Mathematical Modelling and Analysis
Volume 18 Number 3, June 2013, 325–345
http://dx.doi.org/10.3846/13926292.2013.804462

Publisher: Taylor&Francis and VGTU
http://www.tandfonline.com/TMMA
Print ISSN: 1392-6292
Online ISSN: 1648-3510

Periodic Orbits of Single Neuron Models with Internal Decay Rate $0 < \beta \leq 1$

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Received September 15, 2012; revised April 10, 2013; published online June 1, 2013

Abstract. In this paper we consider a discrete dynamical system

$$x_{n+1} = \beta x_n - g(x_n), \quad n = 0, 1, \dots,$$

arising as a discrete-time network of a single neuron, where $0 < \beta \leq 1$ is an internal decay rate, g is a signal function. A great deal of work has been done when the signal function is a sigmoid function. However, a signal function of McCulloch–Pitts nonlinearity described with a piecewise constant function is also useful in the modelling of neural networks. We investigate a more complicated step signal function (function that is similar to the sigmoid function) and we will prove some results about the periodicity of solutions of the considered difference equation. These results show the complexity of neurons behaviour.

Keywords: dynamical system, fixed point, iterative process, nonlinear problem, stability.

AMS Subject Classification: 39A10; 39A23; 92B20.

1 Introduction

Dynamical characteristics of neural networks recently have become a subject of intense research activities. Neural networks have been constructed from a wide range of different views and this has been reflected in a variety of results as well as in mathematical techniques used in their derivation. Although many models have been constructed, the task of modelling neural networks is not complete yet and being able to understand human behaviour and brain functions is still a great motivation for modelling and analyzing neural networks. In the last decades scientists have found that oscillations are an important feature in the brain processing. Oscillations are temporal periodic changes in the state of a system. In nonlinear systems like brain, oscillations define a stable state.

In the literature [8] a delay differential equation

$$x'(t) = -g\big(x(t-\tau)\big) \tag{1.1}$$

is used as a model for a single neuron with no internal decay where $g : R \to R$ is either a sigmoid or a piecewise linear signal function and $\tau \leq 0$ is a synaptic transmission delay. From (1.1) we obtain a model for a single neuron with no internal decay as the following equation

$$x'(t) = -g\big(x([t])\big), \tag{1.2}$$

where $[t]$ denotes a greatest integer function. When we integrate (1.2) from n to $t \in [n,\ n+1[$ we get

$$x(t) = x(n) - \int_n^t g\big(x([s])\big)\, ds = x(n) - g\big(x(n)\big)(t-n).$$

Letting $t \to n+1$ and denoting $x(n) = x_n$, we obtain a difference equation

$$x_{n+1} = x_n - g(x_n).$$

Typical signal functions (activation functions, amplification functions or input–output functions) are of the following types: a) step functions, b) piecewise linear functions, c) sigmoid functions.

A step function can be defined as follows

$$g(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Models involving such signal function are referred as McCulloch–Pitts models, in recognition of the pioneering work of McCulloch and Pitts, 1943. This function describes an all-or-none property of a neuron.

A piecewise linear function is given by formula

$$g(x) = \begin{cases} 0, & x \leq 0, \\ \mu x, & 0 < x < 1/\mu, \\ 1, & x \geq 1/\mu. \end{cases}$$

This function describes the nonlinear off–on characteristics of neurons. Parameter μ is called a neural gain. Note that a piecewise linear function can be reduced to the step function if μ tends to infinity. Such a function has been widely used in cellular neural network models.

A sigmoid function is the most common form of a signal function. It is defined as a strictly increasing smooth bounded function satisfying certain concavity and asymptotic properties. Examples of a sigmoid function are an arctangent function and a logistic function given by formula

$$g(x) = \frac{1}{1+e^{-4\mu x}}, \quad x \in \mathbf{R},$$

where μ is a neuron gain. A step function takes only two values 0 or 1, whereas a sigmoid function takes a continuous range of values in $]0,1[$. But as $\mu \to \infty$, a sigmoid function becomes a step function. Therefore we can consider signal functions as step functions. In the literature several types of such functions have been considered.

The signal function

$$g(x) = \begin{cases} 1, & x \in]0,\sigma[, \\ 0, & x \in]-\infty,0] \cup]\sigma,+\infty[\end{cases}$$

shows that if the activation of one neuron is between 0 and σ, then it has a constant active affection to another neuron, or else it has no affection to another neuron [12]. The function

$$g(x) = \begin{cases} -\rho, & x > \sigma, \\ \rho, & x \le \sigma \end{cases}$$

is a McCulloch–Pitts signal function with the threshold σ and the synaptic weight $\rho > 0$ [11]. In [2, 7, 9] the signal function

$$g(x) = \begin{cases} 1, & x > \sigma, \\ -1, & x \le \sigma \end{cases}$$

has been considered. Here, $\sigma \in \mathbf{R}$ is referred as the threshold. The McCulloch–Pitts nonlinearity reflects the fact that the signal transmission is of digital nature: a neuron is either fully active or completely inactive.

In [10] a single neuron model has been considered

$$x_{n+1} = \beta x_n - g(x_n), \quad n = 0,1,2,\ldots, \tag{1.3}$$

with a signal function given in the very simple form

$$g(x) = \begin{cases} 1, & x \ge 0, \\ -1, & x < 0. \end{cases} \tag{1.4}$$

Equation (1.3) arises as a discrete-time network of single neuron, where β is the internal decay rate, g is a signal function. In our work we consider the single neuron model (1.3) but we propose to consider a more complicated step signal function (a function that is similar to the sigmoid function). We will prove some results about the periodicity of solutions of difference equation (1.3). These results were also presented in the International Conference on Differential Equations, Loughborough, UK, 2011 and International Conference Mathematical Modelling and Analysis, Tallinn, Estonia, 2012. They generalize results of [10] and show the complexity of neurons behaviour.

2 Basic Concepts and Definitions of Difference Equations

To analyze the behaviour of the model (1.3) some basic concepts of difference equation theory (see [4, 5, 10]) are required.

We consider a first-order difference equation

$$x_{n+1} = f(x_n), \quad n = 0, 1, \dots, \tag{2.1}$$

where $f : \mathbf{R} \to \mathbf{R}$ is a given function. A solution of equation (2.1) is a sequence $(x_n)_{n\in\mathbf{N}}$ that satisfies equation (2.1) for all $n = 0, 1, \dots$. If an initial condition $x_0 \in \mathbf{R}$ is given, then the *orbit* $O(x_0)$ of a point x_0 is defined as a set of points

$$O(x_0) = \{x_0, x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), x_3 = f(x_2) = f^3(x_0), \dots\}.$$

Definition 1. A point x_s is said to be a *fixed point* of the map f or a *stationary state* of equation (2.1) if $f(x_s) = x_s$.

Note that for a stationary state x_s the orbit consists only of the point x_s.

Definition 2. A *stationary state* of (2.1) is *stable* if

$$\forall\varepsilon > 0 \ \exists\delta > 0 \ \forall x_0 \in \mathbf{R} \ \forall n \in \mathbf{N} \quad |x_0 - x_s| < \delta \quad \Rightarrow \quad \big|f^n(x_0) - x_s\big| < \varepsilon.$$

Otherwise, the stationary state x_s is called *unstable*.

Definition 3. An orbit $O(x_0)$, $x_0 \in \mathbf{R}$, is said to be *eventually stationary state* if

$$\exists N \ \forall n \geq N \quad x_{n+1} = x_n.$$

Definition 4. An orbit $O(x_0) = \{x_0, x_1, x_2, \dots\}$ of an initial point x_0 of equation (2.1) is said to be *periodic of period* $p \geq 2$ if

$$x_p = x_0 \quad \text{and} \quad x_i \neq x_0, \quad 1 \leq i \leq p - 1.$$

Definition 5. A *periodic orbit* $\{x_0, x_1, x_2, \dots, x_{p-1}, \dots\}$ of period p is *stable* if each point x_i, $i = 0, 1, \dots, p-1$, is a stable stationary state of the difference equation $x_{n+1} = f^p(x_n)$. A *periodic orbit* of period p which is not stable is said to be *unstable*.

Definition 6. A point z is said to be a *limit point* of $O(x_0)$ if there exists a subsequence $(x_{n_k})_{k=0,1,2,\dots}$ of $O(x_0)$ such that $\lim_{k\to+\infty} |x_{n_k} - z| = 0$. The *limit set* $L(x_0)$ of the orbit $O(x_0)$ is the set of all limit points of the orbit.

