Effective Conductivity of 2D Disk – Ring Composite Material

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Abstract. For 2D bounded composite material geometrically composed by a disk of variable radius r and an outer ring it is determined in an analytic form the x-component of the effective conductivity tensor. Namely, it is shown that the x-component is a sum of geometrical progression with respect to powers of r^2 for all sufficiently small r.

Keywords: bounded 2D composites, disk-ring, heat conduction, $\mathbb{R}$-linear problem, effective conductivity.

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1 Introduction

Circular type 2D mechanical structures are subject of long investigations. In the pioneering work [9] it had been found a complete solution of the problem of torsion and bending of elastic cylindric body reinforced by parallel cylindric beams S_k, $k = 1, \ldots, m$, of different materials. The author showed that this problem is equivalent to the problem, which can be formulated in the following form.

Problem A. Let S_k, $k = 0, 1, \ldots, m$, are simply connected non-overlapping domains of the complex plane $\mathbb{C}$ with smooth boundaries C_0, $C_1, \ldots, C_m$, such that the contour C_0 embraces any contour C_k, $k = 1, \ldots, m$. Let $S := S_0 \setminus \cup_{k=1}^m S_k$. The problem is to find a function φ harmonic in each domain S_k, $k = 0, 1, \ldots, m$, continuous in cl S_0 and such that its normal derivative on each contour C_k, $k = 0, \ldots, m$ has a jump satisfying given conditions.

Here and anywhere cl(G) denotes a closure of a domain $G \subset \mathbb{C}$.

It is proved in [9] that the given problem is equivalent to a system of $m+1$ Fredholm integral equations of the second kind. He showed that under some solvability conditions this system has a solution and this solution is unique up to an additive constant. The main aim of the above article is to calculate certain elastic characteristics of the considered elastic cylindric body reinforced by parallel cylindric beams. One of these is, so-called, elastic torsional rigidity. Mathematically speaking, it is a real valued functional depending on the solution of the latter problem.

The solution of the Problem A is uniquely determined from a system of Fredholm integral equations of the second kind. Anyway, from the practical point of view it is important to find an explicit expression of the solution. Therefore, any solution' method for Problem A which leads to a closed form solution (even in certain special cases), is interesting for the practice. For example, in case when $m = k = 1$ and S_0 and S_1 are disks (or what is equivalent S_0' and S_1' are disks with a common origin) the Problem A, is solved by the direct method based on expansions in Taylor's and Laurent's series of a function $f(z)$, which is analytic in $S_1' \cup S'$, $S' = S_0' \setminus \mathrm{cl}\, S_1'$. The function $f(z)$ is such that its real part $\mathrm{Re}\, f$ equals to an unknown function φ in $S_1' \cup S'$. By using this solution the elastic torsional rigidity is calculated in an explicit form in [10]. The case when $m = k = 1$ and S_0 and S_1 are confocal ellipses is considered in [11]. The discussion on recent contributions to the explicit solution of Problem A in the case of non-circular geometry is presented in [7].

Our main aim is to solve a problem similar to Problem A when $m = k = 1$ and S_0 and S_1 are disks with a common origin, geometrically representing a bounded 2D composite material. Here, the annulus $S = S_0 \setminus \mathrm{cl}\, S_1$ and the disk S_1 are domains corresponding to the matrix and the inclusion of the composite, respectively. On the base of this solution we calculate an analog of the elastic torsional rigidity, which is an x-component of, so-called, Effective Conductivity Tensor (ECT). In spite of the fact, that usually the term "effective conductivity" is used for composite materials with a rich microstructure, we utilize the same term here, meaning effective conductivity of a bounded composite material, see, e.g., [2, 3, 5].

More precisely, we consider the problem of determination of the temperature distribution under perfect contact condition in the described inhomogeneous media, which is loaded by a simple heat flow. Note, that this problem is equivalent to the $\mathbb{R}$-linear conjugation problem on the complex plane (for more details on the solution of similar problems we refer to [4, 6], [8, p. 45]). The formula for the x-component of the effective conductivity tensor is represented in an analytic form in terms of radius r of the internal disk and so-called contrast parameter ρ introduced by Bergmann [1].

