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Simulation of Perspective by Nonlinear Transformations

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Abstract. A feature of the brain processing the visualization of objects is such that objects that are much farther away from the eye look smaller than closer objects to the eye. We show that a family of nonlinear transformations, also to be called compactifications, simulate qualitatively this property of keeping objects in perspective. These transformations project objects in a plane on a spherical shell. It is shown then that an observer located at a fixed point on the axis of the sphere visualizes the projected objects on the sphere in perspective. Namely, that objects that are farther away from the observation point seem smaller. Examples are provided. This is a departure from the traditional approaches using linearity and projections of objects from one plane into another plane.

Keywords: perspective, compactification, nonlinear transformations, sphere, infinity.

AMS Subject Classification: 14M27.

1 Introduction

The manner that the eye views objects requires a better understanding of perception and perspective viewing. Geometries that take this into consideration are important. It is noteworthy, that in perspective and projective geometry, some basic mathematical objects are lines and planes. This is understandable as the light is assumed to travel in straight lines. However, the light does not pass through and does not get collected on plane shaped objects. The eye is an organ that detects light and converts it into electro-chemical impulses. It employs nonlinear objects and structures. The eye has a shape of a ball, the surface of the lens in the eye is approximated by two portions of two spherical surfaces and the receptive part of the eye, the retina, is located on an idealized spherical bowl. It therefore behooves us to look for nonlinear mathematical tools that could result in a better or different understanding of perspective viewing. The retina is just one example, where images of objects are created on a spherical like shape. A planetarium and certain special theaters are two more examples where images of objects are viewed on a spherical surface.

It should come as no surprise if the geometry of a sphere plays a unique role in mathematics and its applications. The mapping of the plane onto a sphere leads to the celebrated compactification known as the stereographic projection. Compare e.g. with [12], and the texts [3, 11, 13].

One of the main characteristics of perspective viewing is an approximate representation on a surface of images as seen by the eye. A main characteristic of such drawing is that objects are perceived as smaller if their distance from the observer increases. The thrust of this article is to present a family of nonlinear transformations that mimic this phenomenon. Such transformations are to be called compactifications for reasons that will be apparent in the sequel. Using one of these transformations we project objects in a plane onto a spherical Bowl. We provide quantitative measures for the rate of decrease of sizes of images of objects projected on a spherical bowl.

This perspective could be useful to certain designs. A user could draw manually or with the help of a computer various objects or animations in a plane. For example a backdrop of a scene, could be programmed to draw a large number of objects on a real scale on a plane and then we could use these nonlinear transformations to obtain a perspective from a distance. For example, non-linear projection knowingly used in classical art forms such as *ukiyo-e* [9]. Non-linear projection has also been studied in the context of adjusting linear projection to appear more natural [16].

It is noteworthy that projective geometry tools are a well accepted mean in viewing projected images on a plane. Della Pittura (1435) and Girard Desargues (1591–1661) are credited respectively for the stimulation and the creation of seminal work in perspective and projective geometry. For references on projective geometry see e.g. [10, 14, 15]. Projective geometry differs in the following fundamental manners from the geometry proposed in here. Projective geometry does not distinguish among all directions at infinity. For example, the point $-\infty$ and $+\infty$ are considered the same point in projective geometry. Projective geometry uses projections of objects from a plane into another plane rather than from a plane into a sphere.

It is interesting to note the similarities and differences between the family of compactifications studied in here and other compactifications related to the venerable sphere that are given in the literature. Poincaré's compactification [12] uses the projection of $\mathbb{R}^2$ on a half sphere. Poincaré's half sphere is also utilized in the realization of non-Euclidean Geometry. See [7]. The relevance of these transformations to dynamical systems is documented e.g. in [2, 11, 13]. For different families of radial compactifications and their relevance to approximation theory and to dynamical systems, see [4, 5].

This family of compactifications that is studied in [6] serves as a main tool in this article. The compactifications act as bijections (one to one mapping) as well as projections of a certain extension of $\mathbb{R}^2$ onto certain spherical bowls.

A fixed projection point P is utilized to simulate the location of the eye of an observer.

