

A Joint Limit Theorem for Periodic Hurwitz Zeta-Functions with Algebraic Irrational Parameters

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Abstract. In the paper, a joint limit theorem for weakly convergent probability measures in \mathbb{C}^r for periodic Hurwitz zeta-functions with algebraic irrational parameters satisfying certain independence conditions is obtained.

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1 Introduction

Let $\mathbf{a} = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with minimal period $k \in \mathbb{N}$, and let α , $0 < \alpha \leq 1$, be a fixed parameter. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathbf{a})$, $s = \sigma + it$, is defined, for $\sigma > 1$, by the series

$$\zeta(s, \alpha; \mathbf{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s},$$

and is meromorphically continued to the whole complex plane by using the equality

$$\zeta(s, \alpha; \mathbf{a}) = \frac{1}{k^s} \sum_{l=0}^{k-1} a_l \zeta\left(s, \frac{\alpha + l}{k}\right),$$

where $\zeta(s, \alpha)$ denotes the classical Hurwitz zeta-function. The point $s = 1$ is the unique possible simple pole of $\zeta(s, \alpha; \mathbf{a})$.

In [6], the second author began to characterize the asymptotic behaviour of the function $\zeta(s, \alpha; \mathbf{a})$ by limit theorems on the weak convergence of probability

measures. We discussed the cases of transcendental, rational and algebraic irrational parameter α . The simplest of them is the case of transcendental α because of the linear independence over the field of rational numbers \mathbb{Q} of the set $L(\alpha) = \{\log(m + \alpha) : m \in \mathbb{N}_0\}$. The case of rational α is based on the linear independence over \mathbb{Q} of the set $\{\log p : p \in \mathcal{P}\}$, where \mathcal{P} denotes the set of all prime numbers. The most complicated case is that of algebraic irrational α . In this case, there is no precise information on the linear independence of the set $L(\alpha)$. We know only a very deep theorem of Cassels which asserts that at least 51 percent of elements of the set $L(\alpha)$ in the sense of density are linear independent over \mathbb{Q} . The latter theorem allows to prove a limit theorem for weakly convergent probability measures with explicitly given limit measure.

The paper [7] is devoted to joint limit theorems for periodic Hurwitz zeta-functions. Let $\zeta(s, \alpha_1; \mathbf{a}_1), \dots, \zeta(s, \alpha_r; \mathbf{a}_r)$ be a collection of periodic Hurwitz zeta-functions. In [7], two cases of the parameters $\alpha_1, \dots, \alpha_r$ were discussed. The first case is of algebraically independent $\alpha_1, \dots, \alpha_r$. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ be the unit circle on the complex plane, and $\Omega_1 = \prod_{m=0}^{\infty} \gamma_m$, where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. Then Ω_1 is a compact topological group. Define

$$\underline{\Omega}_1 = \prod_{j=1}^r \Omega_{1j},$$

where $\Omega_{1j} = \Omega_1$ for $j = 1, \dots, r$. Then $\underline{\Omega}_1$ is also a compact topological group. Therefore, on $(\underline{\Omega}_1, \mathcal{B}(\underline{\Omega}_1))$ ($\mathcal{B}(S)$ denotes the class of Borel sets of the space S) the probability Haar measure \underline{m}_{1H} exists, and we obtain the probability space $(\underline{\Omega}_1, \mathcal{B}(\underline{\Omega}_1), \underline{m}_{1H})$. Denote by $\underline{\omega}_1 = (\omega_{11}, \dots, \omega_{1r})$ the elements of the group $\underline{\Omega}_1$, and, on the probability space $(\underline{\Omega}_1, \mathcal{B}(\underline{\Omega}_1), \underline{m}_{1H})$, define \mathbb{C}^r -valued random element $\underline{\zeta}_1(\underline{\sigma}, \underline{\alpha}, \underline{\omega}_1; \underline{\mathbf{a}})$ by the formula

$$\underline{\zeta}_1(\underline{\sigma}, \underline{\alpha}, \underline{\omega}_1; \underline{\mathbf{a}}) = (\zeta_1(\sigma_1, \alpha_1, \omega_{11}; \mathbf{a}_1), \dots, \zeta_1(\sigma_r, \alpha_r, \omega_{1r}; \mathbf{a}_r)),$$