The obtained exact formula allows us to apply it for the description of certain physical phenomenon, these results will be presented in a separate paper. We can to consider also an asymptotic behavior of the solution and components of the effective conductivity tensor in the case when inclusion is either shrinking to a point or expanding up to the whole domain S_0.

2 Formulation of the Problem

Let $\mathbb{C}$ be the complex plane, and $\mathbb{R}$ be the field of real numbers. For any ξ from $\mathbb{C}$ or $\mathbb{R}$ let us denote by $|\xi|$ its modulus. For any $r \in (0,1)$ we consider the following sets of the complex plane: $\mathrm{D}_r := \{z \in \mathbb{C} \colon |z| < r\}$, $\gamma_r := \{t \in \mathbb{C} \colon |t| = r\}$, $\mathrm{A}_r := \{z \in \mathbb{C} \colon r < |z| < 1\}$, $\Gamma := \{t \in \mathbb{C} \colon |t| = 1\}$. The boundary $\partial \mathrm{D}_r$ of D_r is denoted by γ_r and $\partial \mathrm{A}_r := \gamma_r \cup \Gamma$. For any $z = x + iy \in \mathbb{C}$ we denote by (η, θ), $0 < \eta < \infty$, $0 \le \theta \le 2\pi$, its polar coordinates, thus, $z = \eta \exp\{i\theta\}$; $\mathrm{Re}\, z$ is a real part of a complex number z.

Let Ω be an open or closed domain in $\mathbb{C}$ and let $\tau : \Omega \longrightarrow \mathbb{R}$ be a real valued function, then $\mathcal{C}^k(\Omega)$, $k \in 1, \ldots$, denotes a class of functions having continuous partial derivatives up to k-th order. If $\tau \in \mathcal{C}^2(\Omega)$ then

$$\Delta \tau(x + iy) := \frac{\partial^2 \tau}{\partial x^2} + \frac{\partial^2 \tau}{\partial y^2}.$$

For any function $\varphi : \mathrm{D}_r \cup \mathrm{A}_r \longrightarrow \mathbb{R}$, by φ_1 and φ_2 we mean its restrictions to D_r and A_r, respectively.

Let us consider 2D circular composite material consisting of two components, represented geometrically by the annulus A_r (a host material or matrix) and by the disk D_r (inclusion). Each component of the composite is filled in by the homogeneous material of constant thermal conductivity λ_2 and λ_1, respectively. We suppose that a normalized external heat flow is put in the direction opposite to x-axes. Then the main problem is to determine the potential temperature distribution in such a composite in the stationary case.

Mathematically this problem consists in determination of a pair of real-valued functions $\varphi = (\varphi_1, \varphi_2)$, with $\varphi_1 \in C^2(\mathrm{D}_r) \cap C^1(\mathrm{cl}\, \mathrm{D}_r)$, $\varphi_2 \in C^2(\mathrm{A}_r) \cap C^1(\mathrm{cl}\, \mathrm{A}_r)$ representing temperature distribution in the corresponding domains, satisfying the following system of relations

$$\begin{cases} \Delta \varphi(z) = 0, & \forall z \in \mathrm{D}_r \cup \mathrm{A}_r, \\ \varphi_2(t) = -\mathrm{Re}\, t, & \forall t \in \Gamma, \\ \varphi_1(t) = \varphi_2(t), & \forall t \in \gamma_r, \\ \lambda_1 \dfrac{\partial \varphi_1}{\partial \mathbf{n}}(t) = \lambda_2 \dfrac{\partial \varphi_2}{\partial \mathbf{n}}(t), & \forall t \in \gamma_r, \end{cases} \tag{2.1}$$

where λ_1 and λ_2 are some positive constants such that $\rho := \frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1}$ does not equal to zero, $\varphi_1(t)$ and $\varphi_2(t)$ denote the boundary values of, respectively, $\varphi_1(z)$, $z \in \mathrm{D}_r$, and $\varphi_2(z)$, $z \in \mathrm{A}_r$, $\frac{\partial \varphi_k}{\partial \mathbf{n}}$, $k = 1, 2$, is respectively an outward and inward normal derivatives with respect to the boundary of the domain of the function φ_k, $k = 1, 2$. Since by definition domains of the functions φ_1 and φ_2 are D_r and A_r, then for any $t \in \gamma_r$ the directions of the normal $\mathbf{n}$ at the left and at the right side of the fourth equality of the system (2.1) coincide (see Fig. 1).