Details are provided in the following order. In Section 2 we elaborate on the family of compactifications that map $\mathbb{R}^2$ on a spherical bowl. In Section 3 we elaborate on the Euclidean metric induced by a compactification. The metric is then used to prove that the Euclidean distance between the images of the end points of a fixed Euclidean distance D in $\mathbb{R}^2$, appears smaller to a viewer, as the line segment of length D is moved farther away from the viewing point. In Section 4 we consider a family of concentric circles in the set $\{(x, y)\colon x^2 + y^2 > 1\}$. We show that the length of the image of a circle on a spherical bowl that is farther away from the viewing point P, appears smaller to the viewer. We calculate the rate of decrease of the length of the image. In Section 5 we consider a family of parallel lines in a plane. We show that the length of the image of a line on a spherical bowl, that is farther away from the viewing point P, appears smaller to the viewer. We calculate the rate of decrease.

It is noteworthy that the derivations here uncover an abundance of expressions that are "invariant with respect to independent rotations". Essentially, these are mappings that are functions of certain moduli. Quite a few of them turn out to be positive definite. Thanks to these, a generalization of the theorem that asserts the similarity of two triangles in the setting of the stereographic projection is obtained. A discussion of these can be found in [6].

2 A Family of Nonlinear Transformations

The following adaptation from [6] is brought for the sake of a self contained presentation. Denote by $Z = (x_1, x_2, x_3)$ a point in the Euclidean (3 dimensional) space $\mathbb{R}^3$, where x_j satisfy $-\infty < x_j < \infty$, $j = 1, 2, 3$. Denote by ID the continuum of ideal points $\mathrm{ID} := \{\infty(\cos\theta, \sin\theta) \mid 0 \le \theta < 2\pi\}$. Call the set $\mathbb{C} \cup \mathrm{ID}$ the ultra extended complex plane. Denote by $z = (x, y) \in \mathbb{C}$ a point in the (plane) $\mathbb{R}^2$ which is to be identified with the point $Q = (x, y, 0)$ in $\mathbb{R}^3$.

Let $P = (0,\ 0,\ \gamma)$ be a fixed point on the x_3 coordinate, $0 < \gamma \le 1$. Consider γ as a parameter giving rise to a family of nonlinear transformations that contain the so called stereographic projection, namely, $\gamma = 1$, see e.g. [3,11,13], as a particular case.

Put $r^2 = x^2 + y^2$, $R^2 = x_1^2 + x_2^2$ and $\omega = \gamma^2 + (1 - \gamma^2)r^2$. The word Bowl stands for the following set of points:

$$\mathrm{Bowl} := \left\{(x_1, x_2, x_3) \;\middle|\; x_1^2 + x_2^2 + x_3^2 = 1 \text{ and } -1 \le x_3 \le \gamma\right\}.$$

Denote by $G(z)$ a mapping that matches each point $z \in \mathbb{C} \cup \mathrm{ID}$ with a point Z on the Bowl. Its derivation is as follows. If P, Z, and Q lie on the same straight line, then the vectors $\overrightarrow{PZ}$ and $\overrightarrow{PQ}$ are collinear. This is if and only if $\overrightarrow{PZ} = t\overrightarrow{PQ}$ for some real number t. Namely, iff

$$x_1 = tx, \qquad x_2 = ty, \qquad x_3 = (1 - t)\gamma. \tag{2.1}$$

Since x_1, x_2, and x_3 are points on the unit sphere, we have

$$x_1^2 + x_2^2 + x_3^2 = 1. \tag{2.2}$$

We substitute the values of x_1, x_2, and x_3 from Equations (2.1) into Equation (2.2) to obtain $t^2x^2 + t^2y^2 + (1-t)^2\gamma^2 = 1$. Solving for t, we obtain

$$t_{+,-} = \frac{\gamma^2 \pm \sqrt{\gamma^4 - (\gamma^2-1)(r^2+\gamma^2)}}{\gamma^2 + r^2}.$$

Because we want $Z = G(z)$ to map to the "lower" Bowl, so that $x_3 \le \gamma$, we put

$$\omega = \gamma^2 + (1-\gamma^2)r^2, \qquad t = \frac{\gamma^2 + \sqrt{\omega}}{\gamma^2 + r^2} = 1 - \frac{\sqrt{1-R^2}}{\gamma}.$$

We now define a mapping from $\mathbb{C} \cup \mathrm{ID}$ into the Bowl as follows.