where $\underline{\sigma} = (\sigma_1, \dots, \sigma_r)$, $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ and $\underline{\mathbf{a}} = (\mathbf{a}_1, \dots, \mathbf{a}_r)$, and

$$\zeta_1(\sigma_j, \alpha_j, \omega_{1j}; \mathbf{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj} \omega_{1j}(m)}{(m + \alpha_j)^{\sigma_j}}, \quad \sigma_j > \frac{1}{2}, \quad j = 1, \dots, r.$$

Here $\mathbf{a}_j = \{a_{mj} : m \in \mathbb{N}_0\}$, $j = 1, \dots, r$, are periodic sequences of complex numbers, and $\omega_{1j}(m)$ denotes the projection of $\omega_{1j} \in \Omega_{1j}$ to the coordinate space γ_m . Let $P_{\underline{\zeta}_1}$ be the distribution of the random element $\underline{\zeta}_1(\underline{\sigma}, \underline{\alpha}, \underline{\omega}_1; \underline{\mathbf{a}})$ and $\zeta(\underline{\sigma} + it, \underline{\alpha}; \underline{\mathbf{a}}) = (\zeta(\sigma_1 + it_1, \alpha_1; \mathbf{a}_1), \dots, \zeta(\sigma_r + it_r, \alpha_r; \mathbf{a}_r))$. Then the first theorem of [7] is the following statement.

Theorem 1. *Suppose that $\min_{1 \leq j \leq r} \sigma_j > \frac{1}{2}$, and that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over \mathbb{Q} . Then*

$$\frac{1}{T} \text{meas}\{t \in [0, T] : \underline{\zeta}(\underline{\sigma} + it, \underline{\alpha}; \underline{\mathbf{a}}) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}^r),$$

converges weakly to $P_{\underline{\zeta}_1}$ as $T \rightarrow \infty$.

Here and in the sequel, $\text{meas}\{A\}$ stands for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. The second joint limit theorem of [7] deals with rational parameters $\alpha_1, \dots, \alpha_r$. In this case, we use $\Omega_2 = \prod_{p \in \mathbb{P}} \gamma_p$, where $\gamma_p = \gamma$ for all primes p . The torus Ω_2 is also a compact topological group, and we obtain a new probability space $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$, where m_{2H} is the probability Haar measure on $(\Omega_2, \mathcal{B}(\Omega_2))$. Denote by $\omega_2(p)$ the projection of $\omega_2 \in \Omega_2$ to the coordinate space $\gamma_p, p \in \mathbb{P}$, and extend the function $\omega_2(p)$ to the set \mathbb{N} by the formula

$$\omega_2(m) = \prod_{p^l \parallel m} \omega_2^l(p), \quad m \in \mathbb{N}.$$

Suppose that $\alpha_j = a_j/q_j, 0 < a_j < q_j, (a_j, q_j) = 1, j = 1, \dots, r$, and, on the probability space $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$, define the \mathbb{C}^r -valued random element $\zeta_2(\underline{\sigma}, \underline{\alpha}, \omega_2; \underline{\mathbf{a}})$ by the formula

$$\zeta_2(\underline{\sigma}, \underline{\alpha}, \omega_2; \underline{\mathbf{a}}) = (\zeta_2(\sigma_1, \alpha_1, \omega_2; \mathbf{a}_1), \dots, \zeta_2(\sigma_r, \alpha_r, \omega_2; \mathbf{a}_r)),$$

where, for $\sigma_j > \frac{1}{2}$,

$$\zeta_2(\sigma_j, \alpha_j, \omega_2; \mathbf{a}_j) = \omega_2(q_j) q_j^{\sigma_j} \sum_{\substack{m=1 \\ m \equiv \mathbf{a}_j \pmod{q_j}}}^{\infty} \frac{a_{(m-a_j)/q_j} \omega_2(m)}{m^{\sigma_j}}, \quad j = 1, \dots, r.$$

Let P_{ζ_2} be the distribution of the random element $\zeta_2(\underline{\sigma}, \underline{\alpha}, \omega_2; \underline{\mathbf{a}})$. Then the second theorem of [7] is of the form.