Problem (2.1) (cf., e.g., [8, p. 45] with $\lambda_2 = 1$) can be reduced to the following $\mathbb{R}$-linear conjugation problem: *find two functions* $\phi : \mathrm{D_r} \cup \mathrm{A}_r \longrightarrow \mathbb{C}$, $\phi_0 : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C}$, $\phi_0(\infty) = 0$, *such that* $\phi = (\phi_1, \phi_2)$ *is analytic in* $\mathrm{D}_r \cup \mathrm{A}_r$ *with* ϕ_1, ϕ_2 *being continuous in* $\mathrm{cl}\, \mathrm{D}_r$, $\mathrm{cl}\, \mathrm{A}_r$, *respectively,* ϕ_0 *is analytic*
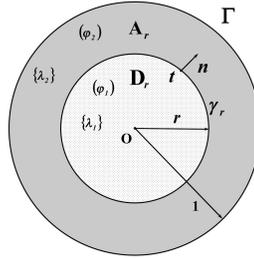
Figure 1. A section of a bounded 2D composite.

in $\text{ext}\,\mathrm{D}_1 = \{z \in \mathbb{C} \colon |z| > 1\}$ *and continuous in* $\text{cl}\{\text{ext}\,\mathrm{D}_1\}$, *functions* ϕ *and* ϕ_0 *satisfy the following conjugation conditions*

$$\phi^+(t) = \phi^-(t) - \rho\overline{\phi^-(t)} \quad \forall t \in \gamma_r, \quad \phi^-(t) = \phi_0(t) - \overline{\phi_0(t)} - t \quad \forall t \in \Gamma, \quad (2.2)$$

where $\phi^+(t)$ *and* $\phi^+(t)$, $t \in \gamma_r$, *is boundary values of* $\phi(z)$, *accordingly, from* $\mathrm{D}_r, \mathrm{A}_r$, $\overline{u + iv} := u - iv$ *for* $u, v \in \mathbb{R}$.

Note, that the problems similar to (2.2) have been considered in [4, 6, 8]. Our aim is to calculate x-component of the effective conductivity tensor (see, e.g. [5]), i.e. a quantity λ^x_{eff} satisfying the following equality

$$F^x \lambda^x_{eff}(r) = \lambda_1 \iint\limits_{\mathrm{D}_r} \frac{\partial \varphi_1}{\partial x}\, dx\, dy + \lambda_2 \iint\limits_{\mathrm{A}_r} \frac{\partial \varphi_2}{\partial x}\, dx\, dy, \qquad (2.3)$$

where $\varphi(x,y) := (\varphi_1(x+iy), \varphi_2(x+iy)) = \text{Re}\,(\phi_1(x+iy), \phi_2(x+iy))$ with $\phi_1(x+iy), \phi_2(x+iy)$ being the solution to (2.2), F^x is a complete flux in x-direction.

The first step is to find a solution of (2.1) in an explicit form.

3 Solution of Problem 2.1

In order to solve boundary value problem (2.1) we propose a method based on expansions in the Taylor's and the Laurent's power series of f_1 and f_2. Here f_1 and f_2 are restrictions of a function $f(z) : \mathrm{D}_r \cup \mathrm{A}_r \longrightarrow \mathbb{C}$, such that $f \equiv f_1$ and f_1 is analytic in D_r, $\text{Re}\,f_1(z) = \varphi_1(x,y)$ for all $z = x + iy \in \mathrm{D}_r$; $f \equiv f_2$ and f_2 is analytic in A_r, $\text{Re}\,f_2(z) = \varphi_2(x,y)$ for all $z = x + iy \in \mathrm{A}_r$.