Definition 7. An orbit $O(x_0)$ is said to be *asymptotically periodic* if its limit set is a periodic orbit. An orbit $O(x_0)$ such that $x_{n+p} = x_n$ for some $n \geq 1$ and some $p \geq 2$ is said to be eventually periodic.

3 Results

In [10] a single neuron model (1.3)

$$x_{n+1} = \beta x_n - g(x_n)$$

with a signal function (1.4) has been considered. Z. Zhou has proved two theorems about the periodicity.

Theorem A. **[10]** *Assume that* $\beta \in]0,1[$. *Then the periodic orbit* $O(\frac{1}{\beta+1})$ *is a stable periodic orbit with period* 2. *And for every* $x_0 \in \mathbf{R}$, *the orbit* $O(x_0)$ *is asymptotically periodic with* $L(x_0) = \{\frac{1}{\beta+1}, \frac{-1}{\beta+1}\}$.

Theorem B. **[10]** *Assume that* $\beta = 1$. *Then for every* $x_0 \in \mathbf{R}$ *the orbit* $O(x_0)$ *is eventually periodic with period* 2.

We consider a model (1.3) with a new signal function in the following form

$$g(x) = \begin{cases} -b, & x \le -\alpha, \\ -a, & -\alpha < x < 0, \\ 0, & x = 0, \\ a, & 0 < x < \alpha, \\ b, & \alpha \le x, \end{cases} \qquad (b > a > 0 \text{ and } \alpha > 0), \tag{3.1}$$

where $-\alpha$, 0, α are three thresholds. Accordingly to the signal functions given in the Introduction constants a and b could be in the interval $]0,1[$, but in this paper we consider arbitrary large constants a and b that satisfy conditions $0 < a < b$. This signal function (3.1) is more similar to the sigmoid function (see Figure 1) than simple step functions that have been discussed in the Introduction.

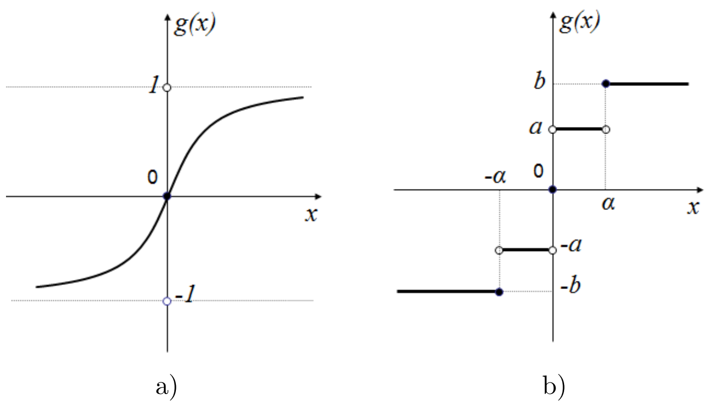

Figure 1. Simple step functions: a) sigmoid function $g(x) = \frac{2}{\pi}\arctan x$, b) step signal function $g(x)$.

Further we analyze model (1.3) with a signal function (3.1) depending on the internal decay rate β. We consider two situations: $\beta = 1$ and $0 < \beta < 1$.

3.1 Model with $\beta = 1$

In this part we prove two theorems, one about periodicity and one about stability.

Theorem 1. *Assume that* $\beta = 1$. *Then*

1) 0 *is a stationary state.*

2) *If there exists a positive integer* k_1 *such that* $0 < x_0 = k_1 a < \alpha$, *then the point* x_0 *is an eventually stationary state, i.e.,*

$$O(x_0) = \{k_1 a, (k_1 - 1)a, \ldots, 2a, a, 0, 0, \ldots\}.$$

3) *If there exists positive integers* k_1 *and* k_2 *such that* $x_0 = k_1 a + k_2 b \geq \alpha$ *and* $k_1 a < \alpha$ *but* $\alpha \leq k_1 a + b$, *then the point* x_0 *is an eventually stationary state, i.e.,*

$$O(x_0) = \{k_1 a + k_2 b, k_1 a + (k_2 - 1)b, \ldots, k_1 a + b, k_1 a, (k_1 - 1)a, \ldots, 2a, a, 0, 0, \ldots\}.$$

4) *The cases* $-\alpha < x_0 = -k_1 a < 0$ *and* $x_0 = -k_1 a - k_2 b \leq -\alpha$ *are similar to Statement* 2) *and Statement* 3) *respectively.*

5) *If* 1)–4) *are not fulfilled, then the orbit* $O(x_0)$ *is periodic with period* 2 *or eventually periodic with period* 2.

Proof. The first four statements are obvious. We will prove the Statement 5).

Case 1. We assume that $0 < x_0 < \alpha$. Then $g(x_0) = a$ and $x_1 = x_0 - a$. Three cases are possible:

Case A. $0 < x_1 = x_0 - a < \alpha$. Since $0 < x_0 < \alpha$ and $0 < x_0 - a < \alpha$ then $a < \alpha$. Therefore if we consider $x_2 = x_0 - a - a = x_0 - 2a$, then only two cases are possible: $-\alpha < x_2 = x_0 - 2a < 0$ or $0 < x_2 = x_0 - 2a < \alpha$. In the first case $x_3 = x_0 - 2a + a = x_0 - a = x_1$ and we obtain an eventually periodic orbit with period 2:

$$O(x_0) = \{x_0, x_0 - a, x_0 - 2a, x_0 - a, x_0 - 2a, \ldots\}.$$

In the second case $x_3 = x_0 - 3a$. Similar to the previous case we have two possibilities: $-\alpha < x_3 < 0$ or $0 < x_3 < \alpha$. In the first case we obtain an eventually periodic orbit with period 2:

$$O(x_0) = \{x_0, x_0 - a, x_0 - 2a, x_0 - 3a, x_0 - 2a, x_0 - 3a, \ldots\}.$$

In the second case $x_4 = x_0 - 4a$. Since $0 < x_0 < \alpha$ is fixed then $\exists n_0 \in \mathbf{N}$:

$$0 < x_{n_0 - 1} = x_0 - (n_0 - 1)a < \alpha \quad \text{and} \quad -\alpha < x_{n_0} = x_0 - n_0 a < 0.$$

This means that in Case A the orbit of a point x_0 is eventually periodic with period 2:

$$O(x_0) = \{x_0, x_0 - a, \ldots, x_0 - (n_0 - 1)a, x_0 - n_0 a, x_0 - (n_0 - 1)a, x_0 - n_0 a, \ldots\}.$$

Case B. $-\alpha < x_1 = x_0 - a < 0$. In this case $x_2 = x_0 - a + a = x_0$. We have a periodic orbit with period 2: $O(x_0) = \{x_0, x_0 - a, x_0, x_0 - a, \ldots\}$.

Case C. $x_1 = x_0 - a \leq -\alpha$. If $x_1 = x_0 - a \leq -\alpha$, then $a > \alpha$. But since $b > a$ and $x_0 > 0$ then $x_2 = x_0 - a + b > 0$. Two cases are possible: $\alpha \leq x_2$ or

$0 < x_2 < \alpha$. If $\alpha \le x_2$, then $x_3 = x_0 - a + b - b = x_0 - a = x_1$ and we obtain an eventually periodic orbit with period 2:

$$O(x_0) = \{x_0, x_0 - a, x_0 - a + b, x_0 - a, x_0 - a + b, \dots\}.$$

If $0 < x_2 < \alpha$, then $x_3 = x_2 - a = x_0 - 2a + b$. Since $a > \alpha$ then $x_3 = x_2 - a < \alpha - a < 0$. Again we have two possibilities: $-\alpha < x_3 < 0$ or $x_3 \le -\alpha$. If $-\alpha < x_3 < 0$, then $x_4 = x_0 - 2a + b + a = x_0 - a + b = x_2$. We obtain an eventually periodic orbit with period 2:

$$O(x_0) = \{x_0, x_0 - a, x_0 - a + b, x_0 - 2a + b, x_0 - a + b, x_0 - 2a + b, \dots\}.$$

