)\,d\sigma(x) &= \int_{\partial\Omega} \overline{\nu}_A(x)\,d\sigma(x)\int_{(0,t)\times\Omega} k_2(t,x,r,y)u(r,y)\,dr\,dy\\ &= \int_{(0,t)\times\Omega} \widetilde{k}_2(t,r,y)u(r,y)\,dr\,dy,\end{aligned}$$

where

$$\widetilde{k}_1(t,r,y) = \int_{\Omega} k_1(t,x,r,y)\,dx, \tag{3.3}$$

$$\widetilde{k}_2(t,r,y) = \int_{\partial\Omega} \overline{\nu}_A(x)k_2(t,x,r,y)\,d\sigma(x). \tag{3.4}$$

We easily deduce the equation for α:

$$-\int_{Q_t} \widetilde{k}(t,r,x)u(r,y)\,dr\,dy + \gamma(\beta,f)(t) = \alpha(t)\int_{\Omega} f(t,x)\,dx,$$

where

$$\widetilde{k}(t,r,y) = \sum_{j=1}^{2} \widetilde{k}_j(t,r,y), \tag{3.5}$$

$$\gamma(\beta,g,h)(t) = \beta'(t) - \int_{\partial\Omega} \nu_A(x)g(t,x)\,d\sigma(x) - \int_{\Omega} h(t,x)\,dx.$$

Assume now

$$\Big|\int_{\Omega} f(t,x)\,dx\Big| \geq m > 0, \quad \forall t \in [0,T] \tag{3.6}$$

and set $\chi(f)(t)^{-1} := \int_{\Omega} f(t,x)\,dx$. Then α can be represented as an operator of u:

$$\begin{aligned}\alpha(t) &= -\chi(f)(t)\int\limits_{(0,t)\times\Omega} \widetilde{k}(t,r,y)u(r,y)\,dr\,dy + \chi(f)(t)\gamma(\beta,g,h)(t)\\ &=: -\Lambda(u)(t) - I(\beta,f,g,h)(t).\end{aligned} \tag{3.7}$$

Consequently, u has to solve the strongly ill-posed integro-differential problem

$$\text{(IP4)}\quad \begin{cases} u \in H^1\big((0,T);L^2(\Omega)\big) \cap L^2\big((0,T);H^2(\Omega)\big), & \\ D_t u(t,x) - A(x,D)u(t,x) & \\ \quad = \mathcal{B}u(t,x) + \widetilde{h}(t,x), & (t,x)\in Q_T,\\ u(t,x) = 0, & (t,x)\in \Sigma_T,\\ D_n u(t,x) = g(t,x) + B_2 u(t,x), & (t,x)\in\Sigma_T, \end{cases} \tag{3.8}$$

where

$$\begin{aligned}\mathcal{B}u(t,x) &= B_1u(t,x) - \Lambda(u)(t)f(t,x),\\ \widetilde{h}(t,x) &= h(t,x) - I(\beta,f,g,h)(t)f(t,x), \quad (t,x)\in Q_T.\end{aligned} \tag{3.9}$$

4 An Estimate of Weak Carleman Type

Let u be a solution to problem (IP4). Then u solves problem (IP2) with $q=\mathcal{B}+\widetilde{h}$, $g_0=0$, and $g_1=g+B_2u$. Consequently, from (2.5), with $s=s_0\geq\widehat{s}_0$ to be chosen later on, we deduce the following estimate

$$\begin{aligned}&\int_{Q_T} l(t)^{-3}\big|u(t,x)\big|^2\exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,dx\\
&\quad + s_0\int_{Q_T} l(t)^{-1}\big|\nabla u(t,x)\big|^2\exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,dx + s_0^{-1}e^{-\lambda\|\psi\|_\infty}\\
&\quad \times\int_{Q_T} l(t)\Big[\big|D_tu(t,x)\big|^2+\sum_{i,j=1}^n\big|D_{x_i}D_{x_j}u(t,x)\big|^2\Big]\exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,dx\\
&\leq 2C_1\int_{Q_T}\big|\mathcal{B}u(t,x)\big|^2\exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,dx\\
&\quad + 2C_1\int_{Q_T}\big|\widetilde{h}(t,x)\big|^2\exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,dx\\
&\quad + 2s_0C_1\int_{\Sigma_T}\big[\big|g(t,x)\big|^2+\big|B_2u(t,x)\big|^2\big]\exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,d\sigma(x).\end{aligned} \tag{4.1}$$

To prove our *weak* Carleman estimate,[1] related to a *suitable* s_0, for our ill-posed integro-differential problem (IP4) we need four positive constants K_j, $j=0,\dots,3$, and choose s_0 satisfying the inequalities

$$p_0(s_0,f) := s_0\big[s_0^2-2C_1K_4(f)\big]>0, \quad s_0\geq\widehat{s}_0, \quad s_0\geq 1, \tag{4.2}$$

where

$$K_4(f)=K_0K_1+m^{-2}\|f\|^2_{L^\infty(Q_T)}m(\Omega)[K_0+\mu_1K_2][K_1+\mu_1K_3]>0. \tag{4.3}$$

Once we have fixed s_0, we are allowed to make the following assumptions concerning k_1 and k_2:

$$\begin{aligned}&\int_{Q_t}\big|k_j(t,x,r,y)\big|\,dr\,dy\leq K_{2(j-1)},\\
&\quad (t,x)\in Q_T,\ \text{if } j=1,\ (t,x)\in\Sigma_T,\ \text{if } j=2,\end{aligned} \tag{4.4}$$

$$\begin{aligned}&\int_{(r,T)\times F_j}\big|k_j(t,x,r,y)\big|\,dt\,d\rho_j(x)\leq K_{2j-1}l(r)^{-3}\exp\big[-2s_0c_1(\psi)l(r)^{-1}\big],\\
&\quad (r,y)\in Q_T,\ j=1,2,\ F_1=\Omega,\ F_2=\partial\Omega, \quad \rho_1=m,\ \rho_2=\sigma,\end{aligned} \tag{4.5}$$

m and σ denoting, respectively, the n-dimensional Lebesgue measure and the related Lebesgue surface measure.

[1] Weak means here that the Carleman estimate holds for a *fixed* $s=s_0$.