Theorem 2. For $j = 1, \dots, r$, suppose that $\alpha_j = \frac{a_j}{q_j}, 0 < a_j < q_j, (a_j, q_j) = 1$, and that $\sigma_j > \frac{1}{2}$. Then

$$\frac{1}{T} \text{meas}\{t \in [0, T]: \zeta(\underline{\sigma} + it, \underline{\alpha}; \underline{\mathbf{a}}) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}^r),$$

converges weakly to P_{ζ_2} as $T \rightarrow \infty$.

The aim of this paper is to obtain a joint limit theorem for periodic Hurwitz zeta-functions with algebraic irrational parameters. The motivation for this is a possible application of limit theorems in the investigation of the universality for periodic Hurwitz zeta-functions.

For $j = 1, \dots, r$, let $I(\alpha_j)$ be a maximal linearly independent over \mathbb{Q} subset of the set $L(\alpha_j)$. Suppose that $L(\alpha_j) \neq I(\alpha_j)$, and define $D(\alpha_j) = L(\alpha_j) \setminus I(\alpha_j)$. If $d_{jm} = \log(m + \alpha_j) \in D(\alpha_j)$, then the set $I(\alpha_j) \cup \{d_{jm}\}$ is already linearly dependent over \mathbb{Q} . Thus, there exist elements $i_{jm_1}, \dots, i_{jm_n} \in I(\alpha_j)$ and $k_{j0}, \dots, k_{jn} \in \mathbb{Z} \setminus \{0\}$ such that

$$d_{jm} = -\frac{k_{j1}}{k_{j0}} i_{jm_1} - \dots - \frac{k_{jn}}{k_{j0}} i_{jm_n}.$$

From this we find that

$$m + \alpha_j = (m_1 + \alpha_j)^{-\frac{k_{j1}}{k_{j0}}} \dots (m_n + \alpha_j)^{-\frac{k_{jn}}{k_{j0}}}. \tag{1.1}$$

Define the sets

$$\mathcal{M}(\alpha_j) = \{m \in \mathbb{N}_0 : \log(m + \alpha_j) \in I(\alpha_j)\}$$

and

$$\mathcal{N}(\alpha_j) = \{m \in \mathbb{N}_0 : \log(m + \alpha_j) \in D(\alpha_j)\}.$$

Now let

$$\Omega_{3j} = \prod_{m \in \mathcal{M}(\alpha_j)} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathcal{M}(\alpha_j)$, and $\underline{\Omega}_3 = \prod_{j=1}^r \Omega_{3j}$. Then $\underline{\Omega}_3$ is a compact topological Abelian group. Therefore, on $(\underline{\Omega}_3, \mathcal{B}(\underline{\Omega}_3))$, the probability Haar measure \underline{m}_{3H} can be defined, and we obtain the probability space $(\underline{\Omega}_3, \mathcal{B}(\underline{\Omega}_3), \underline{m}_{3H})$. Denote by $\omega_{3j}(m)$ the projection of $\omega_{3j} \in \Omega_{3j}$ to γ_m , $m \in \mathcal{M}(\alpha_j)$, and extend the function $\omega_{3j}(m)$ to the set \mathbb{N}_0 by the formula

$$\omega_{3j}(m) = \omega_{3j}(m_1)^{-\frac{k_{j1}}{k_{j0}}} \cdots \omega_{3j}(m_n)^{-\frac{k_{jn}}{k_{j0}}}, \quad m \in \mathcal{N}(\alpha_j),$$

if equality (1.1) holds. Here the principal values of roots are taken. Denote by $\underline{\omega}_3 = (\omega_{31}, \dots, \omega_{3r})$ the elements of the group $\underline{\Omega}_3$, and, on the probability space $(\underline{\Omega}_3, \mathcal{B}(\underline{\Omega}_3), \underline{m}_{3H})$, define the \mathbb{C}^r -valued random element $\underline{\zeta}_3(\underline{\sigma}, \underline{\alpha}, \underline{\omega}_3; \underline{\mathbf{a}})$ by the formula