Note, that a similar idea was realized in the paper [10].

Theorem 1. *For any* $0 < r < 1$ *the solution of the boundary value problem* (2.1) *is given by the following formulas*

$$\varphi_1(z) = -\eta \frac{1 - \rho^{-1}}{r^2 - \rho^{-1}} \cos\theta \quad \forall z = \eta \exp\{i\theta\} \in \mathrm{D}_r, \qquad (3.1)$$

$$\varphi_2(z) = \left(\eta \frac{\rho^{-1}}{r^2 - \rho^{-1}} - \eta^{-1} \frac{r^2}{r^2 - \rho^{-1}}\right) \cos\theta \quad \forall z = \eta \exp\{i\theta\} \in \mathrm{A}_r. \qquad (3.2)$$

Furthermore,

$$\varphi_1(t) = \varphi_2(t) = -(1 - \rho^{-1})\frac{r}{r^2 - \rho^{-1}} \cos\theta \quad \forall t = r \exp\{i\theta\} \in \gamma_r, \qquad (3.3)$$

$$\varphi_2(t) = -\cos\theta \quad \forall t = \exp\{i\theta\} \in \Gamma. \qquad (3.4)$$

Proof. Let a function $f_1 : D_r \longrightarrow \mathbb{C}$ be an analytic in D_r and such that $\mathrm{Re}f_1 = \varphi_1$ in $D_r \cup \gamma_r$. Then the function f_1 has an expansion in the Taylor's power series in D_r and we have

$$f_1(z) = a'_0 + ib'_0 + \sum_{k=1}^{\infty}(a'_k + ib'_k)z^k \quad \forall z \in D_r. \qquad (3.5)$$

Let a function $f_2 : A_r \longrightarrow \mathbb{C}$ be a single-valued analytic function in A_r, such that $\mathrm{Re}f_2 = \varphi_2$ in $A_r \cup \gamma_r \cup \Gamma$. Then it can be expanded in the Laurent's series in A_r and the following equality holds

$$f_2(z) = a''_0 + ib''_0 + \sum_{k=-\infty,\, k\neq 0}^{\infty}(a''_k + ib''_k)z^k \quad \forall z \in A_r. \qquad (3.6)$$

It follows from (3.5), (3.6) and the choice of functions f_1 and f_2 that the functions in the right-hand side of (3.1) and (3.2) are proper candidates to be solutions of the problem (2.1). We have to check later whether the functions φ_1, φ_2, satisfying conditions (2.1), belongs to considered functional spaces. Taking real part of (3.5), (3.6) we obtain the following representations of φ_1, φ_2

$$\varphi_1(z) = a'_0 + \sum_{k=1}^{\infty}\left(a'_k\eta^k\cos k\theta - b'_k\eta^k\sin k\theta\right) \quad \forall z = \eta\exp\{i\theta\} \in D_r, \qquad (3.7)$$

$$\varphi_2(z) = a''_0 + \sum_{k=1}^{\infty}(a''_k\eta^k + a''_{-k}\eta^{-k})\cos k\theta$$

$$+ \sum_{k=1}^{\infty}(b''_{-k}\eta^{-k} - b''_k\eta^k)\sin k\theta \quad \forall z = \eta\exp\{i\theta\} \in A_r. \qquad (3.8)$$

Any pair of functions $\varphi = (\varphi_1, \varphi_2)$ given by (3.7), (3.8), respectively, satisfy the first equation of the system (2.1) in the corresponding domains. The second condition of (2.1) and the formula (3.8) yields

$$a''_0 + \sum_{k=1}^{\infty}(a''_k + a''_{-k})\cos k\theta + \sum_{k=1}^{\infty}(b''_{-k} - b''_k)\sin k\theta = -\cos\theta \quad \forall\theta \in [0, 2\pi].$$