Remark 1. To exhibit an example of functions k_j, $j = 1, 2$, satisfying conditions (4.4), (4.5) we choose the functions

$$k_j(t, x, r, y) = \sum_{i=1}^{+\infty} h_{1,i,j}(t, x) h_{2,i,j}(r, y), \quad j = 1, 2, \tag{4.6}$$

where the functions $h_{1,i,j} \in L^\infty((0, T) \times F_j)$ and $h_{2,i,j} \in L^\infty(Q_T)$ satisfy the following bounds for all $i \in \mathbb{N}$, $j = 1, 2$:

$$\bigl|h_{2,i,j}(r, y)\bigr| \le \kappa_{i,j} l(t)^{-3} \exp\bigl[-2s_0 c_1(\psi) l(t)^{-1}\bigr], \quad (r, y) \in Q_T,\ j = 1, 2, \tag{4.7}$$

$$\sum_{i=1}^{+\infty} \kappa_{i,j} \|h_{1,i,j}\|_{L^\infty((0,T)\times F_j)} < +\infty, \quad j = 1, 2, \tag{4.8}$$

for some nonnegative constants $\kappa_{i,j}$, $i \in \mathbb{N}$, $j = 1, 2$. Indeed, assumptions (4.7)–(4.8) imply

$$\begin{aligned}
\int_{Q_T} \bigl|k_j(t, x, r, y)\bigr|\, dr\, dy &\le \sum_{i=1}^{\infty} \bigl|h_{1,i,j}(t, x)\bigr| \|h_{2,i,j}\|_{L^1(Q_T)} \\
&\le 4m(\Omega) \sum_{i=1}^{\infty} \kappa_{i,j} \|h_{1,i,j}\|_{L^\infty((0,T)\times F_j)} \\
&\quad \times \int_{4/T^2}^{\infty} s^{3/2} (T^2 s - 4)^{-1/2} \exp\bigl[-2s_0 c_1(\psi) s\bigr]\, ds =: K_{2(j-1)}, \quad j = 1, 2,
\end{aligned}$$

$$\begin{aligned}
\int_{(r,T)\times F_1} \bigl|k_1(t, x, r, y)\bigr|\, dt\, dx &\le \sum_{i=1}^{\infty} \|h_{1,i,j}\|_{L^1((0,T)\times F_1)} \bigl|h_{2,i,j}(r, y)\bigr| \\
&\le \frac{27}{8} \bigl[c_1(\psi)\bigr]^{-3} m(\Omega) \sum_{i=1}^{\infty} \kappa_{i,j} \|h_{1,i,j}\|_{L^\infty((0,T)\times\Omega)} =: K_1,
\end{aligned}$$

$$\begin{aligned}
\int_{(r,T)\times F_2} \bigl|k_2(t, x, r, y)\bigr|\, dt\, d\sigma(x) &\le \sum_{i=1}^{\infty} \|h_{1,i,j}\|_{L^1((0,T)\times F_2)} \bigl|h_{2,i,j}(r, y)\bigr| \\
&\le \frac{27}{8} \bigl[c_1(\psi)\bigr]^{-3} \sigma(\partial\Omega) \sum_{i=1}^{+\infty} \kappa_{i,j} \|h_{1,i,j}\|_{L^\infty((0,T)\times\partial\Omega)} =: K_3,
\end{aligned}$$

since $s_0 \ge 1$. We stress that, according to assumption (4.7) and definition (2.4), functions $h_{2,i,j}$ must vanish exponentially as $t \to 0+$ and $t \to T-$.

We can now state the main result of this section.

Theorem 1. *Let* $f \in L^\infty(Q_T)$, $h \in L^2(Q_T)$, $g \in H^1((0, T); L^2(\Omega)) \cap L^2((0, T); H^2(\Omega))$, $\beta \in H^1(0, T)$. *Moreover, let* f *and the kernels* $k_j : E_j \to \mathbb{R}$, $j = 1, 2$, *satisfy, respectively, conditions* (4.4) *and* (4.5). *Then the solution* u *to problem* (IP4) *satisfies the weak Carleman estimate*

$$\begin{aligned}
p(s_0, f) &\int_{Q_T} l(t)^{-3} \exp\bigl[-2s_0 c_1(\psi) l(t)^{-1}\bigr] \bigl|u(t, x)\bigr|^2\, dt\, dx \\
&+ s_0 \int_{Q_T} l(t)^{-1} \bigl|\nabla u(t, x)\bigr|^2 \exp\bigl[2s_0 \alpha_\lambda(t, x)\bigr]\, dt\, dx + s_0^{-1} e^{-\lambda\|\psi\|_\infty}
\end{aligned}$$

$$\times \int_{Q_T} l(t)\Big[\big|D_t u(t,x)\big|^2 + \sum_{i,j=1}^{n} \big|D_{x_i} D_{x_j} u(t,x)\big|^2\Big] \exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,dx$$

$$\le 2C_1 \int_{Q_T} \big|\widetilde{h}(t,x)\big|^2\,dt\,dx + 2s_0 C_1 \int_{\Sigma_T} \big|g(t,x)\big|^2\,dt\,d\sigma(x), \tag{4.9}$$

where $c_1(\psi) = e^{2\lambda\|\psi\|_\infty}$*. Moreover, the solution* (u,α) *to problem* (IP3), (3.2) *satisfies estimate* (4.9) *and the following*

$$\int_0^T \big|\alpha(t)\big|^2 \exp\big[-2s_0 c_1(\psi) l(t)^{-1}\big]\,dt \le 2C_1 m^{-2} p(s_0,f)^{-1} \|f\|^2_{L^\infty(Q_T)} m(\Omega)$$

$$\times [K_0 + \mu_1 K_2][K_1 + \mu_1 K_3]\Big[\int_{Q_T} \big|\widetilde{h}(t,x)\big|^2\,dt\,dx$$

$$+ s_0 \int_{\Sigma_T} \big|g(t,x)\big|^2\,dt\,d\sigma(x)\Big] + 2\int_0^T \big|I(\beta,f,g,h)(t)\big|^2\,dt. \tag{4.10}$$

In particular, if $(g,h,\beta) = (0,0,0)$*, the linear identification problem* (IP3), (3.2) *admits the null solution, only.*

Remark 2. As a by-product, we have proved the uniqueness of the solution to the direct ill-posed problem (IP3) with $\alpha = 0$, i.e. to problem (IP4) with $\Lambda = O$ and $I = 0$.

To the proof of Theorem 1 we premise two lemmata.

Lemma 1. *Let* $\omega = \{(t,r) \in (0,T)^2 \colon r < t\}$ *and* $\widetilde{k} : \omega \times \Omega \to \mathbb{C}$ *be a measurable function satisfying the following inequalities for some positive constants* $K_0(\widetilde{k})$, $K_1(\widetilde{k})$ *and* $s_0 \ge \widehat{s}_0$:

$$K_0(\widetilde{k}) = \operatorname*{ess\,sup}_{t\in(0,T)} \int_{Q_t} \big|\widetilde{k}(t,r,y)\big|\,dr\,dy < \infty,$$

$$\int_r^T \big|\widetilde{k}(t,r,y)\big|\,dt \le K_1(\widetilde{k}) l(r)^{-3} \exp\big[-2s_0 c_1(\psi) l(r)^{-1}\big], \quad (r,y) \in Q_T, \tag{4.11}$$

where

$$c_1(\psi) = e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi_m}, \quad \psi_m = \min_{x\in\overline{\Omega}} \psi(x). \tag{4.12}$$

Then the linear operator

$$\widetilde{B}u(t) = \int_{Q_t} \widetilde{k}(t,r,y) u(r,y)\,dr\,dy, \quad t \in (0,T)$$

satisfies the weighted inequality

$$m(\Omega) \int_0^T \big|\widetilde{B}u(t)\big|^2 \exp\big[2s_0 c_1(\psi) l(t)^{-1}\big]\,dt$$

$$\le \int_{Q_T} \big|\widetilde{B}u(t)\big|^2 \exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,dx$$

$$\le m(\Omega) K_0(\widetilde{k}) K_1(\widetilde{k}) \int_{Q_T} l(t)^{-3} \big|v(t,x)\big|^2 \exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,dx. \tag{4.13}$$