Definition 1. The mapping $G(z)$ from $\mathbb{C} \cup \mathrm{ID}$ into the Bowl is defined by

$$G(z) = \left\{ \begin{array}{ll} (x_1 = tx,\ x_2 = ty,\ x_3 = \gamma(1-t)) & \text{if } z \in \mathbb{C} \\ \begin{pmatrix} x_1 = \sqrt{1-\gamma^2}\cos\theta, \\ x_2 = \sqrt{1-\gamma^2}\sin\theta, \\ x_3 = \gamma \end{pmatrix} & \text{if } z = \infty(\cos\theta, \sin\theta) \end{array} \right\}. \tag{2.3}$$

The following theorem formalizes the previous discussion.

Theorem 1. *G is a bijection (a one to one transformation) from the ultra extended $\mathbb{R}^2$ to the Bowl.*

Remark 1. For $\gamma = 1$, we obtain in (2.3), $\omega \equiv 1$, $t = \frac{2}{1+r^2}$, $x_1 = tx$, $x_2 = ty$, and $x_3 = 1 - t$, the formulas of the stereographic projection.

The definition of G given above is consistent with asymptotics and continuity. Indeed, let $0 < \gamma < 1$. Given a sequence $z_n = r_n(\cos\theta_n, \sin\theta_n) = (x_n, y_n)$ where $r_n \to \infty$ and $(\cos\theta_n, \sin\theta_n) \to (\cos\theta, \sin\theta)$ as $n \to \infty$, the sequence is such that $\omega_n = \gamma^2 + (1-\gamma^2)r_n^2$ and $t_n = \frac{\gamma^2+\sqrt{\omega_n}}{\gamma^2+r_n^2}$, we have $\sqrt{\omega_n} \sim \sqrt{1-\gamma^2}r_n$, $t_n \sim \frac{\sqrt{1-\gamma^2}}{r_n}$ as $n \to \infty$. Hence $x_{1n} \sim \sqrt{1-\gamma^2}\cos\theta_n$, $x_{2n} \sim \sqrt{1-\gamma^2}\sin\theta_n$.

For $\gamma = 1$ and $z = \infty(\cos\theta, \sin\theta)$ we obtain $t = 0$, $x_1 = 0$, $x_2 = 0$ and $x_3 = 1$. This is consistent with the sequence $Z_n = G(z_n)$ being such that $t_n \to 0$, $x_{1n} \to 0$, $x_{2n} \to 0$, $x_{3n} \to 1$, as $n \to \infty$. For $\gamma = 0$ we obtained a translation of the Poincaré half sphere. Compare e.g. with the texts, Ahlfors [1] and Hille [8].

Remark 2. The difference between each member of our family of compactifications, with $0 < \gamma^2 < 1$ and the stereographic projection, that corresponds to $\gamma^2 = 1$, is substantial indeed. First and foremost, the stereographic projection of $\mathbb{R}^2$ is obtained by adding a single ideal point infinity. Our compactification augments $\mathbb{R}^2$ with a continuum of points ID. The stereographic projection treats the point infinity like any other finite point. Our compactification recognizes among all the different arguments or directions of infinity.

It is noteworthy that the stereographic projection preserves angles, namely, it is conformal. An angle between any two curves in the plane is the same as the angle that is mapped on images of these curves on the sphere. This property does not hold if $0 < \gamma^2 < 1$. The interested reader may want to verify this by an independent calculation. Geometrically, a mapping with $0 < \gamma^2 < 1$ may be classified as a diffeomorphism that takes unbounded sets into bounded sets.

3 The Induced Metric and the Image of a Fixed Length

The manner that we show that objects that are farther from the viewing point P appear smaller on the spherical bowl requires the introduction of a certain metric that is associated with the compactification. This metric, to be denoted by $\chi(z,\hat{z})$, for the ultra extended $\mathbb{R}^2$ is provided below.