Hence, by uniqueness of the Fourier series, coefficients of (3.8) satisfy the following relations

$$a''_0 = 0, \quad a''_{-1} = -(1 + a''_1),$$
$$a''_{-k} = -a''_k, \quad k = 2, 3, \ldots, \quad b''_{-k} = b''_k, \quad k = 1, 2, \ldots.$$

Then we can rewrite the relation (3.8) in the following form

$$\varphi_2(z) = \left(a_1'' \eta - (1 + a_1'')\eta^{-1}\right)\cos\theta + \sum_{k=2}^{\infty} a_k'' \left(\eta^k - \eta^{-k}\right)\cos k\theta$$

$$+ \sum_{k=1}^{\infty} b_k'' \left(\eta^{-k} - \eta^k\right)\sin k\theta \quad \forall z = \eta\exp\{i\theta\} \in \mathrm{cl}\,A_r, \qquad (3.9)$$

The third condition of (2.1) implies the equality

$$a_0' + \left((a_1' - a_1'')r + (1 + a_1'')r^{-1}\right)\cos\theta + \sum_{k=2}^{\infty} \left((a_k' - a_k'')r^k + a_k''r^{-k}\right)\cos k\theta$$

$$- \sum_{k=1}^{\infty} \left((b_k' - b_k'')r^k + b_k''r^{-k}\right)\sin k\theta = 0 \quad \forall\theta \in [0, 2\pi].$$

Thus, we have

$$a_0' = 0, \quad a_1' = a_1'' - (1 + a_1'')r^{-2}, \qquad (3.10)$$
$$a_k' = a_k''(1 - r^{-2k}), \quad k = 2, 3\ldots, \quad b_k' = b_k''(1 - r^{-2k}), \quad k = 1, 2, \ldots.$$

Hence, we can rewrite the representation (3.7) in the following form

$$\varphi_1(z) = \eta(a_1'' - (1 + a_1'')r^{-2})\cos\theta + \sum_{k=2}^{\infty} a_k'' \eta^k \left(1 - r^{-2k}\right)\cos k\theta$$

$$- \sum_{k=1}^{\infty} b_k'' \eta^k (1 - r^{-2k})\sin k\theta \quad \forall z = \eta\exp\{i\theta\} \in \mathrm{cl}\,D_r. \qquad (3.11)$$

Taking into account, that normal derivatives on the circle centered at the origin coincide with the derivatives with respect to polar radius, we have the following relations $\frac{\partial\varphi_k}{\partial\mathbf{n}} = \frac{\partial\varphi_k}{\partial\eta}|_{\eta=r}$, $k = 1, 2$. Hence, dividing by $\lambda_1 + \lambda_2$ the last condition of (2.1) we obtain the following relation

$$\left(-\rho a_1'' + (1 + a_1'')r^{-2}\right)\cos\theta - \rho\sum_{k=2}^{\infty} k a_k'' r^{k-1} \left(1 - r^{-2k}\right)\cos k\theta$$

$$- \sum_{k=1}^{\infty} k b_k'' r^{k-1} \left(-\rho + r^{-2k}\right) = 0 \quad \forall\theta \in [0, 2\pi], \qquad (3.12)$$

where $\rho = (\lambda_1 - \lambda_2)/(\lambda_1 + \lambda_2)$. Let

$$0 < r < \min\{|\rho|^{-1/2}, 1\}, \qquad (3.13)$$

then the following inequalities hold

$$\left(1 - r^{-2k}\right) \neq 0, \quad k = 2, 3, \ldots, \qquad -\rho + r^{-2k} \neq 0, \quad k = 1, 2, \ldots,$$

thus, the equality (3.12) is true for such r if and only if the following relations hold

$$a_1'' = \rho^{-1}(r^2 - \rho^{-1})^{-1}, \quad a_k'' = 0, \quad k = 2, 3, \ldots, \quad b_k'' = 0, \quad k = 1, 2, \ldots. \qquad (3.14)$$

Taking into account that for any $0 < \lambda_1, \lambda_2 < +\infty$ the Bergmann contrast parameter ρ satisfies the inequality $-1 < \rho < 1$, clarify that the inequality (3.13) holds for any $r \in (0,1)$, therefore, relations (3.14) hold for all $r \in (0,1)$.