Proof. Consider first the inequalities

$$\big|\widetilde{B}u(t)\big|^2 \le \int_{Q_t} \widetilde{k}(t,r,y)\,dr\,dy \int_{Q_t} \big|\widetilde{k}(t,r,y)\big|\big|u(r,y)\big|^2\,dr\,dy$$
$$\le K_0(\widetilde{k}) \int_{Q_t} \big|\widetilde{k}(t,r,y)\big|\big|u(r,y)\big|^2\,dr\,dy, \quad (t,x)\in Q_T.$$

Then we have the following chain of inequalities

$$\int_{Q_T} \big|\widetilde{B}u(t)\big|^2 \exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,dx$$
$$\le K_0(\widetilde{k}) \int_{Q_T} \exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,dx \int_{Q_t} \big|\widetilde{k}_\lambda(t,r,y)\big|\big|u(r,y)\big|^2\,dr\,dy$$
$$\le K_0(\widetilde{k}) \int_{Q_T} \overline{k}_\lambda(r,y) l(r)^{-3} \exp\big[2s_0\alpha_\lambda(r,y)\big]\big|u(r,y)\big|^2\,dr\,dy, \tag{4.14}$$

where, for all $(r,y)\in Q_T$, we have set

$$\overline{k}_\lambda(r,y) = \int_{(r,T)\times\Omega} l(r)^3 \exp\big[2s_0(\alpha_\lambda(t,x)-\alpha_\lambda(r,y))\big]\big|\widetilde{k}(t,r,y)\big|\,dt\,dx.$$

Observe now the identities

$$\alpha_\lambda(t,x)-\alpha_\lambda(r,y) = \big[\alpha_\lambda(t,x)-\alpha_\lambda(r,x)\big] + \big[\alpha_\lambda(r,x)-\alpha_\lambda(r,y)\big] \tag{4.15}$$
$$= \frac{l(t)-l(r)}{l(t)l(r)}\big[e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi(x)}\big] + \frac{1}{l(r)}\big[e^{\lambda\psi(x)} - e^{\lambda\psi(y)}\big].$$

Note that, when $t>r$,

$$l(t)-l(r) = (t-r)(T-t-r) > 0$$

if and only if $r\in(0,T/2)$ and $t\in(r,T-r)$, so that, in this case, we have $[l(t)-l(r)][l(t)l(r)]^{-1} \le l(r)^{-1}$. Since

$$e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi(x)} \le e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi_m} = c_1(\psi), \tag{4.16}$$
$$e^{\lambda\psi(x)} - e^{\lambda\psi(y)} \le e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi_m} \le c_1(\psi), \tag{4.17}$$

we easily deduce the inequalities

$$\alpha_\lambda(t,x)-\alpha_\lambda(r,y) \le l(r)^{-1}\begin{cases}\big[e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi(y)}\big], & \text{if } t\in(\max\{r,T-r\},T),\\ \big[e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi(y)}\big], & \text{otherwise,}\end{cases}$$
$$\le c_1(\psi)l(r)^{-1}. \tag{4.18}$$

Then, according to assumption (4.11) and (4.12), for all $(r,y)\in Q_T$ we conclude that

$$\overline{k}_\lambda(r,y) \le l(r)^3 \exp\big[2s_0c_1(\psi)l(r)^{-1}\big] \int_{(r,T)\times\Omega} \big|\widetilde{k}(t,r,y)\big|\,dt\,dx$$
$$\le m(\Omega)l(r)^3 \exp\big[2s_0c_1(\psi)l(r)^{-1}\big] \int_r^T \big|\widetilde{k}(t,r,y)\big|\,dt$$
$$\le m(\Omega)K_1(\widetilde{k}). \tag{4.19}$$

Finally, the latter inequality in (4.13) easily follows from (4.14), (4.19), while the first is an easy consequence of (4.16). □

We prove now the following Lemma 2 concerning the kernels k_j, $j = 1, 2$.

Lemma 2. *Let* $E_{1,t_1,t_2} = (t_1, t_2) \times \Omega$ *and* $E_{2,t_1,t_2} = (t_1, t_2) \times \partial\Omega$, $0 \le t_1 < t_2 \le T$. *Let* $k_j : E_{j,0,T} \times Q_T \to \mathbb{C}$, $j = 1, 2$, *be two measurable functions satisfying inequalities* (4.4) *and* (4.5). *Then the linear operator*

$$B_j u(t,x) = \int_{Q_t} k_1(t,x,r,y)u(r,y)\,dr\,dy, \quad (t,x) \in E_{j,0,T},$$

satisfies the weighted inequality

$$\begin{aligned}
&\int_{E_{j,0,T}} \big|B_j u(t,x)\big|^2 \exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,d\rho_j(x) \\
&\quad \le K_{2(j-1)}K_{2j-1} \int_{Q_T} l(t)^{-3}\big|u(t,x)\big|^2 \exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,d\rho_j(x), \quad j = 1, 2,
\end{aligned}$$

where $\rho_1 = m$ *and* $\rho_2 = \sigma$.

Proof. Consider first the inequalities

$$\begin{aligned}
\big|B_j u(t,x)\big|^2 &\le \int_{Q_t} k_j(t,x,r,y)\,dr\,dy \int_{Q_t} \big|k_j(t,x,r,y)\big|\big|u(r,y)\big|^2\,dr\,dy \\
&\le K_{2(j-1)} \int_{Q_t} \big|k_j(t,x,r,y)\big|\big|u(r,y)\big|^2\,dr\,dy, \quad (t,x) \in E_{j,0,T},\ j = 1, 2.
\end{aligned}$$

Then we have the following chain of inequalities

$$\begin{aligned}
&\int_{E_{j,0,T}} \big|B_j u(t,x)\big|^2 \exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,d\rho_j(x) \\
&\quad \le K_{2(j-1)} \int_{E_{j,0,T}} \exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,d\rho_j(x) \int_{Q_t} \big|k_{j,\lambda}(t,r,x,y)\big|\big|u(r,y)\big|^2\,dr\,dy \\
&\quad \le K_{2(j-1)} \int_{Q_T} \overline{k}_j(r,y)l(r)^{-3}\exp\big[2s_0\alpha_\lambda(r,y)\big]\big|u(r,y)\big|^2\,dr\,dy, \qquad (4.20)
\end{aligned}$$

where, for all $(t,x) \in E_{j,0,T}$, we have set

$$\overline{k}_{j,\lambda}(r,y) = \int_{E_{j,r,T}} l(r)^3 \exp\big[2s_0(\alpha_\lambda(t,x) - \alpha_\lambda(r,y))\big]\big|k_j(t,x,r,y)\big|\,dt\,d\rho_j(x).$$

Now from (4.15)–(4.18) and assumptions (4.4) and (4.5), we conclude that

$$\begin{aligned}
\overline{k}_j(r,y) &\le l(r)^3 \exp\big[2c_1(\psi)l(r)^{-1}\big] \int_{E_{j,0,T}} \big|k_1(t,x,r,y)\big|\,dt\,d\rho_j(x) \\
&\le K_{2j-1}, \quad (r,y) \in Q_T. \qquad (4.21)
\end{aligned}$$

Finally, the latter inequality in (4.13) easily follows from (4.20), (4.21), while the first easily follows from the inequality

$$e^{\lambda\|\psi\|_\infty} - e^{\lambda\psi(x)} \le e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi_m} = c_1(\psi). \tag{4.22}$$

This concludes the proof of the lemma. □

Proof of Theorem 1. From definition (3.5) and Lemmata 1 and 2 we easily deduce that $K_0(\widetilde{k}) = m(\Omega)[K_0 + \mu_1 K_2]$ and $K_1(\widetilde{k}) = m(\Omega)[K_1 + \mu_1 K_3]$.