$$\underline{\zeta}_3(\underline{\sigma}, \underline{\alpha}, \underline{\omega}_3; \underline{\mathbf{a}}) = (\zeta_3(\sigma_1, \alpha_1, \omega_{31}; \mathbf{a}_1), \dots, \zeta_3(\sigma_r, \alpha_r, \omega_{3r}; \mathbf{a}_r)),$$

where, for $\sigma_j > \frac{1}{2}$,

$$\zeta_3(\sigma_j, \alpha_j, \omega_{3j}; \mathbf{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj} \omega_{3j}(m)}{(m + \alpha_j)^{\sigma_j}}, \quad j = 1, \dots, r.$$

Denote by $P_{\underline{\zeta}_3}$ the distribution of the random element $\underline{\zeta}_3(\underline{\sigma}, \underline{\alpha}, \underline{\omega}_3; \underline{\mathbf{a}})$. The main result of the paper is the following theorem.

Theorem 3. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraic irrational, the set $\bigcup_{j=1}^r I(\alpha_j)$ is linearly independent over \mathbb{Q} , and that $\min_{1 \leq j \leq r} \sigma_j > \frac{1}{2}$. Then*

$$P_T(A) \stackrel{\text{def}}{=} \frac{1}{T} \text{meas}\{t \in [0, T] : \underline{\zeta}(\underline{\sigma} + it, \underline{\alpha}; \underline{\mathbf{a}}) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}^r),$$

converges weakly to $P_{\underline{\zeta}_3}$ as $T \rightarrow \infty$.

Theorem 3 is the first attempt to obtain probabilistic limit theorems used in proofs of universality theorems for zeta-functions. On the other hand, Theorem 3 characterizes the asymptotic behaviour of a collection of periodic Hurwitz zeta-functions with algebraic irrational parameters. This is a motivation of the paper.

2 A Limit Theorem on the Torus $\underline{\Omega}_3$

In this section, we consider the weak convergence of

$$Q_T(A) = \frac{1}{T} \text{meas}\{t \in [0, T]: (((m + \alpha_1)^{-it}: m \in \mathcal{M}(\alpha_1)), \dots, (m + \alpha_r)^{-it}: m \in \mathcal{M}(\alpha_r))) \in A\}, \quad A \in \mathcal{B}(\underline{\Omega}_3).$$

Theorem 4. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ satisfy the hypotheses of Theorem 3. Then Q_T converges weakly to the Haar measure \underline{m}_{3H} as $T \rightarrow \infty$.*

Proof. The dual group of $\underline{\Omega}_3$ is isomorphic to

$$\mathcal{D} \stackrel{\text{def}}{=} \left(\bigoplus_{m \in \mathcal{M}(\alpha_1)} \mathbb{Z}_m \right) \oplus \dots \oplus \left(\bigoplus_{m \in \mathcal{M}(\alpha_r)} \mathbb{Z}_m \right),$$

where $\mathbb{Z}_m = \mathbb{Z}$ for all $m \in \mathcal{M}(\alpha_j)$, $j = 1, \dots, r$. An element $\underline{k} = (k_{1m}: m \in \mathcal{M}(\alpha_1), \dots, k_{rm}: m \in \mathcal{M}(\alpha_r))$ of \mathcal{D} , where only a finite number of integers k_{jm} , $j = 1, \dots, r$, are distinct from zero, acts on $\underline{\Omega}_3$ by

$$\underline{x} \rightarrow \underline{x}^{\underline{k}} = \prod_{m \in \mathcal{M}(\alpha_1)} x_{1m}^{k_{1m}} \dots \prod_{m \in \mathcal{M}(\alpha_r)} x_{rm}^{k_{rm}},$$

where $\underline{x} = ((x_{1m}: m \in \mathcal{M}(\alpha_1)), \dots, (x_{rm}: m \in \mathcal{M}(\alpha_r))) \in \underline{\Omega}_3$. Therefore, the Fourier transform $g_T(\underline{k})$ of the measure Q_T is of the form