If $x_3 \le -\alpha$, then $x_4 = x_0 - 2a + 2b$. Since $b > a$ then

$$0 < x_2 = x_0 + b - a < x_4 = x_0 + 2(b - a).$$

Again two cases are possible: $\alpha \le x_4$ or $0 < x_4 < \alpha$. If $\alpha \le x_4$, then $x_5 = x_0 - 2a + b = x_3$ and we obtain an eventually periodic orbit with period 2:

$$O(x_0) = \{x_0, x_0 - a, x_0 - a + b, x_0 - 2a + b, x_0 - 2a + 2b,\\ x_0 - 2a + b, x_0 - 2a + 2b, \dots\}.$$

If $0 < x_4 < \alpha$, then $x_5 = x_0 - 3a + 2b < 0$ (note that $a > \alpha$). Two cases are possible: $-\alpha < x_5 < 0$ or $x_5 \le -\alpha$. If $-\alpha < x_5 < 0$, then we obtain an eventually periodic orbit with period 2:

$$O(x_0) = \{x_0, x_0 - a, x_0 - a + b, x_0 - 2a + b, x_0 - 2a + 2b, x_0 - 3a + 2b,\\ x_0 - 2a + 2b, x_0 - 3a + 2b, \dots\}.$$

If $x_5 \le -\alpha$, then we consider x_6 and sort two possibilities. We have observation: the orbit $O(x_0)$ is not eventually periodic orbit with period 2 only if

$$0 < x_2 = x_0 + b - a < x_4 = x_0 + 2(b - a) < x_6 = x_0 + 3(b - a) < \dots < \alpha \quad \text{and}\\ x_1 = x_0 - a < x_3 = x_0 - a + (b - a) < x_5 = x_0 - a + 2(b - a) < \dots \le -\alpha.$$

Since $b - a > 0$ there are $n_1, n_2 \in \mathbf{N}$ such that

$$x_{2n_1} = x_0 + n_1(b - a) \ge \alpha \quad \text{or} \quad -\alpha < x_{2n_2+1} = x_0 - a + n_2(b - a) < 0.$$

Therefore we obtain an eventually periodic orbit with period 2 in both cases:

$$O(x_0) = \bigl\{x_0, x_0 - a, x_0 - a + b, x_0 - 2a + b, \dots, x_0 - n_1 a + (n_1 - 1)b,\\ x_0 + n_1(b - a), x_0 - n_1 a + (n_1 - 1)b, x_0 + n_1(b - a), \dots\bigr\} \quad \text{or}\\ O(x_0) = \bigl\{x_0, x_0 - a, x_0 - a + b, x_0 - 2a + b, \dots, x_0 + n_2(b - a),\\ x_0 - a + n_2(b - a), x_0 + n_2(b - a), x_0 - a + n_2(b - a), \dots\bigr\}.$$

Case 2. Now we assume that $\alpha \le x_0$. Then $g(x_0) = b$ and $x_1 = x_0 - b$. Four cases are possible:

Case A. $x_1 = x_0 - b \le -\alpha$. In this case $x_2 = x_0 - b + b = x_0$ and we have a periodic orbit with period 2: $O(x_0) = \{x_0, x_0 - b, x_0, x_0 - b, \dots\}$.

Case B. $-\alpha < x_1 = x_0 - b < 0$. Since $b > a$, then $-\alpha < x_2 = x_0 - b + a < x_0$ and three situations are possible: $-\alpha < x_2 < 0$ or $0 < x_2 < \alpha$, or $\alpha \le x_2 < x_0$.

Case B1. Inequalities $-\alpha < x_2 < 0$ hold only if $b > \alpha > a$. Therefore $-\alpha < x_3 = x_0 - b + 2a < 0$ or $0 < x_3 = x_0 - b + 2a < \alpha$. If $0 < x_3 < \alpha$, then $x_4 = x_0 - b + a = x_2$ and we obtain an eventually periodic orbit with period 2:

$$O(x_0) = \{x_0, x_0 - b, x_0 - b + a, x_0 - b + 2a, x_0 - b + a, x_0 - b + 2a, \ldots\}.$$

If $-\alpha < x_3 < 0$, then $x_4 = x_0 - b + 3a$. Two cases are possible: $-\alpha < x_4 < 0$ or $0 < x_4 < \alpha$. If $0 < x_4 < \alpha$, then $x_5 = x_0 - b + 2a = x_3$ and again we obtain an eventually periodic orbit with period 2:

$$O(x_0) = \{x_0, x_0 - b, x_0 - b + a, x_0 - b + 2a, x_0 - b + 3a,\\ x_0 - b + 2a, x_0 - b + 3a, \ldots\}.$$

If $-\alpha < x_4 < 0$, then $x_5 = x_0 - b + 4a$ and we have two cases. Since $a > 0$ there is $k_0 \in \mathbf{N}$ such that

$$-\alpha < x_{k_0-1} < 0 \quad \text{and} \quad 0 < x_{k_0} = x_0 - b + k_0 a < \alpha.$$

Therefore we obtain an eventually periodic orbit with period 2:

$$O(x_0) = \big\{x_0, x_0 - b, x_0 - b + a, x_0 - b + 2a, \ldots, x_0 - b + (k_0 - 1)a,\\ x_0 - b + k_0 a, x_0 - b + (k_0 - 1)a, x_0 - b + k_0 a, \ldots\big\}.$$

Case B2. If $0 < x_2 < \alpha$, then $x_3 = x_0 - b = x_1$ and we obtain a periodic orbit with period 2: $O(x_0) = \{x_0, x_0 - b, x_0, x_0 - b, \ldots\}$.

Case B3. If $\alpha \le x_2 < x_0$ then we consider next element $x_3 = x_0 - 2b + a$. Since $b > a$ and $b > \alpha$ then two cases are possible: $x_3 \le -\alpha$ or $-\alpha < x_3 < 0$. If $x_3 \le -\alpha$, then $x_4 = x_0 - b + a = x_2$ and we obtain an eventually periodic orbit with period 2:

$$O(x_0) = \{x_0, x_0 - b, x_0 - b + a, x_0 - 2b + a, x_0 - b + a, x_0 - 2b + a, \ldots\}.$$

If $x_3 \le -\alpha$, then $x_4 = x_0 - 2b + 2a$ and we have two cases: $0 < x_4 < \alpha$ or $\alpha \le x_4$. If $0 < x_4 < \alpha$, then $x_5 = x_0 - 2b + a = x_3$ and we obtain an eventually periodic orbit with period 2:

$$O(x_0) = \{x_0, x_0 - b, x_0 - b + a, x_0 - 2b + a, x_0 - 2b + 2a,\\ x_0 - 2b + a, x_0 - 2b + 2a, \ldots\}.$$

If $\alpha \le x_4$, then $x_5 = x_0 - 3b + 2a$ and again we have two possibilities. The orbit $O(x_0)$ is not eventually periodic orbit with period 2 only if

$$0 > x_1 = x_0 - b > x_3 = x_0 - b + (a - b) > x_5 = x_0 - b + 2(a - b) > \cdots > -\alpha$$

and

$$x_0 > x_2 = x_0 - b + a > x_4 = x_0 - 2b + 2a > x_6 = x_0 - 3b - 3a > \cdots \ge \alpha.$$

Since $b-a>0$ there are $k_1, k_2 \in \mathbf{N}$ such that

$$x_{2k_1+1} = x_0 - b + k_1(a-b) \le -\alpha \quad \text{or} \quad x_{2k_2} = x_0 + k_2(a-b) < \alpha.$$

Therefore we obtain an eventually periodic orbit with period 2 in both cases

$$\begin{aligned} O(x_0) = \{&x_0, x_0-b, x_0-b+a, x_0-2b+a, \ldots, x_0+k_1(a-b), \\ &x_0-b+k_1(b-a), x_0+k_1(a-b), x_0-b+k_1(a-b), \ldots\} \quad \text{or} \\ O(x_0) = \{&x_0, x_0-b, x_0-b+a, x_0-2b+a, \ldots, x_0-b+k_2(a-b), \\ &x_0+k_2(a-b), x_0-b+k_2(a-b), x_0+k_2(a-b), \ldots\}. \end{aligned}$$

Case C. $0 < x_1 < \alpha$. This case can be reduced to the case that is considered at the beginning of the proof with $0 < x_0 < \alpha$ (Case 1).