Then we estimate $\Lambda(u)f$:

$$\begin{aligned}
&\int_{Q_T} \exp\bigl[2s_0\alpha_\lambda(t,x)\bigr]\bigl|\Lambda(u)(t)f(t,x)\bigr|^2\,dt\,dx \\
&\quad \le m^{-2}\|f\|^2_{L^\infty(Q_T)} \int_{Q_T} \exp\bigl[2s_0\alpha_\lambda(t,x)\bigr]\bigl|\widetilde{B}v(t)\bigr|^2\,dt \\
&\quad \le m^{-2}\|f\|^2_{L^\infty(Q_T)} m(\Omega)[K_0 + \mu_1 K_2][K_1 + \mu_1 K_3] \\
&\qquad \times \int_{Q_T} l(r)^{-3}\exp\bigl[2s_0\alpha_\lambda(r,y)\bigr]\bigl|u(r,y)\bigr|^2\,dr\,dy.
\end{aligned} \tag{4.23}$$

From Lemmata 1 and 2 and (4.23) we deduce the estimate

$$\begin{aligned}
&\int_{Q_T} \bigl|\mathcal{B}v(t,x)\bigr|^2 \exp\bigl[2s_0\alpha_\lambda(t,x)\bigr]\,dt\,dx \\
&\quad \le K_4(f)\int_{Q_T} l(t)^{-3}\bigl|v(t,x)\bigr|^2\exp\bigl[2s_0\alpha_\lambda(t,x)\bigr]\,dt\,dx,
\end{aligned}$$

where we have set

$$K_4(f) = K_0K_1 + m^{-2}\|f\|^2_{L^\infty(Q_T)} m(\Omega)[K_0 + \mu_1 K_2][K_1 + \mu_1 K_3].$$

Consequently, according to assumptions (4.2) and estimates (4.13), estimate (4.1) simplifies to the following

$$\begin{aligned}
&p(s_0,f)\int_{Q_T} l(t)^{-3}\bigl|u(t,x)\bigr|^2\exp\bigl[2s_0\alpha_\lambda(t,x)\bigr]\,dt\,dx \\
&\quad + s_0\int_{Q_T} l(t)^{-1}\bigl|\nabla u(t,x)\bigr|^2\exp\bigl[2s_0\alpha_\lambda(t,x)\bigr]\,dt\,dx + s_0^{-1}e^{-\lambda\|\psi\|_\infty} \\
&\quad \times \int_{Q_T} l(t)\Bigl[\bigl|D_t u(t,x)\bigr|^2 + \sum_{i,j=1}^n \bigl|D_{x_i}D_{x_j}u(t,x)\bigr|^2\Bigr] \\
&\quad \times \exp\bigl[2s_0\alpha_\lambda(t,x)\bigr]\,dt\,dx \\
&\le 2C_1\int_{Q_T}\bigl|\widetilde{h}(t,x)\bigr|^2\,dt\,dx + 2s_0C_1\int_{\Sigma_T}\bigl|g(t,x)\bigr|^2\,dt\,d\sigma(x).
\end{aligned} \tag{4.24}$$

Taking advantage of the inequality

$$\begin{aligned}
&\int_{Q_T} l(t)^{-3}\bigl|u(t,x)\bigr|^2\exp\bigl[2s_0\alpha_\lambda(t,x)\bigr]\,dt\,dx \\
&\quad \ge \int_{Q_T} l(t)^{-3}\exp\bigl[-2s_0c_1(\psi)l(t)^{-1}\bigr]\bigl|u(t,x)\bigr|^2\,dt\,dy,
\end{aligned}$$

from (4.24) we deduce the *weak* Carleman estimate (4.9).

To find out the corresponding estimate for the unknown function α, from the representation (3.7), i.e. $\alpha(t) = -\Lambda(u)(t) - I(\beta, f, g, h)(t)$, and from Lemma 1, with $\widetilde{B} = \Lambda(u)$, and the Carleman estimate (4.9) we easily deduce the estimates

$$\begin{aligned}
&\int_0^T \big|\alpha(t)\big|^2 \exp\big[-2s_0 c_1(\psi) l(t)^{-1}\big]\, dt \\
&\quad \le 2\int_{Q_T} \big|\Lambda u(t)\big|^2 \exp\big[2s_0\alpha_\lambda(t,x)\big]\, dt\, dx \\
&\qquad + 2\int_0^T \big|I(\beta, f, g, h)(t)\big|^2 \exp\big[2s_0\alpha_\lambda(t,x)\big]\, dt \\
&\quad \le 2C_1 m^{-2} p(s_0, f)^{-1} \|f\|^2_{L^\infty(Q_T)} m(\Omega)[K_0 + \mu_1 K_2][K_1 + \mu_1 K_3] \\
&\qquad \times \Big[\int_{Q_T} \big|\widetilde{h}(t,x)\big|^2\, dt\, dx + s_0 \int_{\Sigma_T} \big|g(t,x)\big|^2\, dt\, d\sigma(x)\Big] \\
&\qquad + 2\int_0^T \big|I(\beta, f, g, h)(t)\big|^2\, dt.
\end{aligned} \tag{4.25}$$

Observe now that $g = h = 0$ in Q_T and $\beta = 0$ in $(0,T)$ imply successively $\gamma(0,0,0) = 0$, $I(0,f,0,0) = 0$ for any f satisfying (3.6) and $\widetilde{h} = 0$ in Q_T. Consequently, from estimates (4.9) and (4.10) we deduce $u = 0$ in Q_T and $\alpha = 0$ in $(0,T)$. We have thus shown that problem (IP3), (3.2) admits at most one solution. □

5 A Continuous Dependence Result for the Solution to the Identification Problem Related to (IP3)

The aim of this section is to estimate first the solution u of the ill-posed problem (IP4) in $C((0,T]; L^2(\Omega)) \cap L^2_{loc}((0,T]; H^1(\Omega))$ and the related function α in $L^2_{loc}((0,T]; \mathbb{R})$. For this task we need that the kernels k_1 and k_2 satisfy the additional properties

$$K_5 := \int_{Q_T\times Q_T} l(r)^3 \exp\big[-2s_0\alpha_\lambda(r,y)\big]\big|k_1(t,x,r,y)\big|^2\, dt\, dx\, dr\, dy < \infty, \tag{5.1}$$

$$K_6 := \int_{\Sigma_T\times Q_T} l(r)^3 \exp\big[-2s_0\alpha_\lambda(r,y)\big]\big|k_2(t,x,r,y)\big|^2\, dt\, d\sigma(x)\, dr\, dy < \infty. \tag{5.2}$$