$$\begin{aligned} g_T(\underline{k}) &= \int_{\underline{\Omega}_3} \left(\prod_{m \in \mathcal{M}(\alpha_1)} x_{1m}^{k_{1m}} \dots \prod_{m \in \mathcal{M}(\alpha_r)} x_{rm}^{k_{rm}} \right) dQ_T \\ &= \frac{1}{T} \int_0^T \left(\prod_{m \in \mathcal{M}(\alpha_1)} (m + \alpha_1)^{-itk_{1m}} \dots \prod_{m \in \mathcal{M}(\alpha_r)} (m + \alpha_r)^{-itk_{rm}} \right) dt \\ &= \frac{1}{T} \int_0^T \exp \left\{ -it \left(\sum_{m \in \mathcal{M}(\alpha_1)} k_{1m} \log(m + \alpha_1) + \dots \right. \right. \\ &\quad \left. \left. + \sum_{m \in \mathcal{M}(\alpha_r)} k_{rm} \log(m + \alpha_r) \right) \right\} dt, \end{aligned} \tag{2.1}$$

where only a finite number of integers k_{jm} , $j = 1, \dots, r$, are distinct from zero. Since the set $\bigcup_{j=1}^r I(\alpha_j)$ is linearly independent over \mathbb{Q} , we have that

$$l(\underline{k}) \stackrel{\text{def}}{=} \sum_{m \in \mathcal{M}(\alpha_1)} k_{1m} \log(m + \alpha_1) + \dots + \sum_{m \in \mathcal{M}(\alpha_r)} k_{rm} \log(m + \alpha_r) = 0$$

if and only if $\underline{k} = \underline{0}$. Therefore, after integration in (2.1), we find that

$$g_T(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ \frac{\exp\{-iTl(\underline{k})\} - 1}{-iTl(\underline{k})} & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

Thus,

$$\lim_{T \rightarrow \infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}, \end{cases}$$

and it follows from a continuity theorem for probability measures on compact groups (see, for example, Theorem 1.4.2 from [3]) that the measure Q_T converges weakly to \underline{m}_{3H} as $T \rightarrow \infty$. \square

3 Limit Theorems for Absolutely Convergent Series

For fixed $\hat{\sigma} > \frac{1}{2}$, and $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, let

$$v(m, n, \alpha_j) = \exp\left\{-\left(\frac{m + \alpha_j}{n + \alpha_j}\right)^{\hat{\sigma}}\right\}, \quad j = 1, \dots, r.$$

Define

$$\zeta_n(s, \alpha_j; \mathbf{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj} v(m, n, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \dots, r.$$

Then the series for $\zeta_n(s, \alpha_j; \mathbf{a}_j)$ converges absolutely for $\sigma > \frac{1}{2}$ independently on the arithmetical nature of α_j [4]. For $A \in \mathcal{B}(\mathbb{C}^r)$, we set

$$P_{T,n}(A) = \frac{1}{T} \text{meas}\{t \in [0, T]: \underline{\zeta}_n(\underline{\sigma} + it, \underline{\alpha}; \underline{\mathbf{a}}) \in A\},$$

where $\underline{\zeta}_n(\underline{\sigma} + it, \underline{\alpha}; \underline{\mathbf{a}}) = (\zeta_n(\sigma_1 + it, \alpha_1; \mathbf{a}_1), \dots, \zeta_n(\sigma_r + it, \alpha_r; \mathbf{a}_r))$. Moreover, for $\underline{\omega}_3 = (\omega_{31}, \dots, \omega_{3r}) \in \underline{\Omega}_3$, let

$$\zeta_n(s, \alpha_j, \omega_{3j}; \mathbf{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj} \omega_{3j}(m) v(m, n, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \dots, r.$$