We denote by $\|G(z)-G(\hat{z})\|$ the Euclidean distance between two points on the spherical bowl. Denote by $\hat{Z}=(\hat{x}_1,\hat{x}_2,\hat{x}_3)$ a point in the Euclidean space $\mathbb{R}^3$, where $\hat{x}_j$ satisfy $-\infty < \hat{x}_j < \infty$, $j=1,2,3$. Denote by $\hat{z}=(\hat{x},\hat{y})$ a point in the ultra extended $\mathbb{R}^2$ which is identified with the point $\hat{Q}=(\hat{x},\hat{y},0)$ such that $G(\hat{z})=\hat{Z}$. We also put

$$\hat{r}^2=\hat{x}^2+\hat{y}^2, \qquad \hat{\omega}=\gamma^2+\left(1-\gamma^2\right)\hat{r}^2, \quad \text{and} \quad \hat{t}=\frac{\gamma^2+\sqrt{\hat{\omega}}}{\gamma^2+\hat{r}^2}.$$

The following theorem, [6], plays an important role in this article.

Theorem 2. *The ultra extended $\mathbb{R}^2$ is a complete metric space with respect to the chordal metric χ defined below as the Euclidean distance $\|G(z)-G(\hat{z})\|$.*

$$\chi(z,\hat{z})\equiv\left\|G(z)-G(\hat{z})\right\|=\sqrt{\left(x_1-\hat{x}_1\right)^2+\left(x_2-\hat{x}_2\right)^2+\left(x_3-\hat{x}_3\right)^2}.$$

Specifically, the square of the metric χ^2 is given by

$$\chi^2(z,\hat{z})=F\left(D^2-\Delta\right), \tag{3.1}$$

where F is a contraction $($dilation$)$ factor

$$F=\frac{(\gamma^2+\sqrt{\omega})(\gamma^2+\sqrt{\hat{\omega}})}{(\gamma^2+r^2)(\gamma^2+\hat{r}^2)}.$$

D^2 is the square of the Euclidean distance between z and $\hat{z}$

$$D^2=(x-\hat{x})^2+(y-\hat{y})^2.$$

The expression for Δ is given by

$$\Delta=\frac{(1-\gamma^2)(r^2-\hat{r}^2)^2}{(\sqrt{\omega}+\sqrt{\hat{\omega}})(\gamma^2+\sqrt{\omega})(\gamma^2+\sqrt{\hat{\omega}})}A,$$

where

$$A=\left[\gamma^2+\frac{(1-\gamma^2)r^2\hat{r}^2+\gamma^4(1+\gamma^2)+\gamma^2(r^2+\hat{r}^2)}{(\gamma^2+r^2)\sqrt{\hat{\omega}}+(\gamma^2+\hat{r}^2)\sqrt{\omega}}\right]. \tag{3.2}$$

F, Δ, and D^2 are non negative functions. Specifically, $\Delta \geq 0$ for $\gamma^2 < 1$ and $\Delta = 0$ iff $\gamma^2 = 1$ or $r^2 = \hat{r}^2$. For $z = \infty(\cos\theta, \sin\theta)$, $\hat{z} = (\hat{x}, \hat{y})$,

$$\chi^2(z, \hat{z}) = 2\gamma^2 \frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2} + 2\big(1 - \gamma^2\big) - 2\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\sqrt{1 - \gamma^2}(\hat{x}\cos\theta + \hat{y}\sin\theta)$$

and for $z = \infty(\cos\theta, \sin\theta)$, $\hat{z} = \infty(\cos\hat{\theta}, \sin\hat{\theta})$,

$$\chi^2(z, \hat{z}) = 4\big(1 - \gamma^2\big)\sin^2\left(\frac{\theta - \hat{\theta}}{2}\right) = \left(2\sqrt{1 - \gamma^2}\left|\sin\left(\frac{\theta - \hat{\theta}}{2}\right)\right|\right)^2. \tag{3.3}$$

Proof. Details are given in [6]. The interested reader may want to reproduce a lengthy but straightforward calculation that leads to the desired formulas (3.1) to (3.2). □

The image of a distance between two points at infinity on the horizon may be converted by a viewer to an image of a finite distance. This property is captured mathematically by the induced metric of the above compactifications. The distance $D = \infty$ between two points $z = \infty(\cos\theta, \sin\theta)$ and $\hat{z} = \infty(\cos\hat{\theta}, \sin\hat{\theta})$ is compacted into the distance $2\sqrt{1 - \gamma^2}|\sin(\frac{\theta - \hat{\theta}}{2})|$ according to formula (3.3) in Theorem 5 above.