Remark 3. Since $\exp[-2s_0\alpha_\lambda(r,y)] \le \exp[2s_0 c_1(\psi) l(r)^{-1}]$, it suffices to replace K_5 and K_6 by

$$K_5' = \int_{Q_T\times Q_T} l(r)^3 \exp\big[2s_0 c_1(\psi) l(r)^{-1}\big]\big|k_1(t,x,r,y)\big|^2\, dt\, dx\, dr\, dy < \infty,$$

$$K_6' = \int_{\Sigma_T\times Q_T} l(r)^3 \exp\big[2s_0 c_1(\psi) l(r)^{-1}\big]\big|k_2(t,x,r,y)\big|^2\, dt\, d\sigma(x)\, dr\, dy < \infty.$$

If we restrict condition (4.5) to the following

$$\int_{(0,T)\times F_j}\big|k_j(t,x,r,y)\big|^2\,dt\,dx \le K_{2j-1}l(r)^{-3}\exp\big[-2s_0c_1(\psi)l(r)^{-1}\big],$$
$$(r,y)\in Q_T,\ j=1,2,\ F_1=\Omega,\ F_2=\partial\Omega, \tag{5.3}$$

it is easy to observe that K_5' and K_6' satisfy the inequalities

$$K_5' \le K_1\int_{Q_T} dr\,dy = K_1Tm(\Omega) =: K_5'',$$
$$K_6' \le K_3\int_{Q_T} dr\,dy = K_3Tm(\Omega) =: K_6''.$$

Therefore, under assumptions (5.3) we can replace the pair (K_5, K_6) with (K_5'', K_6''). In particular, when functions k_j, $j = 1,2$ are defined by formula (4.6), then conditions (5.3) is satisfied, if conditions (4.7) are replaced with the stricter one

$$\big|h_{2,i,j}(r,y)\big| \le \kappa_{i,j}l(r)^{-3/2}\exp\big[-s_0c_1(\psi)l(r)^{-1}\big],\quad (r,y)\in Q_T,\ j=1,2$$

and conditions (4.8) are implemented with

$$\sum_{i=1}^{\infty}\kappa_{i,j}^2 < +\infty,\quad \sum_{i=1}^{+\infty}\|h_{1,i,j}\|^2_{L^2((0,T)\times F_j)} < +\infty,\quad j=1,2. \tag{5.4}$$

Indeed, we get

$$\begin{aligned}
&\int_{(0,T)\times F_j\times Q_T} l(r)^3\exp\big[2s_0c_1(\psi)l(r)^{-1}\big]\big|k_1(t,x,r,y)\big|^2\,dt\,dx\,dr\,dy\\
&\le \int_{(0,T)\times F_j}\sum_{i=1}^{+\infty}\big|h_{1,i,j}(t,x)\big|^2\,dt\,dx\\
&\quad\times\int_{Q_T}\sum_{i=1}^{+\infty} l(r)^3\exp\big[2s_0c_1(\psi)l(r)^{-1}\big]\big|h_{2,i,j}(r,y)\big|^2\,dr\,dy\\
&\le \sum_{i=1}^{+\infty}\|h_{1,i,j}\|^2_{L^2((0,T)\times F_j)}\sum_{i=1}^{+\infty}\kappa_{i,j}^2,\quad j=1,2.
\end{aligned}$$

Finally, we note that conditions (4.8) are implied by the stronger (5.4).

We now state our continuous dependence theorem.

Theorem 2. *Let $f\in L^\infty(Q_T)$, $h\in L^2(Q_T)$, $g\in H^1((0,T);L^2(\Omega))\cap L^2((0,T);H^2(\Omega))$, $\beta\in H^1(0,T)$. Moreover, let f and the kernels $k_1 : Q_T\times Q_T\to\mathbb{R}$ and $k_1 : Q_T\times Q_T\to\mathbb{R}$ satisfy, respectively, conditions* (3.6) *and* (4.4), (4.5), (5.1), (5.2). *Then the solution (u,α) to problem* (IP3), (3.2) *satisfies, for all $\varepsilon\in(0,1/4)$ and $\tau\in(2\varepsilon T,T)$, the continuous dependence estimates*

$$\begin{aligned}
&\big\|u(\tau,\cdot)\big\|^2_{L^2(\Omega)} + 2\mu_0\int_{2\varepsilon T}^{\tau}\big\|\nabla u(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt\\
&\quad\le \Phi_1(\varepsilon,m,f)\big\{\|g\|^2_{H^1((0,T);L^2(\Omega))} + \|g\|^2_{L^2((0,T);H^2(\Omega)}\\
&\qquad + \|h\|^2_{L^2(Q_T)} + \|\beta\|^2_{H^1((0,T)}\big\},
\end{aligned} \tag{5.5}$$

$$\|\alpha\|_{L^2(0,T)} \leq \Phi_2(\varepsilon, m, f)\big\{\|g\|_{H^1((0,T);L^2(\Omega))} + \|g\|_{L^2((0,T);H^2(\Omega)}$$
$$+ \|h\|_{L^2(Q_T)} + \|\beta\|_{H^1((0,T)}\big\}, \tag{5.6}$$

where Φ_j, $j = 1, 2$, are two positive functionals depending continuously on the triplet $(\varepsilon, m, f) \in (0, 1/2) \times \mathbb{R}_+ \times L^\infty(Q_T)$.

Proof. First we introduce the family of functions $\sigma_\varepsilon \in W^{1,\infty}([0,T];[0,1])$, $\varepsilon \in (0, 1/4)$, defined by

$$\sigma_\varepsilon(t) = \begin{cases} 0, & t \in [0, \varepsilon T], \\ (\varepsilon T)^{-1}(t - \varepsilon T), & t \in (\varepsilon T, 2\varepsilon T), \\ 1, & t \in [2\varepsilon T, T]. \end{cases} \tag{5.7}$$

Introduce also the function $u_\varepsilon = \sigma_\varepsilon u$. It is a simply task to show that u_ε solves the following initial and boundary-value problem:

$$\text{(DP1)} \quad \begin{cases} u_\varepsilon \in H^1\big((0,T);L^2(\Omega)\big) \cap L^2\big((0,T);H^2(\Omega)\big), \\ D_t u_\varepsilon(t,x) - A(x,D)u_\varepsilon(t,x) \\ \quad = \sigma_\varepsilon(t)\mathcal{B}u(t,x) + \sigma_\varepsilon'(t)u(t,x) + \widetilde{h}_\varepsilon(t,x), & (t,x) \in Q_T, \\ u_\varepsilon(0,x) = 0, & x \in \Omega, \\ u_\varepsilon(t,x) = 0, & (t,x) \in \Sigma_T. \end{cases}$$

Recall now that $-A(\cdot, D)$ satisfies the following estimate for all $v \in H^2(\Omega) \cap H_0^1(\Omega)$:

$$\begin{aligned} -\big\langle A(\cdot,D)v, v\big\rangle &= \int_\Omega \sum_{i,j=1}^n a_{i,j}(x) D_{x_i} v D_{x_j} v \, dx - \int_\Omega \sum_{j=1}^n a_j v D_{x_j} v \, dx \\ &= \int_\Omega \sum_{i,j=1}^n a_{i,j}(x) D_{x_i} v D_{x_j} v \, dx + \frac{1}{2}\int_\Omega v^2 \sum_{j=1}^n D_{x_j} a_j \, dx \\ &\geq \mu_0 \|\nabla v\|_{L^2(\Omega)}^2 - \mu_3 \|v\|_{L^2(\Omega)}^2, \end{aligned}$$

where $\mu_3 = \|(\sum_{j=1}^n D_{x_j} a_j)^-\|_{L^\infty(\Omega)}/2$, h^- denoting the negative part of function h.