Obviously, the series for $\zeta_n(s, \alpha_j, \omega_{3j}; \mathbf{a}_j)$ also converges absolutely for $\sigma > \frac{1}{2}$. Let $\underline{\zeta}_n(\underline{\sigma} + it, \underline{\alpha}, \underline{\omega}_3; \underline{\mathbf{a}}) = (\zeta_n(\sigma_1 + it, \alpha_1, \omega_{31}; \mathbf{a}_1), \dots, \zeta_n(\sigma_r + it, \alpha_r, \omega_{3r}; \mathbf{a}_r))$, and, for $A \in \mathcal{B}(\mathbb{C}^r)$ and a fixed $\hat{\omega}_3 = (\hat{\omega}_{31}, \dots, \hat{\omega}_{3r})$,

$$\hat{P}_{T,n}(A) = \frac{1}{T} \text{meas}\{t \in [0, T]: \underline{\zeta}_n(\underline{\sigma} + it, \underline{\alpha}, \hat{\omega}_3; \underline{\mathbf{a}}) \in A\}.$$

Theorem 5. *Suppose that $\min_{1 \leq j \leq r} \sigma_j > \frac{1}{2}$. Then, on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$, there exists a probability measure P_n such that the measures $P_{T,n}$ and $\hat{P}_{T,n}$ both converge weakly to P_n as $T \rightarrow \infty$.*

Proof. Define a function $h_n : \underline{\Omega}_3 \rightarrow \mathbb{C}^r$ by the formula

$$h_n(\underline{\omega}_3) = \left(\sum_{m=0}^{\infty} \frac{a_{m1} \omega_{31}(m) v(m, n, \alpha_1)}{(m + \alpha_1)^{\sigma_1}}, \dots, \sum_{m=0}^{\infty} \frac{a_{mr} \omega_{3r}(m) v(m, n, \alpha_r)}{(m + \alpha_r)^{\sigma_r}} \right).$$

Since the series in the definition of h_n converge absolutely, the function h_n is continuous, moreover,

$$\begin{aligned} h_n(\left((m + \alpha_1)^{-it} : m \in \mathcal{M}(\alpha_1) \right), \dots, \left((m + \alpha_r)^{-it} : m \in \mathcal{M}(\alpha_r) \right)) \\ = (\zeta_n(\sigma_1 + it, \alpha_1; \mathbf{a}_1), \dots, \zeta_n(\sigma_r + it, \alpha_r; \mathbf{a}_r)) = \underline{\zeta}_n(\underline{\sigma} + it, \underline{\alpha}; \underline{\mathbf{a}}). \end{aligned}$$

Hence, we have that

$$P_{T,n}(A) = Q_T h_n^{-1}(A), \quad A \in \mathcal{B}(\mathbb{C}^r).$$

Therefore, Theorem 4 together with Theorem 5.1 from [1] show that the measure $P_{T,n}$ converges weakly to $\underline{m}_{3H} h_n^{-1}$ as $T \rightarrow \infty$.

It remains to prove that the measure $\hat{P}_{T,n}$ also converges weakly to $\underline{m}_{3H} h_n^{-1}$ as $T \rightarrow \infty$. Let a function $h : \underline{\Omega}_3 \rightarrow \underline{\Omega}_3$ be given by the formula $h(\underline{\omega}_3) = \underline{\omega}_3 \hat{\omega}_3$. Then we have that

$$\begin{aligned} & h_n(h(\underbrace{((m + \alpha_1)^{-it} : m \in \mathcal{M}(\alpha_1)), \dots, ((m + \alpha_r)^{-it} : m \in \mathcal{M}(\alpha_r))}_{\mathbf{a}}))) \\ &= (\zeta_n(\sigma_1 + it, \alpha_1, \omega_{31}; \mathbf{a}_1), \dots, \zeta_n(\sigma_r + it, \alpha_r, \omega_{3r}; \mathbf{a}_r)) \\ &= \zeta_n(\underline{\sigma} + it, \underline{\alpha}, \hat{\omega}_3; \mathbf{a}). \end{aligned}$$

Thus, the above arguments show that the measure $\hat{P}_{T,n}$ converges weakly to $\underline{m}_{3H}(h_n h)^{-1}$ as $T \rightarrow \infty$. However, the invariance of the Haar measure \underline{m}_{3H} implies the equality $\underline{m}_{3H}(h_n h)^{-1} = (\underline{m}_{3H} h^{-1})h_n^{-1} = \underline{m}_{3H} h_n^{-1}$. \square