Case D. $\alpha \le x_1$. In this case

$$\exists k_3 \in \mathbf{N} \quad x_{k_3-1} = x_0 - (k_3-1)b \ge \alpha \quad \text{and} \quad x_{k_3} = x_0 - k_3 b < \alpha.$$

Therefore $x_{k_3} \le -\alpha$ or $-\alpha < x_{k_3} < 0$, or $0 < x_{k_3} < \alpha$. These three situations are discussed above.

Case 3. Situations with fixed $x_0 \le -\alpha$ or $-\alpha < x_0 < 0$ are similar to the previously considered. $\square$

Theorem 2. *Assume that* $\beta = 1$. *Then*

1) 0 *is an unstable stationary state.*

2) *Periodic orbit* $\{x_0, x_0-a, x_0, x_0-a, \ldots\}$ *with* $0 < x_0 < \alpha$ *is stable if* $-\alpha < x_0 - a < 0$.

3) *Periodic orbit* $\{x_0, x_0+a, x_0, x_0+a, \ldots\}$ *with* $-\alpha < x_0 < 0$ *is stable if* $0 < x_0 + a < \alpha$.

4) *Periodic orbit* $\{x_0, x_0-b, x_0, x_0-b, \ldots\}$ *with* $\alpha < x_0$ *is stable if* $x_0 - b < -\alpha$.

5) *Periodic orbit* $\{x_0, x_0+b, x_0, x_0+b, \ldots\}$ *with* $x_0 < \alpha$ *is stable if* $\alpha < x_0+b$.

Proof. *Proof of* 1). We need to prove that there exists some $\varepsilon > 0$ such that for any $\delta > 0$ there is some $|x_0| < \delta$ satisfying $|x_n| \ge \varepsilon$ for some $n \ge 1$. We fix $\varepsilon = \frac{a}{2}$. For an arbitrary chosen $\delta > 0$ we fix x_0 such that

$$|x_0| < \delta \quad \text{and} \quad 0 < x_0 < \alpha \quad \text{and} \quad 0 < x_0 < \frac{a}{2}.$$

Then $x_1 = x_0 - a < \frac{a}{2} - a = -\frac{a}{2}$, that is, for $n = 1$ we have $|x_1| > \frac{a}{2} = \varepsilon$.

Proof of 2). We fix $0 < x_0 < \alpha$. For an arbitrary chosen $\varepsilon > 0$ we fix

$$\delta < \min\{\varepsilon, x_0, \alpha - x_0, a - x_0, x_0 - a + \alpha\}.$$

Then the orbit for every $x_0' \in [x_0 - \delta, x_0 + \delta]$ is periodic with period 2: $O(x_0') = \{x_0', x_0' - a, x_0', x_0' - a, \ldots\}$ and therefore

$$x_0' = x_{2n}', \qquad x_1' = x_0' - a = x_{2n+1}', \quad n = 1, 2, \ldots.$$

This leads to

$$|x_0' - x_0| = |x_{2n}' - x_0| < \delta < \varepsilon \quad \text{and} \quad |x_1' - x_1| = |x_{2n+1}' - x_1| < \delta < \varepsilon.$$

The proof of cases 3)–5) is similar. □

3.2 Model with $0 < \beta < 1$

The situation with $0 < \beta < 1$ is much more complicated. We begin with the study of the stability of periodic points.

Theorem 3. *Let us assume that* $0 < \beta < 1$. *Then*

1) 0 *is an unstable stationary state.*

2) *If* $0 < \frac{a}{\beta+1} < \alpha$, *then the periodic orbit* $O(\frac{a}{\beta+1})$ *is a stable periodic orbit with period* 2.

3) *If* $\alpha < \frac{b}{\beta+1}$, *then the periodic orbit* $O(\frac{b}{\beta+1})$ *is a stable periodic orbit with period* 2.

Proof. *Proof of* 1). The proof is similar as in Theorem 2 for Statement 1).
Proof of 2). Let $h(x) = \beta x - g(x)$. Since $0 < \frac{a}{\beta+1} < \alpha$ and $g(\frac{a}{\beta+1}) = a$ then

$$h\left(\frac{a}{\beta+1}\right) = \frac{\beta a}{\beta+1} - a = \frac{\beta a - \beta a - a}{\beta+1} = \frac{-a}{\beta+1}.$$

Since $-\alpha < \frac{-a}{\beta+1} < 0$ and $g(\frac{-a}{\beta+1}) = -a$ then

$$h\left(\frac{-a}{\beta+1}\right) = \frac{-\beta a}{\beta+1} + a = \frac{-\beta a + \beta a + a}{\beta+1} = \frac{a}{\beta+1}.$$

This implies that $O(\frac{a}{\beta+1}) = \{\frac{a}{\beta+1}, \frac{-a}{\beta+1}, \ldots\}$ is a periodic orbit with period 2.

Next we show that $O(\frac{a}{\beta+1})$ is stable. For an arbitrary chosen $\varepsilon > 0$ let

$$0 < \delta < \min\left\{\frac{a}{\beta+1}, \alpha - \frac{a}{\beta+1}, \varepsilon\right\} \leq \varepsilon.$$

If x_0 satisfies the inequality $|x_0 - \frac{a}{\beta+1}| < \delta$ then $x_0 \in\,]0, \alpha[$ and therefore $g(x_0) = a$. Then

$$\begin{aligned} x_1 = h(x_0) = \beta x_0 - a &= \beta x_0 - \frac{a(\beta+1)}{\beta+1} \\ &= \beta\left(x_0 - \frac{a}{\beta+1}\right) - \frac{a}{\beta+1} \in \left]\frac{-a}{\beta+1} - \beta\delta, \frac{-a}{\beta+1} + \beta\delta\right[. \end{aligned}$$

We conclude that $-\alpha < x_1 < 0$. Then $g(x_1) = -a$ and

$$\begin{aligned} x_2 = h(x_1) = h^2(x_0) &= \beta x_1 + a = \beta^2 x_0 - \beta a + a \\ &= \beta^2 x_0 + \frac{a(1-\beta)(1+\beta)}{1+\beta} = \beta^2 x_0 + \frac{a(1-\beta^2)}{\beta+1} \\ &= \beta^2\left(x_0 - \frac{a}{\beta+1}\right) + \frac{a}{\beta+1} \in \left]\frac{a}{\beta+1} - \beta^2\delta, \frac{a}{\beta+1} + \beta^2\delta\right[. \end{aligned}$$

We conclude that $0 < x_2 < \alpha$. Then $g(x_2) = a$ and

$$\begin{aligned}
x_3 = h(x_2) = h^3(x_0) &= \beta^3 x_0 - \beta^2 a + \beta a - a \\
&= \beta^3 x_0 - a(\beta^2 - \beta + 1) = \beta^3 x_0 - a\frac{\beta^3 + 1}{\beta + 1} \\
&= \beta^3\Big(x_0 - \frac{a}{\beta+1}\Big) - \frac{a}{\beta+1} \in \Big]\frac{-a}{\beta+1} - \beta^3\delta, \frac{-a}{\beta+1} + \beta^3\delta\Big[.
\end{aligned}$$

We conclude that $-\alpha < x_3 < 0$. Further we obtain that

$$\begin{aligned}
h^{2n}(x_0) = x_{2n} &= \beta^{2n}x_0 - \beta^{2n-1}a + \beta^{2n-2}a - \cdots + a = \beta^{2n}x_0 + a\frac{1-\beta^{2n}}{\beta+1} \\
&= \beta^{2n}\Big(x_0 - \frac{a}{\beta+1}\Big) + \frac{a}{\beta+1} \in \Big]\frac{a}{\beta+1} - \beta^{2n}\delta, \frac{a}{\beta+1} + \beta^{2n}\delta\Big[\\
&\Rightarrow \quad 0 < x_{2n} < \alpha, \quad n = 1, 2, \ldots, \\
h^{2n+1}(x_0) = x_{2n+1} &= \beta^{2n+1}x_0 - \beta^{2n}a + \cdots - a = \beta^{2n+1}x_0 - a\frac{\beta^{2n+1}+1}{\beta+1} \\
&= \beta^{2n+1}\Big(x_0 - \frac{a}{\beta+1}\Big) - \frac{a}{\beta+1} \in \Big]\frac{-a}{\beta+1} - \beta^{2n+1}\delta, \frac{-a}{\beta+1} + \beta^{2n+1}\delta\Big[\\
&\Rightarrow \quad -\alpha < x_{2n+1} < 0, \quad n = 0, 1, 2, \ldots.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Big|h^{2n}(x_0) - \frac{a}{\beta+1}\Big| &= \Big|\beta^{2n}\Big(x_0 - \frac{a}{\beta+1}\Big) + \frac{a}{\beta+1} - \frac{a}{\beta+1}\Big| < \beta^{2n}\delta < \delta < \varepsilon, \\
\Big|h^{2n+1}(x_0) + \frac{a}{\beta+1}\Big| &= \Big|\beta^{2n+1}\Big(x_0 - \frac{a}{\beta+1}\Big) - \frac{a}{\beta+1} + \frac{a}{\beta+1}\Big| \\
&< \beta^{2n+1}\delta < \delta < \varepsilon.
\end{aligned}$$

We obtain that $O(\frac{a}{\beta+1}) = \{\frac{a}{\beta+1}, \frac{-a}{\beta+1}, \ldots\}$ is a stable periodic orbit with period 2.