It is well-known that the solution u_ε to (DP1) satisfies the following estimate – obtained by multiplying scalarly by u_ε both sides in the differential equation and then integrating by parts:

$$\begin{aligned} &\frac{1}{2} D_t \big\|u_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)}^2 + \mu_0 \big\|\nabla u_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)}^2 - \mu_3 \big\|u_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)}^2 \\ &\quad \leq \big\|u_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)} \big\{\big\|\mathcal{B}u(t,\cdot)\big\|_{L^2(\Omega)} + \big\|\sigma_\varepsilon'(t)u(t,\cdot)\big\|_{L^2(\Omega)} + \big\|\widetilde{h}_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)}\big\}. \end{aligned}$$

Integrating over $(0, \tau)$, $\tau \in (0, T]$, we get

$$\begin{aligned} &\frac{1}{2}\big\|u_\varepsilon(\tau,\cdot)\big\|_{L^2(\Omega)}^2 + \mu_0 \int_0^\tau \big\|\nabla u_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)}^2 \, dt - \mu_3 \int_0^\tau \big\|u_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)}^2 \, dt \\ &\leq \int_0^\tau \big\|u_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)} \big\|\mathcal{B}u(t,\cdot)\big\|_{L^2(\Omega)} dt + \int_0^\tau \big\|u_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)} \big\|\sigma_\varepsilon'(t)u(t,\cdot)\big\|_{L^2(\Omega)} dt \\ &\quad + \int_0^\tau \big\|u_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)} \big\|\widetilde{h}_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)} \, dt. \end{aligned} \tag{5.8}$$

Observe then that operator $\mathcal{B}$ admits the representation

$$\mathcal{B}u(t,x) = \int_{Q_t} k_3(t,x,r,y)u(r,y)\,dr\,dy, \quad (t,x) \in Q_T, \tag{5.9}$$

where the kernel $k_3 : Q_T \times Q_T \to \mathbb{C}$ is defined, for all $(t,x,r,y) \in Q_T \times Q_T$, by

$$k_3(t,x,r,y) = k_1(t,x,r,y) - \chi(f)(t)f(t,x)\widetilde{k}(t,r,y). \tag{5.10}$$

Note now that from definition (3.3) and inequalities (3.6), (5.1) we deduce

$$\begin{aligned}
&\int_{Q_T\times Q_T} l(r)^3 \exp\bigl[-2s_0\alpha_\lambda(r,y)\bigr]\bigl|\chi(f)(t)\bigr|^2\bigl|f(t,x)\bigr|^2\bigl|\widetilde{k}_1(t,r,y)\bigr|^2\,dt\,dx\,dr\,dy \\
&\quad \le m^{-2}\|f\|^2_{L^\infty(Q_T)} \int_{Q_T\times Q_T} l(r)^3 \exp\bigl[-2s_0\alpha_\lambda(r,y)\bigr]\,dt\,dx\,dr\,dy \\
&\qquad \times \int_\Omega \bigl|k_1(t,\xi,r,y)\bigr|^2\,d\xi \le m^{-2}m(\Omega)\|f\|^2_{L^\infty(Q_T)} \\
&\qquad \times \int_{Q_T\times Q_T} l(r)^3 \exp\bigl[-2s_0\alpha_\lambda(r,y)\bigr]\bigl|k_1(t,\xi,r,y)\bigr|^2\,dt\,d\xi\,dr\,dy \\
&\quad \le m^{-2}m(\Omega)\|f\|^2_{L^\infty(Q_T)}K_5.
\end{aligned} \tag{5.11}$$

Similarly, from (3.4) and (5.2) we get

$$\begin{aligned}
&\int_{\Sigma_T\times Q_T} l(r)^3 \exp\bigl[-2s_0\alpha_\lambda(r,y)\bigr]\bigl|\chi(f)(t)\bigr|^2\bigl|f(t,x)\bigr|^2\bigl|\widetilde{k}_2(t,r,y)\bigr|^2\,dt\,d\sigma(x)\,dr\,dy \\
&\quad \le m^{-2}\|f\|^2_{L^\infty(Q_T)} \int_{\Sigma_T\times Q_T} l(r)^3 \exp\bigl[-2s_0\alpha_\lambda(r,y)\bigr]\,dt\,d\sigma(x)\,dr\,dy \\
&\qquad \times \int_{\partial\Omega} \overline{\nu}_A(\xi)\bigl|k_2(t,\xi,r,y)\bigr|^2\,d\xi \le m^{-2}\mu_1\sigma(\partial\Omega)\|f\|^2_{L^\infty(Q_T)} \\
&\qquad \times \int_{\Sigma_T\times Q_T} l(r)^3 \exp\bigl[-2s_0\alpha_\lambda(r,y)\bigr]\bigl|k_2(t,\xi,r,y)\bigr|^2\,dt\,d\sigma(\xi)\,dr\,dy \\
&\quad \le m^{-2}\mu_1\sigma(\partial\Omega)\|f\|^2_{L^\infty(Q_T)}K_6.
\end{aligned} \tag{5.12}$$

Hence, from definitions (5.10), (3.3) and (3.4) we deduce the estimate

$$\begin{aligned}
&\int_{Q_T\times Q_T} l(r)^3 \exp\bigl[-2s_0\alpha_\lambda(r,y)\bigr]\bigl|k_3(t,x,r,y)\bigr|^2\,dt\,dx\,dr\,dy \\
&\quad \le 3\int_{Q_T\times Q_T} l(r)^3 \exp\bigl[-2s_0\alpha_\lambda(r,y)\bigr] \\
&\qquad \times \Bigl[\bigl|k_1(t,x,r,y)\bigr|^2 + m^{-2}\bigl|f(t,x)\bigr|^2 \sum_{j=1}^{2}\bigl|\widetilde{k}_2(t,r,y)\bigr|^2\Bigr]\,dt\,dx\,dr\,dy \\
&\quad \le \bigl\{3K_5 + 3m^{-2}\bigl[m(\Omega)K_5 + \mu_1\sigma(\partial\Omega)K_6\bigr]\|f\|^2_{L^\infty(Q_T)}\bigr\} =: K_7(m,f).
\end{aligned}$$