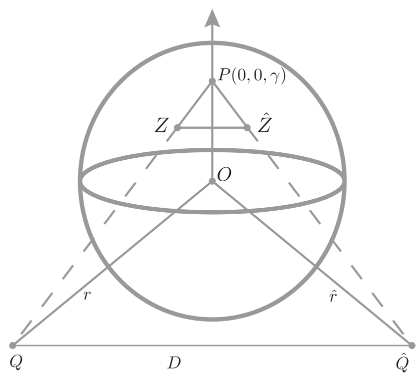

Figure 1. Q and $\hat{Q}$ projected on the unit sphere. The projected points are z and $\hat{z}$, respectively. $z\hat{z} = \chi(z, \hat{z})$.

For the next description of the perspective provided by our compactification we analyze the Euclidean distance $Z\hat{Z}$, where Z and $\hat{Z}$ correspond to Q and $\hat{Q}$ in $\mathbb{R}^2$ under the compactification (Figure 1). Denote by O the center of the sphere. We position the line segment $Q\hat{Q} = D$ so that $OQ\hat{Q}$ is an isosceles triangle. Thus $OQ = O\hat{Q} = r = \hat{r}$, and (although the length of the equal sides vary) the length of the third side of the triangle, $Q\hat{Q}$, has a constant value D. The formulas developed above show that then $r = \hat{r} \Longrightarrow \Delta = 0$ so that

$$\big\|G(z) - G(\hat{z})\big\|^2 = \chi^2(z, \hat{z}) = F\big(D^2 - \Delta\big) = t^2 D^2.$$

Let us determine the rate of change of t with respect to the variable r^2. By the definition of t we have

$$\frac{\partial t}{\partial(r^2)} = \frac{\partial[\frac{\gamma^2+\sqrt{\omega}}{\gamma^2+r^2}]}{\partial(r^2)} = \frac{[\gamma^2+r^2]\frac{(1-\gamma^2)}{2\sqrt{\omega}} - [\gamma^2+\sqrt{\omega}\,]}{[\gamma^2+r^2]^2}.$$

Expressing r^2 in terms of ω as $r^2 = \frac{\omega-\gamma^2}{(1-\gamma^2)}$ in the numerator of the formula above yields

$$\frac{\partial t}{\partial(r^2)} = -\frac{(\sqrt{\omega}+\gamma^2)^2}{2\sqrt{\omega}[\gamma^2+r^2]^2}.$$

The implications of the latter derivation for perspective viewing is the following. Consider an infinite ladder with equally spaced rungs, situated in the x, y plane, with the middle of the head of the ladder coinciding with the center of the sphere so that a ray emanating from the origin of the sphere divides the ladder into two congruent parts. How would the Euclidean distance, as measured by the above metric, between the projected end points of successive rungs appear to a viewer situated at the origin of the sphere? The formulas above confirm that the same length rungs located farther and farther away from the viewer in the x, y plane will appear smaller and smaller to the viewer.

It is noteworthy, that for $\gamma = 1$, the distance between two objects that are moved far away to the horizon (infinity) become zero. This can be interpreted as the stereographic projection "blurring" an object or rendering two different objects situated far away as indistinguishable. In contrast, for any fixed γ, $-1 < \gamma < 1$, there is a lower bound on the amount of potential blurring of an object that is situated far away. Alternatively, two different objects that are situated far away will be distinguishable.

4 Images of Concentric Circles

We turn now to yet another demonstration that objects that are farther away from the viewing point P, look smaller. Consider the continuum of concentric circles situated in $\mathbb{R}^2$ with center at the origin (that is the center of the sphere) and with radii that increase continuously without bound. Geometrical considerations show that indeed the circumferences of the images of the growing concentric circles that lie outside the unit circle $x^2 + y^2 = 1$ diminish as the concentric circles grow. We provide a quantitative measure to the rate of decrease.