Proof of 3). Since

$$h\Big(\frac{b}{\beta+1}\Big) = \frac{\beta b}{\beta+1} - b = \frac{-b}{\beta+1} \quad \text{and} \quad h\Big(\frac{-b}{\beta+1}\Big) = \frac{-\beta b}{\beta+1} + b = \frac{b}{\beta+1},$$

then $O(\frac{b}{\beta+1}) = \{\frac{b}{\beta+1}, \frac{-b}{\beta+1}, \ldots\}$ is a periodic orbit with period 2.

Next we show that $O(\frac{b}{\beta+1})$ is stable. For an arbitrary chosen $\varepsilon > 0$ we fix

$$0 < \delta < \min\Big\{\frac{b}{\beta+1} - \alpha, \varepsilon\Big\}.$$

The δ is chosen such that $\alpha < \frac{b}{\beta+1} - \delta$, therefore an arbitrary fixed $x_0 \in \,]\frac{b}{\beta+1} - \delta, \frac{b}{\beta+1} + \delta[$ satisfies the inequality $\alpha < x_0$ and $g(x_0) = b$. Then

$$\begin{aligned}
x_1 = h(x_0) = \beta x_0 - b &= \beta x_0 - \frac{b(\beta+1)}{\beta+1} \\
&= \beta\Big(x_0 - \frac{b}{\beta+1}\Big) - \frac{b}{\beta+1} \in \Big]\frac{-b}{\beta+1} - \beta\delta, \frac{-b}{\beta+1} + \beta\delta\Big[\quad \Rightarrow \quad x_1 < -\alpha.
\end{aligned}$$

Then

$$x_2 = h^2(x_0) = \beta^2 x_0 - \beta b + b = \beta^2 x_0 + b(1-\beta) = \beta^2 x_0 + \frac{b(1-\beta^2)}{\beta+1}$$
$$= \beta^2\Big(x_0 - \frac{b}{\beta+1}\Big) + \frac{b}{\beta+1} \in \Big]\frac{b}{\beta+1} - \beta^2\delta, \frac{b}{\beta+1} + \beta^2\delta\Big[\quad \Rightarrow \quad \alpha < x_2.$$

Further we get

$$x_3 = h^3(x_0) = \beta^3 x_0 - \beta^2 b + \beta b - b = \beta^3 x_0 - b\frac{\beta^3+1}{\beta+1}$$
$$= \frac{-b}{\beta+1} + \beta^3\Big(x_0 - \frac{b}{\beta+1}\Big) \in \Big]\frac{-b}{\beta+1} - \beta^3\delta, \frac{-b}{\beta+1} + \beta^3\delta\Big[\quad \Rightarrow \quad x_3 < -\alpha.$$

We conclude that

$$h^{2n}(x_0) = x_{2n} = \beta^{2n} x_0 + b\frac{1-\beta^{2n}}{\beta+1} = \frac{b}{\beta+1} + \beta^{2n}\Big(x_0 - \frac{b}{\beta+1}\Big)$$
$$\in \Big]\frac{b}{\beta+1} - \beta^{2n}\delta, \frac{b}{\beta+1} + \beta^{2n}\delta\Big[\ \Rightarrow \ \alpha < x_{2n} \quad n = 1, 2, \dots,$$
$$h^{2n+1}(x_0) = x_{2n+1} = \beta^{2n+1} x_0 - b\frac{\beta^{2n+1}+1}{\beta+1} = -\frac{b}{\beta+1} + \beta^{2n+1}\Big(x_0 - \frac{b}{\beta+1}\Big)$$
$$\in \Big]\frac{-b}{\beta+1} - \beta^{2n+1}\delta, \frac{-b}{\beta+1} + \beta^{2n+1}\delta\Big[\ \Rightarrow \ x_{2n+1} < -\alpha, \quad n = 0, 1, 2, \dots.$$

Therefore

$$\Big|h^{2n}(x_0) - \frac{b}{\beta+1}\Big| < \beta^{2n}\delta < \delta < \varepsilon, \ \Big|h^{2n+1}(x_0) + \frac{b}{\beta+1}\Big| \ < \beta^{2n+1}\delta < \delta < \varepsilon.$$

The orbit $O(\frac{b}{\beta+1}) = \{\frac{b}{\beta+1}, \frac{-b}{\beta+1}, \dots\}$ is a stable periodic orbit with period 2. □

Corollary 1. If $\frac{a}{\beta+1} < \alpha < \frac{b}{\beta+1}$ then exists two stable periodic orbits with period 2.

Corollary 2. There exists at least one stable periodic orbit with period 2.

In case $0 < \beta < 1$ parameters a, b, α play a very important role. Depending on these parameters, all real numbers are divided into three sets: points whose orbits are asymptotically periodic with the limit set $L(x_0) = \{\frac{a}{\beta+1}, \frac{-a}{\beta+1}\}$, points whose orbits are asymptotically periodic with the limit set $L(x_0) = \{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\}$ and points whose orbits are an eventually stationary state 0.

Theorem 4. *Assume that* $0 < \beta < 1$.

1) *If* $0 < \frac{a}{\beta+1} < \alpha < \frac{b}{\beta+1}$, *then for every*

$$x_0 \in \Omega_b = \Big[\frac{\alpha-b}{\beta}, -\alpha\Big] \cup \bigcup_{k=0}^{\infty}\Big[\frac{\alpha - b\sum_{s=0}^{k+1}\beta^s}{\beta^{k+2}}, \frac{-\alpha - b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}}\Big]$$
$$\cup \Big[\alpha, \frac{b-\alpha}{\beta}\Big] \cup \bigcup_{k=0}^{\infty}\Big[\frac{\alpha + b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}}, \frac{-\alpha + b\sum_{s=0}^{k+1}\beta^s}{\beta^{k+2}}\Big]$$

the orbit $O(x_0)$ *is asymptotically periodic with the limit set* $L(x_0) = \{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\}$.

2) *If* $0 < \frac{a}{\beta+1} < \alpha < \frac{b}{\beta+1}$ *and* $\min\{\frac{a}{\beta+1}, \alpha - \frac{a}{\beta+1}\} = \frac{a}{\beta+1}$, *then for every*

$$x_0 \in \Omega_a = \bigcup_{k=0}^{\infty} \left] \frac{-\alpha - b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}}, \frac{\alpha - b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}} \right[\cup \left]-\alpha, 0\right[\cup \left]0, \alpha\right[\cup \bigcup_{k=0}^{\infty} \left] \frac{-\alpha + b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}}, \frac{\alpha + b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}} \right[\setminus \Omega_0$$

the orbit $O(x_0)$ *is asymptotically periodic with the limit set* $L(x_0) = \{\frac{a}{\beta+1}, \frac{-a}{\beta+1}\}$ *and*

$$\Omega_0 = \Bigl\{ -\frac{b}{\beta}, \frac{b}{\beta}, -\frac{b+\beta b}{\beta^2}, \frac{b+\beta b}{\beta^2}, \ldots, -\frac{b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}}, \frac{b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}}, \ldots \Bigm| k = 0, 1, 2, \ldots \Bigr\}$$

is a set of points whose orbits are eventually stationary state 0.