Then $1 + a_1'' = r^2(r^2 - \rho^{-1})^{-1}$ and (3.10) turns to $a_1' = -(1 - \rho^{-1}) \times (r^2 - \rho^{-1})^{-1}$, therefore, the equality (3.11) coincides with the formula (3.1). By using the last relations, we obtain that (3.9) coincides with (3.2). Formulas (3.3) and (3.4) are simple consequences of the continuity of the functions (3.1) and (3.2), accordingly, in $\operatorname{cl} D_r$ and $\operatorname{cl} A_r$. Therefore, the pair of functions (3.1) and (3.2) defines the unique solution of the problem (2.1). $\square$

4   Effective Conductivity

In the considered case x-component F^x of the complete flux can be normalized as follows $F^x = -\pi$ (see [8]).

Theorem 2. *Under assumptions of Theorem 1 the following formula holds*

$$\lambda_{eff}^x(r) = \lambda_2 \frac{\rho^{-1} + r^2}{\rho^{-1} - r^2}. \tag{4.1}$$

Furthermore, the function (2.3) *is the sum of the following series*

$$\lambda_{eff}^x(r) = \lambda_2 \left(1 + \rho r^2\right) \left(1 + \sum_{k=1}^{\infty} \rho^k r^{2k}\right), \tag{4.2}$$

converging for any $0 < r < 1$.

Proof. By the Green formula

$$\lambda_{eff}^x(r) = -\pi^{-1}\left(\lambda_1 \oint_{\gamma_r} \varphi_1(x,y)\,dy + \lambda_2\left(\oint_{\Gamma} - \oint_{\gamma_r}\right)\varphi_2(x,y)\,dy\right)$$

$$= -\pi^{-1}\left(\oint_{\gamma_r} \left(\lambda_1\varphi_1(x,y) - \lambda_2\varphi_2(x,y)\right)dy + \lambda_2 \oint_{\Gamma} \varphi_2(x,y)\,dy\right),$$

where in the contour integrals the orientation is assumed to be the opposite to the clockwise direction, φ_1 (or φ_2) is a restriction of φ to D_r (or A_r). Therefore, by the formulas (3.3) and (3.4) and using the polar coordinates (η, θ) we rewrite the last equality in the following form

$$\lambda_{eff}^x(r) = -\pi^{-1}\frac{\lambda_2\rho^{-1} - (\lambda_2\rho^{-1} + \lambda_1(1 - \rho^{-1}))r^2}{-\rho^{-1} + r^2} \int_0^{2\pi} (\cos\theta)^2\,d\theta. \tag{4.3}$$

Since

$$\lambda_2\rho^{-1} + \lambda_1\left(1 - \rho^{-1}\right) = -\lambda_2, \quad \int_0^{2\pi} (\cos\theta)^2\,d\theta = \pi,$$

the equality (4.3) turns to the formula (4.1), which holds for all $r \in (0,1)$).

Then the equality (4.2) follows from the obtained formula after substitution to it the following series expansion

$$\frac{1}{-\rho^{-1} + r^2} = -\rho - \rho\sum_{k=1}^{\infty} \rho^k r^{2k} \quad \forall r \in (0,1). \quad \square$$

Corollary 1. $\lambda^x_{eff}(r)$ is an even function in r, continuous for all $r \in [0, 1]$. Furthermore, $\lambda^x_{eff}(1) = \lambda_1$, $\lambda^x_{eff}(0) = \lambda_2$.

Remark 1. The obtained formula (4.1) coincides with the celebrated Clausis-Mossotti approximation formula for x-component of the effective conductivity tensor (see, e.g. [8]).

Remark 2. If one have to construct the composite material with a prescribed value of $\lambda^x_{eff}(r)$, then formula (4.1) gives an exact relation between values of conductivity of matrix and inclusion (in the case of fixed r) or a functional representation of λ_2 in terms of r (in the case of fixed λ_1).

Acknowledgements

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