We compare the arc length of two circles $C(r) := \{(x, y)\colon x^2 + y^2 = r^2\}$ and $C(\hat{r}) := \{(\hat{x}, \hat{y})\colon \hat{x}^2 + \hat{y}^2 = \hat{r}^2\}$, $\hat{r} > r > 1$ with the length of their images on the spherical bowl (Figure 2). Since their circumferences are proportional to their radii, it suffices to compare the radii of the images of $C(r)$ and $C(\hat{r})$ on the spherical bowl. The square of the radius of the image of $C(r)$ is readily found to be

$$R^2 = x_1^2 + x_2^2 = t^2 r^2.$$

We prove now the following

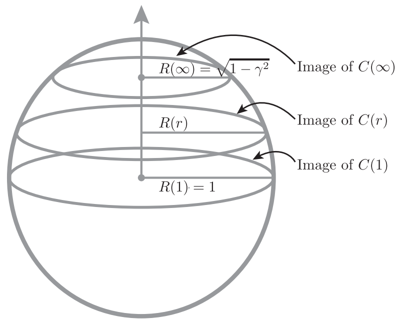

Figure 2. Images of the family of concentric circles. The boundary of the bowl is the circle $x_1^2 + x_2^2 = 1 - \gamma^2$, $x_3 = \gamma$.

Theorem 3. *The radius* $R(r) := tr = \frac{[\gamma^2+\sqrt{\omega}]r}{\gamma^2+r^2}$ *of an image circle* $C(r)$, *is a monotone decreasing function of* r. *The rate of decrease is given by*

$$\frac{\partial R}{\partial r} = \frac{\gamma^2 t(1-\gamma^2)(1-r^2)}{\sqrt{\omega}(1+\sqrt{\omega})(\sqrt{\omega}-\gamma^2)}.$$

Proof. Notice that

$$\frac{\partial t}{\partial r} = -\frac{rt^2}{\sqrt{\omega}} \quad \Longrightarrow \quad \frac{\partial R}{\partial r} = t\left[-\frac{r^2 t}{\sqrt{\omega}} + 1\right].$$

We write the above expression in terms of $\sqrt{\omega}$. First, we express t in terms of $\sqrt{\omega}$. This yields

$$r^2 = \frac{\omega - \gamma^2}{(1-\gamma^2)}, \qquad t = \frac{(1-\gamma^2)}{(\sqrt{\omega}-\gamma^2)}.$$

Using the above the reader can find by a straightforward calculation that

$$\frac{\partial R}{\partial r} = \gamma^2 t\left[\frac{1-\omega}{\sqrt{\omega}(1+\sqrt{\omega})(\sqrt{\omega}-\gamma^2)}\right].$$

The fact that $1 - \omega = (1-\gamma^2)(1-r^2)$ leads to the formula

$$\frac{\partial R}{\partial r} = \frac{\gamma^2 t(1-\gamma^2)(1-r^2)}{\sqrt{\omega}(1+\sqrt{\omega})(\sqrt{\omega}-\gamma^2)}.$$

Notice that $\gamma^2 < 1 \Longrightarrow \gamma^2 > \gamma^4$ and hence $\omega = \gamma^2 + (1-\gamma^2)r^2 > \gamma^4$ which in turn implies that

$$\sqrt{\omega} - \gamma^2 > 0.$$

This in turn implies that the rate of change $\frac{\partial R}{\partial r} < 0$ if $r^2 > 1$ and that $\frac{\partial R}{\partial r} > 0$ if $r^2 < 1$, thus, proving the desired conclusions. □

Remark 3. Notice that in contrast to circles farther away from the viewing point P, circles inside the circle $r = 1$ in $\mathbb{R}^2$, that are closer to the viewing point P, are magnified as their radii increases.

5 Images of a Family of Parallel Lines

Consider a continuum of parallel lines that make up half of the plane in $\mathbb{R}^2$. Assume, without loss of generality, that each line contains the same fixed vector $v = (c, d, 0) \neq (0, 0, 0)$ and passes through a certain point $(a, 0, 0)$ on the x_1 axis so that its vector equation is given by

$$\big(x_1(s), x_2(s), x_3(s)\big) = (a, 0, 0) + s(c, h, 0), \quad -\infty < s < \infty. \tag{5.1}$$

Consequently, the parametric equation of each line, denoted by $l(a)$, is given by

$$x_1(s) \equiv a, \qquad x_2(s) = s, \qquad x_3(s) \equiv 0, \quad -\infty < s < \infty.$$