Consequently, using the Carleman estimate (4.9), we deduce the following weighted estimate for $\mathcal{B}u$, holding for all $\tau \in (0,T)$:

$$\int_{Q_\tau} \bigl|\mathcal{B}u(t,x)\bigr|^2\,dt\,dx \le \int_{Q_\tau} \Bigl[\int_{Q_t} l(r)^{3/2} \exp\bigl[-s_0\alpha_\lambda(r,y)\bigr]$$

$$
\begin{aligned}
&\times \big|k_3(t,x,r,y)\big| l(r)^{-3/2} \exp\big[s_0\alpha_\lambda(r,y)\big]\big|u(r,y)\big|\,dr\,dy\Big]^2\,dt\,dx \\
&\le \int_{Q_\tau}\Big[\int_{Q_t} l(r)^3 \exp\big[-2s_0\alpha_\lambda(r,y)\big]\big|k_3(t,x,r,y)\big|^2\,dr\,dy \\
&\quad \times \int_{Q_t} l(r)^{-3}\exp\big[2s_0\alpha_\lambda(r,y)\big]\big|u(r,y)\big|^2\,dr\,dy\Big]\,dt\,dx \\
&\le \int_{Q_\tau\times Q_t} l(r)^3\exp\big[-2s_0\alpha_\lambda(r,y)\big]\big|k_3(t,x,r,y)\big|^2\,dt\,dx\,dr\,dy \\
&\quad \times \int_{Q_T} l(r)^{-3}\exp\big[2s_0\alpha_\lambda(r,y)\big]\big|u(r,y)\big|^2\,dr\,dy \\
&\le 2C_1K_7(m,f)p(s_0,f)^{-1}\big[\|\widetilde{h}\|^2_{L^2(Q_T)} + \|g\|^2_{L^2(\Sigma_T)}\big] \\
&=: J_1(m,f,g,\widetilde{h}). \qquad (5.13)
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&\int_0^\tau \big\|u_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)}\big\|\mathcal{B}u(t,\cdot)\big\|_{L^2(\Omega)}\,dt \\
&\quad\le \frac12\int_0^\tau \big\|u_\varepsilon(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt + \frac12\int_{Q_\tau}\big|\mathcal{B}u(t,x)\big|^2\,dt\,dx \\
&\quad\le \frac12\int_0^\tau \big\|u_\varepsilon(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt + \frac12 J_1(m,f,g,\widetilde{h}). \qquad (5.14)
\end{aligned}
$$

Using now the inclusion $\operatorname{supp}\sigma'_\varepsilon \subset [\varepsilon T, 2\varepsilon T]$, we get the inequalities

$$
\begin{aligned}
&\int_0^\tau \big|\sigma'_\varepsilon(t)\big|\big\|u_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)}\big\|u(t,\cdot)\big\|_{L^2(\Omega)}\,dt \\
&\quad\le \frac12\int_0^\tau \big\|u_\varepsilon(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt + \frac12\int_0^\tau \big|\sigma'_\varepsilon(t)\big|^2\big\|u(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt \\
&\quad\le \frac12\int_0^\tau \big\|u_\varepsilon(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt + \frac12\int_0^T \big|\sigma'_\varepsilon(t)\big|^2\big\|u(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt \\
&\quad\le \frac12\int_0^\tau \big\|u_\varepsilon(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt + \frac12\|\sigma'_\varepsilon\|^2_{L^\infty(0,T)}\int_{\varepsilon T}^{2\varepsilon T} \big\|u(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt. \qquad (5.15)
\end{aligned}
$$

Observe now that

$$
\begin{aligned}
\rho_{1,\lambda}(t,s_0) &:= \exp\big\{-2s_0\big[e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi_m}\big]l(t)^{-1}\big\} \le \exp\big[2s_0\alpha_\lambda(t,x)\big] \qquad (5.16)\\
&\le \exp\big\{-2s_0\big[e^{2\lambda\|\psi\|_\infty} - e^{\lambda\|\psi\|_\infty}\big]l(t)^{-1}\big\} =: \rho_{2,\lambda}(t,s_0), \quad (t,x)\in Q_T,
\end{aligned}
$$

where $\psi_m = \min_{x\in\overline{\Omega}}\psi(x)$. From (4.24), (5.7) and from the inequality

$$
\begin{aligned}
1 &= l(t)^3\rho_{1,\lambda}(t,s_0)^{-1}l(t)^{-3}\rho_{1,\lambda}(t,s_0) \le 2^{-6}T^6\rho_{1,\lambda}(\varepsilon T,s_0)^{-1}l(t)^{-3}\rho_{1,\lambda}(t,s_0) \\
&=: C_2(s_0,\varepsilon,T)l(t)^{-3}\rho_{1,\lambda}(t,s_0), \quad t\in[\varepsilon T,2\varepsilon T],\ \varepsilon\in(0,1/2)
\end{aligned}
$$

we get

$$
\int_{\varepsilon T}^{2\varepsilon T}\big\|u(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt \le C_2(s_0,\varepsilon,T)\int_{\varepsilon T}^{2\varepsilon T} l(t)^{-3}\rho_{1,\lambda}(t,s_0)\big\|u(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt
$$

$$
\begin{aligned}
&\le C_2(s_0,\varepsilon,T)\int_0^T l(t)^{-3}\rho_{1,\lambda}(t,s_0)\big\|u(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt\\
&\le 3C_1C_2(s_0,\varepsilon,T)p_0(s_0,f)^{-1}\\
&\quad\times\int_{Q_T}\big|\widetilde{h}(t,x)\big|^2\exp\big[2s_0\alpha_\lambda(t,x)\big]\,dt\,dx\\
&\le 3C_1C_2(s_0,\varepsilon,T)p_0(s_0,f)^{-1}\big[\|\widetilde{h}\|^2_{L^2(Q_T)}+\|g\|^2_{L^2(\Sigma_T)}\big]\\
&=: J_2(\varepsilon,m,f,g,\widetilde{h}).
\end{aligned}
\tag{5.17}
$$

Therefore, taking (4.9) and (5.14), (5.15), (5.17) into account, from (5.8) we easily deduce, for all $\tau\in(0,T)$, the following integral inequality:

$$
\begin{aligned}
&\big\|u_\varepsilon(\tau,\cdot)\big\|^2_{L^2(\Omega)}+2\mu_0\int_0^\tau\big\|\nabla u_\varepsilon(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt\\
&\quad\le 2\int_0^\tau\big\|u_\varepsilon(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt+2\int_0^\tau\big\|\widetilde{h}_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)}\big\|u_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)}\,dt\\
&\qquad+(\varepsilon T)^{-2}\int_{\varepsilon T}^{2\varepsilon T}\big\|u(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt+J_1(m,f,g,\widetilde{h}).\\
&\quad\le 2\int_0^\tau\big\|u_\varepsilon(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt+2\int_0^\tau\big\|\widetilde{h}_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)}\big\|u_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)}\,dt\\
&\qquad+J(\varepsilon,m,f,\widetilde{h}),
\end{aligned}
\tag{5.18}
$$

where we have set

$$
J(\varepsilon,m,f,\widetilde{h})=J_1(m,f,g,\widetilde{h})+(\varepsilon T)^{-2}J_2(\varepsilon,m,f,g,\widetilde{h}).
\tag{5.19}
$$

Finally, from (5.18) and (5.17) we deduce the fundamental integro-differential inequality

$$
\begin{aligned}
z_\varepsilon(\tau)&=:\big\|u_\varepsilon(\tau,\cdot)\big\|^2_{L^2(\Omega)}+2\mu_0\int_0^\tau\big\|\nabla u_\varepsilon(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt\\
&\le J(\varepsilon,m,f,\widetilde{h})+2\int_0^\tau z_\varepsilon(t)\,dt+2\int_0^\tau\big\|\widetilde{h}_\varepsilon(t,\cdot)\big\|_{L^2(\Omega)}z_\varepsilon(t)^{1/2}\,dt.
\end{aligned}
\tag{5.20}
$$

Then we need Theorem 4.9 in [2], with $p=1/2$, which we report here as a lemma.