3) *If* $0 < \frac{a}{\beta+1} < \alpha < \frac{b}{\beta+1}$, $\min\{\frac{a}{\beta+1}, \alpha - \frac{a}{\beta+1}\} = \alpha - \frac{a}{\beta+1}$ *and* $\alpha < a \le \frac{b-\alpha}{\beta}$, *then*

(i) *for every* $x_0 \in [\frac{\alpha-a}{\beta}, 0[\cup]0, \frac{a-\alpha}{\beta}]$ *and in general for every* $x_0 \in \Omega_b \cup \Omega_{bb}$ *the orbit* $O(x_0)$ *is asymptotically periodic with the limit set* $L(x_0) = \{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\}$ *where*

$$\Omega_{bb} = \left[\frac{\alpha-a}{\beta}, 0\right[\cup \left]0, \frac{a-\alpha}{\beta}\right] \cup \bigcup_{k=1}^{\infty} \biggl(\left[\frac{\alpha - a + b\sum_{s=1}^{k}\beta^s}{\beta^{k+1}}, \frac{b\sum_{s=0}^{k-1}\beta^s}{\beta^k} \right[\cup \left] \frac{b\sum_{s=0}^{k-1}\beta^s}{\beta^k}, \frac{a - \alpha + b\sum_{s=1}^{k}\beta^s}{\beta^{k+1}} \right] \biggr) \cup \bigcup_{k=1}^{\infty} \biggl(\left[\frac{\alpha - a - b\sum_{s=1}^{k}\beta^s}{\beta^{k+1}}, \frac{-b\sum_{s=0}^{k-1}\beta^s}{\beta^k} \right[\cup \left] \frac{-b\sum_{s=0}^{k-1}\beta^s}{\beta^k}, \frac{a - \alpha - b\sum_{s=1}^{k}\beta^s}{\beta^{k+1}} \right] \biggr).$$

(ii) *for every* $x_0 \in]-\alpha, \frac{\alpha-a}{\beta}[\cup]\frac{a-\alpha}{\beta}, \alpha[$ *and in general for every* $x_0 \in (\Omega_a \setminus \Omega_{bb}) \setminus \Omega_0$ *the orbit* $O(x_0)$ *is asymptotically periodic with the limit set* $L(x_0) = \{\frac{a}{\beta+1}, \frac{-a}{\beta+1}\}$. Ω_0 *is a set of points whose orbits are eventually stationary state* 0.

Proof. *Proof of* 1). At first we show that for every $x_0 \in [\alpha, \frac{b-\alpha}{\beta}]$ the orbit $O(x_0)$ is asymptotically periodic with the limit set $\{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\}$.

If $\alpha \le x_0 \le \frac{b-\alpha}{\beta}$ then $x_1 = \beta x_0 - b$, that is

$$\beta\alpha - b \le x_1 \le \beta\frac{b-\alpha}{\beta} - b = -\alpha.$$

We notice that $\frac{\alpha-b}{\beta} < \beta\alpha - b$ holds if

$$0 < \beta^2\alpha - \alpha - \beta b + b = \alpha\big(\beta^2 - 1\big) + b(1-\beta).$$

Since $\alpha < \frac{b}{\beta+1}$ and $\beta^2 - 1 < 0$, then

$$\alpha\big(\beta^2 - 1\big) + b(1-\beta) > \frac{b}{\beta+1}\big(\beta^2-1\big) + b(1-\beta) = b(\beta-1) + b(1-\beta) = 0.$$

Since $\frac{\alpha-b}{\beta} < x_1 \le -\alpha$ then

$$\begin{aligned} x_2 &= \beta x_1 + b = \beta^2 x_0 - \beta b + b \\ &= \beta^2 x_0 + \frac{b(1-\beta^2)}{\beta+1} = \beta^2\Big(x_0 - \frac{b}{\beta+1}\Big) + \frac{b}{\beta+1}. \end{aligned}$$

But on the other hand

$$\alpha = \frac{\beta(\alpha-b)}{\beta} + b < x_2 = \beta x_1 + b \le \beta(-\alpha) + b < \frac{b-\alpha}{\beta}.$$

In general, we have

$$\begin{aligned} \alpha < x_{2n} &= \beta^{2n}\Big(x_0 - \frac{b}{\beta+1}\Big) + \frac{b}{\beta+1} < \frac{b-\alpha}{\beta}, \quad n = 1, 2, \ldots, \\ \frac{\alpha-b}{\beta} < x_{2n+1} &= \beta^{2n+1}\Big(x_0 - \frac{b}{\beta+1}\Big) - \frac{b}{\beta+1} \le -\alpha, \quad n = 0, 1, 2, \ldots. \end{aligned}$$

Therefore

$$\lim_{n\to+\infty} x_{2n} = \frac{b}{\beta+1} \quad \text{and} \quad \lim_{n\to+\infty} x_{2n+1} = \frac{-b}{\beta+1}.$$

This means that $L(x_0) = \{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\}$.

Proof for initial values $x_0 \in [\frac{\alpha-b}{\beta}, -\alpha]$ is similar. If inequality $\frac{b-\alpha}{\beta} < x_0$ holds and x_0 belongs to the interval

$$\Big[\frac{\alpha + b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}}, \frac{-\alpha + b\sum_{s=0}^{k+1}\beta^s}{\beta^{k+2}}\Big], \quad k \in \{0, 1, 2, \ldots\},$$

then $x_1 = \beta x_0 - b$:

$$\begin{aligned} &\frac{\beta(\alpha + b\sum_{s=0}^{k}\beta^s)}{\beta^{k+1}} - b = \frac{\alpha + b\sum_{s=0}^{k}\beta^s - \beta^k b}{\beta^k} = \frac{\alpha + b\sum_{s=0}^{k-1}\beta^s}{\beta^k} \le x_1 \\ &\le \frac{\beta(b\sum_{s=0}^{k+1}\beta^s - \alpha)}{\beta^{k+2}} - b = \frac{b\sum_{s=0}^{k+1}\beta^s - \alpha - \beta^{k+1}b}{\beta^{k+1}} = \frac{b\sum_{s=0}^{k}\beta^s - \alpha}{\beta^{k+1}}, \end{aligned}$$

that is, if x_0 belongs to the k-th interval, then x_1 belongs to the $(k-1)$-th interval, x_2 belongs to the $(k-2)$-th interval, $\ldots$, $x_k \in [\alpha, \frac{b-\alpha}{\beta}]$. Therefore orbit of x_0 is asymptotically periodic with the limit set $\{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\}$.

Similarly, orbits of initial values that satisfy the inequality $x_0 < \frac{\alpha-b}{\beta}$ and belong to intervals

$$\Big[\frac{\alpha - b\sum_{s=0}^{k+1}\beta^s}{\beta^{k+2}}, \frac{-\alpha - b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}}\Big], \quad k = 0, 1, 2, \ldots,$$

are asymptotically periodic with the limit set $\{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\}$.

Remark. The before mentioned intervals are not empty. We can make the following evaluations:

$$\alpha < \frac{b}{\beta+1} \quad\Leftrightarrow\quad \alpha(\beta+1) < b \quad\Leftrightarrow\quad \alpha < \frac{b-\alpha}{\beta} \quad\Leftrightarrow\quad b - \alpha - \alpha\beta > 0.$$

The length of the k-th ($k = 0, 1, 2, \ldots$) interval is

$$\frac{-\alpha + b\sum_{s=0}^{k+1}\beta^s}{\beta^{k+2}} - \frac{\alpha + b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}} = \frac{1}{\beta^{k+2}}\big(-\alpha + b + \beta b + \cdots + \beta^{k+1}b$$
$$-\alpha\beta - \beta b - \beta^2 b - \cdots - \beta^{k+1}b\big) = \frac{b-\alpha-\alpha\beta}{\beta^{k+2}} > 0.$$

Since $0 < \beta < 1$ then length of intervals tend to infinity as k tends to infinity. Spacings between intervals are $\frac{2\alpha}{\beta^{k+1}}$, $k = 0, 1, 2, \ldots$ that also go to infinity as k goes to infinity.

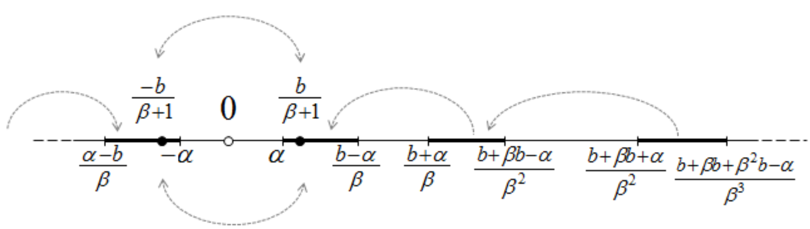

Figure 2. Illustration of Statement 1) of Theorem 4. Presentation of set Ω_b.