Let us determine the point on the sphere that corresponds to the value $s = +\infty$. Then, if θ is the angle that the vector v makes with the positive x axis then by (2.1)

$$Z(\infty) = G\big(z(+\infty)\big) = \big(x_1 = \sqrt{1-\gamma^2}\cos\theta, x_2 = \sqrt{1-\gamma^2}\sin\theta, x_3 = \gamma\big).$$

This is the image of the "vanishing point" at infinity of all parallel rays with the same direction in the continuum family. It lies on the circle

$$x_1^2 + x_2^2 = 1 - \gamma^2, \qquad x_3 = \gamma, \tag{5.2}$$

and is independent of the points $(a, 0, 0)$. The image of the points at infinity on each ray passing through the points $(a, 0, 0)$ and containing the vector $-v$ is independent of a and is given by

$$Z(-\infty) = G\big(z(-\infty)\big) = (x_1 = -\sqrt{1-\gamma^2}\cos\theta, x_2 = -\sqrt{1-\gamma^2}\sin\theta, x_3 = \gamma).$$

Notice that unlike in projective geometry, each line has two points $G(z(-\infty))$ and $G(z(+\infty))$ at infinity, that correspond to two opposite rays emanating from a point on a given line. Although all lines of this family have in common the two diametrically opposite points, $G(z(-\infty))$ and $G(z(+\infty))$ on the circle (5.2), their images $Z(s) = G(z(s))$, $-\infty < s < \infty$, differ. Each image lies in the intersection of the plane containing the point P and the line with the spherical bowl. Therefore, this intersection curve is an arc of a circle that is a common chord. Two different lines differ by the parameter a in (5.1).

Evidently, as $a \geq 0$ increases, the lines in (5.1) are getting farther away from the viewing point P. As a increases from $a = 0$ to $a = \infty$, the image of the parallel lines on the sphere form a family of arcs of circles with decreasing length. They decrease from an arc of a large circle with radius 1 to an arc of the circle with radius $\sqrt{1-\gamma^2}$ that is on the boundary of the Bowl.

It can easily be verified that a normal to the plane containing the vector $(0, 1, 0)$ and the vector $(a, 0, -\gamma)$ (corresponding to the arrow with head at $(a, 0, 0)$ and tail at $(0, 0, \gamma)$) is given by $K := (\gamma, 0, a)$.

The equation of the plane having a normal K and containing the point P is readily found to be

$$\gamma x_1 + a x_3 = a\gamma, \tag{5.3}$$

which implies $x_1 = \gamma^{-1}a(\gamma - x_3)$.

We will determine the center and the radius of the circle that lies in the intersection of the plane (5.3) and the sphere $x_1^2 + x_2^2 + x_3^2 = 1$.

The shortest distance S from the center O to the plane (5.3) is given by

$$S = \frac{|a\gamma|}{\sqrt{\gamma^2 + a^2}}.$$

Hence, the radius λ of the circle that has an arc as the image of the line (5.1) on the compactification sphere, is given by

$$\lambda = \sqrt{1 - S^2} = \sqrt{1 - \frac{a^2\gamma^2}{\gamma^2 + a^2}}.$$

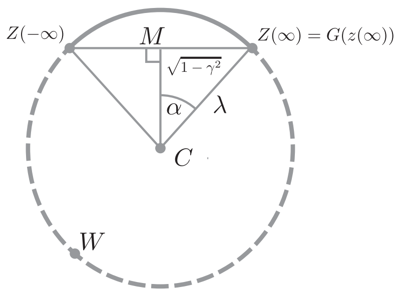

Figure 3. The circle determined by the intersection of the unit sphere and a line $l(a)$ passing through $(a, 0, 0)$ and containing the vector $(0, 1, 0)$. The dashed arc $Z(-\infty)WZ(+\infty)$ is an image of $l(a)$.

All arcs have in common the two points $Z(-\infty)$ and $Z(\infty)$ as the images of the two points at infinity on each straight line of the family of parallel lines. Denote by K the center of the circle. The length of each arc is $L(a^2) := 2\pi\lambda - 2\alpha\lambda$ where $2\alpha = \widehat{Z(-\infty)KZ}(\infty)$ is double the acute angle α, with vertex at C, in the right angle triangle of $KMZ(\infty)$ where M is the middle of the line segment $Z(-\infty)Z(\infty)$. See Figures 3 and 4.