Lemma 3. *Let z be a nonnegative $C([0,T])$–function and let b, k be nonnegative $L^1((0,T))$–functions satisfying*

$$
z(t)\le a+\int_0^t b(s)z(s)\,ds+\int_0^t k(s)z(s)^{1/2}\,ds,\quad t\in[0,T],
$$

where $a\ge 0$ is a given constant. Then for all $t\in[0,T]$

$$
z(t)\le\exp\Big(\int_0^t b(s)\,ds\Big)\Big[a^{1/2}+\frac{1}{2}\int_0^t k(s)\exp\Big(-\frac{1}{2}\int_0^s b(\sigma)\,d\sigma\Big)\,ds\Big]^2.
$$

Then from estimate (5.20), Lemma 3 and the elementary inequality $(a+b)^2 \le 2a^2 + 2b^2$, for $\tau \in [0,T]$, we deduce the estimate

$$\begin{aligned}&\big\|u_\varepsilon(\tau,\cdot)\big\|^2_{L^2(\Omega)} + 2\mu_0 \int_0^\tau \big\|\nabla u_\varepsilon(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt \\ &\quad = z_\varepsilon(\tau) \le 2J(\varepsilon,m,f,g,\widetilde{h})\exp(2\tau) + \Big\{2\int_0^\tau \exp(\tau-t)\big\|\widetilde{h}(t,\cdot)\big\|_{L^2(\Omega)}\,dt\Big\}^2.\end{aligned} \tag{5.21}$$

Observe now that

$$\begin{aligned}\|\widetilde{h}\|_{L^2(Q_T)} \le C_3(\Omega,T,m)\big\{&\|g\|_{H^1((0,T);L^2(\Omega))} + \|g\|_{L^2((0,T);H^2(\Omega))} \\ &+ \|h\|_{L^2(Q_T)} + \|\beta\|_{H^1((0,T)}\big\}.\end{aligned} \tag{5.22}$$

Then from (5.21) and (5.22) we easily deduce the estimate

$$\begin{aligned}&\big\|u_\varepsilon(\tau,\cdot)\big\|^2_{L^2(\Omega)} + 2\mu_0 \int_0^\tau \big\|\nabla u_\varepsilon(t,\cdot)\big\|^2_{L^2(\Omega)}\,dt \\ &\quad \le C_4(\varepsilon,m,f)\big\{\|g\|^2_{H^1((0,T);L^2(\Omega))} + \|g\|^2_{L^2((0,T);H^2(\Omega)} \\ &\qquad + \|h\|^2_{L^2(Q_T)} + \|\beta\|^2_{H^1((0,T)}\big\}.\end{aligned} \tag{5.23}$$

Taking now $\tau \in [2\varepsilon T, T]$, from (5.23) we obtain the desired estimate (5.5) for u.

Then we observe that reasoning as for the proof of estimates (5.11) and (5.12), both with $f \equiv 1$, we get

$$\begin{aligned}&\int_{(0,T)\times Q_T} l(r)^3 \exp\big[-2s_0\alpha_\lambda(r,y)\big]\big|\widetilde{k}_1(t,r,y)\big|^2\,dt\,dr\,dy \\ &\quad \le m(\Omega)\int_{(0,T)\times Q_T} l(r)^3 \exp\big[-2s_0\alpha_\lambda(r,y)\big]\,dt\,dr\,dy \int_\Omega \big|k_1(t,\xi,r,y)\big|^2\,d\xi \\ &\quad \le m(\Omega)\int_{Q_T\times Q_T} l(r)^3 \exp\big[-2s_0\alpha_\lambda(r,y)\big]\big|k_1(t,\xi,r,y)\big|^2\,dt\,d\xi dr\,dy \\ &\quad \le m(\Omega)K_5.\end{aligned}$$

Similarly we get

$$\begin{aligned}&\int_{(0,T)\times Q_T} l(r)^3 \exp\big[-2s_0\alpha_\lambda(r,y)\big]\big|\widetilde{k}_2(t,r,y)\big|^2\,dt\,dr\,dy \\ &\le \sigma(\partial\Omega)\int_{\Sigma_T\times Q_T} l(r)^3 \exp\big[-2s_0\alpha_\lambda(r,y)\big]\,dt\,d\sigma(x)\,dr\,dy \int_{\partial\Omega} \overline{\nu}_A(\xi)\big|k_2(t,\xi,r,y)\big|^2\,d\xi \\ &\le \mu_1\sigma(\partial\Omega)\int_{\Sigma_T\times Q_T} l(r)^3 \exp\big[-2s_0\alpha_\lambda(r,y)\big]\big|k_2(t,\xi,r,y)\big|^2\,dt\,d\sigma(\xi)\,dr\,dy \\ &\le \mu_1\sigma(\partial\Omega)K_6.\end{aligned}$$

Hence, from definitions (5.9), (3.3) and (3.4) we deduce the estimate

$$\begin{aligned}&\int_{(0,T)\times Q_T} l(r)^3 \exp\big[-2s_0\alpha_\lambda(r,y)\big]\big|\widetilde{k}(t,r,y)\big|^2\,dt\,dr\,dy \\ &\quad \le 2\int_{(0,T)\times Q_T} l(r)^3 \exp\big[-2s_0\alpha_\lambda(r,y)\big]\sum_{j=1}^2\big|\widetilde{k}_j(t,r,y)\big|^2\,dt\,dr\,dy \\ &\quad \le 2\big(m(\Omega)K_5 + \mu_1\sigma(\partial\Omega)K_6\big) =: K_8.\end{aligned}$$

Therefore we obtain the estimate

$$\begin{aligned}\int_0^T \big|\Lambda u(t)\big|^2\,dt &\le m^{-2}\int_0^T \Big\{\int_{Q_t} l(r)^{3/2}\exp\big[-s_0\alpha_\lambda(r,y)\big]\big|\widetilde{k}(t,r,y)\big| \\ &\qquad \times l(r)^{-3/2}\exp\big[s_0\alpha_\lambda(r,y)\big]\big|u(r,y)\big|\,dr\,dy\Big\}^2\,dt \\ &\le m^{-2}\int_0^T dt\int_{Q_t} l(r)^3\exp\big[-2s_0\alpha_\lambda(r,y)\big]\big|\widetilde{k}(t,r,y)\big|^2\,dr\,dy \\ &\qquad \times \int_{Q_t} l(r)^{-3}\exp\big[2s_0\alpha_\lambda(r,y)\big]\big|u(r,y)\big|^2\,dr\,dy \\ &\le m^{-2}K_8 p_0(s_0,f)^{-1}\big[\|\widetilde{h}\|^2_{L^2(Q_T)}+\|g\|^2_{L^2(\Sigma_T)}\big].\end{aligned}$$

Then from (3.7) we easily obtain the final estimate (5.6) for α. □

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