In Figure 2 some intervals of the set Ω_b are presented. The set Ω_b is symmetric with respect to 0. Points of bold intervals in Figure 2 belong to Ω_b and orbits of these points are asymptotically periodic with the limit set $L(x_0) = \{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\}$.

Proof of 2). Since $\min\{\frac{a}{\beta+1}, \alpha - \frac{a}{\beta+1}\} = \frac{a}{\beta+1}$ then $\frac{a}{\beta+1} \le \frac{\alpha}{2}$ (or $\frac{2a}{\beta+1} \le \alpha$). If $\frac{a}{\beta+1} \le \frac{\alpha}{2}$, then

$$a \le \frac{\alpha(\beta+1)}{2} < \frac{\alpha\cdot 2}{2} = \alpha.$$

Let $x_0 \in]0, \alpha[$. Then $x_1 = \beta x_0 - a$ and $-\alpha < -a < x_1 < \beta\alpha - a$. We can not guarantee that $\beta\alpha - a < 0$ but

$$\exists k \in \mathbf{N} \quad x_k = \beta^k x_0 - a\big(1 + \beta + \beta^2 + \cdots + \beta^{k-1}\big) < 0.$$

Then $-a < x_k < 0$ and therefore

$$\begin{aligned}
&0 < -\beta a + a < x_{k+1} = \beta x_k + a < a < \alpha, \\
&-\alpha < -a < -\beta^2 a + \beta a - a < x_{k+2} = \beta x_{k+1} - a < \beta a - a < 0.
\end{aligned}$$

In general, we have

$$\begin{aligned}
-a &< -\beta^{2n} a + \beta^{2n-1} a - \cdots - \beta^2 a + \beta a - a < x_{k+2n} \\
&< \beta^{2n-1} a - \beta^{2n-2} a + \cdots + \beta a - a < 0, \\
0 &< -\beta^{2n+1} a + \beta^{2n} a - \cdots + \beta^2 a - \beta a + a < x_{k+2n+1} \\
&< \beta^{2n} a - \beta^{2n-1} a + \cdots - \beta a + a < a.
\end{aligned}$$

Since

$$\begin{aligned}
&\lim_{n\to+\infty} (-a)\big(1 - \beta + \beta^2 - \cdots + \beta^{2n}\big) \\
&\quad = \lim_{n\to+\infty} (-a)\big(1 - \beta + \beta^2 - \beta^3 + \cdots - \beta^{2n-1}\big) = \frac{-a}{\beta+1}, \\
&\lim_{n\to+\infty} a\big(1 - \beta + \beta^2 - \cdots - \beta^{2n+1}\big) \\
&\quad = \lim_{n\to+\infty} a\big(1 - \beta + \beta^2 - \beta^3 + \cdots + \beta^{2n}\big) = \frac{a}{\beta+1},
\end{aligned}$$

then

$$\lim_{n\to+\infty} x_{k+2n} = \frac{-a}{\beta+1}, \qquad \lim_{n\to+\infty} x_{k+2n+1} = \frac{a}{\beta+1}.$$

This means that the orbit $O(x_0)$ is asymptotically periodic with the limit set $\{\frac{a}{\beta+1}, \frac{-a}{\beta+1}\}$.

Proof for $x_0 \in]-\alpha, 0[$ is similar. Similarly to the Statement 1) we can find intervals of points whose orbits are asymptotically periodic with the limit set $\{\frac{a}{\beta+1}, \frac{-a}{\beta+1}\}$. However, it should be noted that there is a set of points whose orbits are eventually stationary state 0, this set is

$$\begin{aligned}
\Omega_0 = \Big\{ &-\frac{b}{\beta}, \frac{b}{\beta}, -\frac{b+\beta b}{\beta^2}, \frac{b+\beta b}{\beta^2}, \ldots, -\frac{b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}}, \frac{b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}}, \ldots \;\Big| \\
&k = 0, 1, 2, \ldots \Big\}.
\end{aligned}$$

Set of points whose orbits are asymptotically periodic with the limit set $\{\frac{a}{\beta+1}, \frac{-a}{\beta+1}\}$ is

$$\begin{aligned}
x_0 \in \Omega_a = &\bigcup_{k=0}^{\infty} \left] \frac{-\alpha - b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}}, \frac{\alpha - b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}} \right[\cup \,]-\alpha, 0[\\
&\cup\,]0, \alpha[\,\cup \bigcup_{k=0}^{\infty} \left] \frac{-\alpha + b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}}, \frac{\alpha + b\sum_{s=0}^{k}\beta^s}{\beta^{k+1}} \right[\setminus \Omega_0.
\end{aligned}$$

Proof of 3). Since $\frac{a}{\beta+1} < \alpha \Leftrightarrow a < \alpha\beta + \alpha \Leftrightarrow \frac{a-\alpha}{\beta} < \alpha$ and $a > \alpha$ then in statements (i) and (ii) considered intervals are not empty.

Proof of (i). We consider points from the interval $]0, \frac{a-\alpha}{\beta}]$. Proof for points from interval $[\frac{\alpha-a}{\beta}, 0[$ is similar. If $0 < x_0 \le \frac{a-\alpha}{\beta}$, then

$$\frac{\alpha - b}{\beta} \le -a < x_1 = \beta x_0 - a \le \frac{\beta(a-\alpha)}{\beta} - a = -\alpha.$$

If $\frac{\alpha-a}{\beta} \le x_0 < 0$, then

$$\alpha = \frac{\beta(\alpha - a)}{\beta} + a \le x_1 = \beta x_0 + a < a \le \frac{b-\alpha}{\beta}.$$

Therefore $x_1 \in [\frac{\alpha-b}{\beta}, -\alpha]$ or $x_1 \in [\alpha, \frac{b-\alpha}{\beta}]$ and by Statement 2) of Theorem 4 point x_0 is asymptotically periodic with the limit set $\{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\}$.

Proof of (ii). Let $\frac{a-\alpha}{\beta} < x_0 < \alpha$. Then

$$-\alpha = \frac{\beta(a-\alpha)}{\beta} - a < x_1 = \beta x_0 - a < \beta\alpha - a < \frac{\alpha - a}{\beta}.$$

The last inequality is true by the following equivalences

$$\begin{aligned}
\beta\alpha - a < \frac{\alpha - a}{\beta} \quad &\Leftrightarrow \quad \beta^2 - \beta a < \alpha - a \\
&\Leftrightarrow \quad 0 < \alpha - a - \beta^2\alpha + \beta a = \alpha\big(1-\beta^2\big) + a(\beta - 1) \\
&\qquad\quad = \alpha(1-\beta)(1+\beta) + a(\beta-1) = (1-\beta)\big(\alpha(1+\beta) - a\big)
\end{aligned}$$

and the expression in second brackets is positive because $\frac{a}{\beta+1} < \alpha \Leftrightarrow a < \alpha(1+\beta)$. Now we can estimate other elements of the orbit $O(x_0)$:

$$\begin{gathered}
(a-\alpha)/\beta < -\beta\alpha + a < x_2 = \beta x_1 + a < \beta^2\alpha - \beta a + a < \alpha, \\
-\alpha < -\beta^2\alpha + \beta a - a < x_3 = \beta x_2 - a < \beta^3\alpha - \beta^2 a + \beta a - a < (\alpha - a)/\beta, \\
\cdots \\
(a-\alpha)/\beta < -\beta^{2n-1}\alpha + \beta^{2n-2}a - \cdots - \beta a + a < x_{2n} \\
< \beta^{2n}\alpha - \beta^{2n-1}a + \cdots - \beta a + a < \alpha, \\
-\alpha < -\beta^{2n}\alpha + \beta^{2n-1}a - \cdots + \beta a - a < x_{2n+1} \\
< \beta^{2n+1}\alpha - \beta^{2n}a + \cdots + \beta a - a < (\alpha - a)/\beta.
\end{gathered}$$

If we let n go to infinity then for the last two inequalities exist limits that are equal to $\frac{a}{\beta+1}$ and $\frac{-a}{\beta+1}$ respectively, then

$$\lim_{n\to+\infty} x_{2n} = \frac{a}{\beta+1} \quad \text{and} \quad \lim_{n\to+\infty} x_{2n+1} = \frac{-a}{\beta+1}.$$

This means that the orbit $O(x_0)$ is asymptotically periodic with the limit set $\{\frac{a}{\beta+1}, \frac{-a}{\beta+1}\}$.