Notice that

$$\alpha = \sin^{-1}\left(\frac{\sqrt{1-\gamma^2}}{\lambda}\right)$$

and

$$L(a^2) := 2\lambda\left[\pi - \sin^{-1}\left(\frac{\sqrt{1-\gamma^2}}{\lambda}\right)\right].$$

We wish to calculate the rate of change of $L(a^2)$ with respect to a^2. To this end we first find that

$$\frac{\partial[\ln(\lambda)]}{\partial(a^2)} = \frac{1}{2}\left[\frac{-\gamma^2}{\gamma^2 + a^2 - a^2\gamma^2}\right] < 0.$$

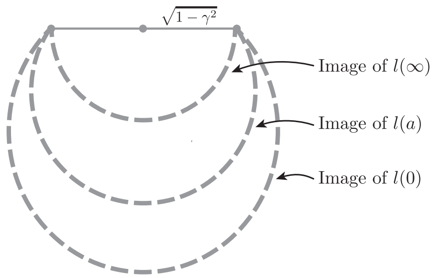

Figure 4. The image of the parallel lines l form a family of arcs of circles. The image of $l(0)$ is an arc of a large circle passing through O.

This fits very well with the geometrical intuition that as a^2 grows the radii of the circles diminishes. Notice that

$$\frac{\partial[\pi - \sin^{-1}(\frac{\sqrt{1-\gamma^2}}{\lambda})]}{\partial(a^2)} = \frac{\frac{\sqrt{1-\gamma^2}}{\lambda^2}}{\sqrt{1-(\frac{\sqrt{1-\gamma^2}}{\lambda})^2}} \frac{\partial\lambda}{\partial(a^2)} \leq 0.$$

Putting together the calculations above we have

$$\begin{aligned}\frac{\partial L(a^2)}{\partial(a^2)} &= 2\frac{\partial\lambda}{\partial(a^2)}\left[\pi - \sin^{-1}\left(\frac{\sqrt{1-\gamma^2}}{\lambda}\right)\right] + 2\lambda\frac{\partial[\pi - \sin^{-1}(\frac{\sqrt{1-\gamma^2}}{\lambda})]}{\partial(a^2)} \\ &= 2\frac{\partial\lambda}{\partial(a^2)}\left[\pi - \sin^{-1}\left(\frac{\sqrt{1-\gamma^2}}{\lambda}\right)\right] + 2\lambda\frac{\frac{\sqrt{1-\gamma^2}}{\lambda^2}}{\sqrt{1-(\frac{\sqrt{1-\gamma^2}}{\lambda})^2}}\frac{\partial\lambda}{\partial(a^2)} \\ &= 2\frac{\partial\lambda}{\partial(a^2)}\left\{\left[\pi - \sin^{-1}\left(\frac{\sqrt{1-\gamma^2}}{\lambda}\right)\right] + \frac{\frac{\sqrt{1-\gamma^2}}{\lambda}}{\sqrt{1-(\frac{\sqrt{1-\gamma^2}}{\lambda})^2}}\right\}.\end{aligned}$$

Observe that for the range of variables the following inequalities hold:

$$0 \leq \sqrt{1-\gamma^2} \leq \lambda \leq 1, \quad 0 \leq \sin^{-1}\left(\frac{\sqrt{1-\gamma^2}}{\lambda}\right) \leq \frac{\pi}{2},$$

$$\frac{\pi}{2} \leq \left\{\left[\pi - \sin^{-1}\left(\frac{\sqrt{1-\gamma^2}}{\lambda}\right)\right] + \frac{\frac{\sqrt{1-\gamma^2}}{\lambda}}{\sqrt{1-(\frac{\sqrt{1-\gamma^2}}{\lambda})^2}}\right\},$$

so the factor

$$\left\{\left[\pi - \sin^{-1}\left(\frac{\sqrt{1-\gamma^2}}{\lambda}\right)\right] + \frac{\frac{\sqrt{1-\gamma^2}}{\lambda}}{\sqrt{1-(\frac{\sqrt{1-\gamma^2}}{\lambda})^2}}\right\}$$

is positive and consequently $\frac{\partial L(a^2)}{\partial(a^2)} < 0$ and the result follows.

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