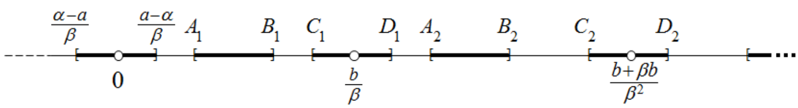

Figure 3. Illustration of Statement 3) of Theorem 4. Notations: $A_1 = \alpha$, $B_1 = \frac{b-\alpha}{\beta}$, $A_2 = \frac{b+\alpha}{\beta}$, $B_2 = \frac{b+\beta b-\alpha}{\beta^2}$.

In Statement 3) of Theorem 4 we need to reduce the set Ω_a in comparison with Statement 2) of Theorem 4. That is, interval $]\frac{b-\alpha}{\beta}, \frac{b+\alpha}{\beta}[=]B_1, A_2[$ (see Figure 3) contains a set of points that in the first iteration go to $[\frac{\alpha-a}{\beta}, 0[\cup]0, \frac{a-\alpha}{\beta}]$ (orbits of points of this interval are asymptotically periodic with the limit set $\{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\}$). This set is

$$[C_1, D_1] \setminus \left\{\frac{b}{\beta}\right\} = \left[\frac{\alpha - a + \beta b}{\beta^2}, \frac{b}{\beta}\right[\cup \left]\frac{b}{\beta}, \frac{a - \alpha + \beta b}{\beta^2}\right].$$

This is not empty set and not intersect with Ω_b: since $\frac{a}{\beta+1} < \alpha$ and $a > \alpha$ then

$$\frac{b-\alpha}{\beta} < \frac{\alpha - a + \beta b}{\beta^2} < \frac{b}{\beta} < \frac{a - \alpha + \beta b}{\beta^2} < \frac{b+\alpha}{\beta}.$$

Then we can find another set of points such first iteration belongs to $[C_1, D_1]$. This set is

$$[C_2, D_2] \setminus \left\{\frac{b+\beta b}{\beta^2}\right\} = \left[\frac{\alpha - a + \beta b + \beta^2 b}{\beta^3}, \frac{b+\beta b}{\beta^2}\right[\\ \cup \left]\frac{b+\beta b}{\beta^2}, \frac{a - \alpha + \beta b + \beta^2 b}{\beta^3}\right].$$

We conclude that the set

$$\Omega_{bb} = \left[\frac{\alpha - a}{\beta}, 0\right[\cup \left]0, \frac{a-\alpha}{\beta}\right] \cup \bigcup_{k=1}^{\infty}\left(\left[\frac{\alpha - a + b\sum_{s=1}^{k}\beta^s}{\beta^{k+1}}, \frac{b\sum_{s=0}^{k-1}\beta^s}{\beta^k}\right[\right. \\ \left.\cup \left]\frac{b\sum_{s=0}^{k-1}\beta^s}{\beta^k}, \frac{a - \alpha + b\sum_{s=1}^{k}\beta^s}{\beta^{k+1}}\right]\right) \\ \cup \bigcup_{k=1}^{\infty}\left(\left[\frac{\alpha - a - b\sum_{s=1}^{k}\beta^s}{\beta^{k+1}}, \frac{-b\sum_{s=0}^{k-1}\beta^s}{\beta^k}\right[\right. \\ \left.\cup \left]\frac{-b\sum_{s=0}^{k-1}\beta^s}{\beta^k}, \frac{a - \alpha - b\sum_{s=1}^{k}\beta^s}{\beta^{k+1}}\right]\right)$$

consists of points whose orbits are asymptotically periodic with the limit set $\{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\}$. This implies that all orbits of points in the set $\Omega_b \cup \Omega_{bb}$ are asymptotically periodic with the limit set $\{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\}$. Orbits of points in the set $(\Omega_a \setminus \Omega_{bb}) \setminus \Omega_0$ are asymptotically periodic with the limit set $\{\frac{a}{\beta+1}, \frac{-a}{\beta+1}\}$.
□

Unfortunately, we have to admit that here we cannot look at all possible situations with parameters. For example, one interesting situation arises when $\frac{a}{\beta+1} < \alpha = \frac{b}{\beta+1}$ holds. Experiments with initial values x_0 show (for example, $a = 9$, $b = 12$, $\alpha = 8$, $\beta = 0.5$) that in this case the set of points whose orbits are asymptotically periodic with the limit set $\{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\}$ are isolated points (in the mentioned example such points are $\{-2, 2, -8, 8, -20, 20, -40, 40, \ldots\}$).

4 Some Illustrative Examples and Conclusions

In this paper we have demonstrated that a step signal function with more than two steps make the analyze of a neuron model $x_{n+1} = \beta x_n - g(x_n)$, $0 < \beta \le 1$, very complicated. With our signal function (3.1) in this model we have found many periodic orbits and eventually periodic orbits with period 2 if $\beta = 1$. If $0 < \beta < 1$, then the situation is even more interesting. If $\frac{a}{\beta+1} < \alpha < \frac{b}{\beta+1}$, then we have two stable periodic orbits with period 2 $\{\frac{a}{\beta+1}, \frac{-a}{\beta+1}\}$ and $\{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\}$ and all other orbits are asymptotically periodic with the limits sets $\{\frac{a}{\beta+1}, \frac{-a}{\beta+1}\}$ or $\{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\}$ or eventually stationary state 0. One interesting case is considered in the following example.

Example 1. If $\beta = 0.5$, $\alpha = 8$, $a = 9$, $b = 15$, then conditions of Statement 3) of Theorem 4 are satisfied:

$$\min\left\{\frac{a}{\beta+1}, \alpha - \frac{a}{\beta+1}\right\} = \min\left\{\frac{9}{1.5}, 8 - \frac{9}{1.5}\right\} = \min\{6, 2\} = 2 = \alpha - \frac{a}{\beta+1},$$

$$\frac{a}{\beta+1} < \alpha < \frac{b}{\beta+1}, \quad \text{that is,} \quad \frac{9}{1.5} < 8 < \frac{15}{1.5} \quad \Leftrightarrow \quad 6 < 8 < 10,$$

$$\alpha < a \le \frac{b-\alpha}{\beta}: \quad 8 < 9 \le \frac{15-8}{0.5} = 14.$$

In this case a real line is divided in to infinity many intervals. Orbits of points of these intervals are asymptotically periodic with the limit set $\{\frac{a}{\beta+1}, \frac{-a}{\beta+1}\} = \{6, -6\}$ or $\{\frac{b}{\beta+1}, \frac{-b}{\beta+1}\} = \{10, -10\}$.

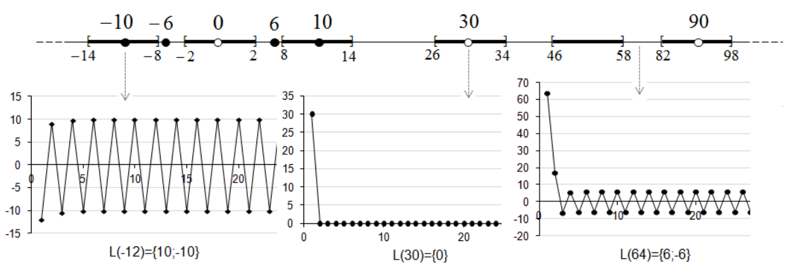

Figure 4. Illustration if $x_0 \in \{-12; 30; 64\}$.

In Figure 4 orbits of bolded intervals are asymptotically periodic with the limit set $\{10, -10\}$, orbits of thin intervals are asymptotically periodic with the limit set $\{6, -6\}$ and orbits of "empty" points are eventually stationary state 0.

We conclude that model (1.3) with our new signal function (3.1) describes more general situation as considered in [10] (also [2, 7, 9, 11, 12]). The model with our signal function shows that it is difficult to simulate even a single neuron behaviour. The reader interested in the other aspects of the neuron simulations is recommend to [1, 3, 6].

Acknowledgment

This work was partially supported by the University of Latvia under ESF Project. No. 2009 /0223 /1DP /1.1.1.2.0. /09 /APIA / VIAA /008, the European Social Fund within the project "Support for Doctoral Studies at University of Latvia" and Latvian Council of Science research project 09.1220.

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