4 Approximation in the Mean

In this section, we approximate $\zeta(\underline{\sigma} + it, \underline{\alpha}; \mathbf{a})$ by $\zeta_n(\underline{\sigma} + it, \underline{\alpha}; \mathbf{a})$, and $\zeta(\underline{\sigma} + it, \underline{\alpha}, \underline{\omega}_3; \mathbf{a})$ by $\zeta_n(\underline{\sigma} + it, \underline{\alpha}, \underline{\omega}_3; \mathbf{a})$. We use the Euclidean metric ρ in \mathbb{C}^r . Let $\underline{z}_j = (z_{j1}, \dots, z_{jr}) \in \mathbb{C}^r$, $j = 1, 2$, and

$$\rho(\underline{z}_1, \underline{z}_2) = \left(\sum_{j=1}^r |z_{1j} - z_{2j}|^2 \right)^{\frac{1}{2}}.$$

Lemma 1. *Suppose that $\min_{1 \leq j \leq r} \sigma_j > \frac{1}{2}$. Then*

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \rho(\zeta(\underline{\sigma} + it, \underline{\alpha}; \mathbf{a}), \zeta_n(\underline{\sigma} + it, \underline{\alpha}; \mathbf{a})) dt = 0.$$

Proof. By Lemma 6 from [6], we have that, for every $j = 1, \dots, r$,

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\zeta(\sigma_j + it, \alpha_j; \mathbf{a}_j) - \zeta_n(\sigma_j + it, \alpha_j; \mathbf{a}_j)| dt = 0.$$

Since

$$\rho(\underline{z}_1, \underline{z}_2) \leq \sum_{j=1}^r |z_{1j} - z_{2j}|, \tag{4.1}$$

this proves the lemma. \square

Lemma 2. *Suppose that $\min_{1 \leq j \leq r} \sigma_j > \frac{1}{2}$. Then, for almost all $\underline{\omega}_3 \in \underline{\Omega}_3$,*

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \rho(\zeta(\underline{\sigma} + it, \underline{\alpha}, \underline{\omega}_3; \mathbf{a}), \zeta_n(\underline{\sigma} + it, \underline{\alpha}, \underline{\omega}_3; \mathbf{a})) dt = 0.$$

Proof. By Lemma 15 from [6], for almost all $\omega_{3j} \in \Omega_{3j}$ and every $j = 1, \dots, r$,

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\zeta(\sigma_j + it, \alpha_j, \omega_{3j}; \mathbf{a}_j) - \zeta_n(\sigma_j + it, \alpha_j, \omega_{3j}; \mathbf{a}_j)| dt = 0. \tag{4.2}$$

Let $\hat{\Omega}_{3j}$ be the subset of Ω_{3j} whose elements satisfy the later relation. Then we have that $m_{3jH}(\hat{\Omega}_{3j}) = 1$, where m_{3jH} is the Haar measure on $(\Omega_{3j}, \mathcal{B}(\Omega_{3j}))$, $j = 1, \dots, r$. Let $\hat{\Omega}_3 = \hat{\Omega}_{31} \times \dots \times \hat{\Omega}_{3r}$. Since the Haar measure \underline{m}_{3H} is the product of the measures m_{31H}, \dots, m_{3rH} , we find that $\underline{m}_{3H}(\hat{\Omega}_3) = 1$. Therefore, the assertion of the lemma follows from inequality (4.1) and relation (4.2). \square

5 Limit Theorems for $\underline{\zeta}(\mathbf{s}, \underline{\alpha}; \underline{\mathbf{a}})$ and $\underline{\zeta}_3(\mathbf{s}, \underline{\alpha}, \underline{\omega}_3; \underline{\mathbf{a}})$

For $A \in \mathcal{B}(\mathbb{C}^r)$, define one more probability measure

$$\hat{P}_T(A) = \frac{1}{T} \text{meas}\{t \in [0, T]: \underline{\zeta}_3(\underline{\sigma} + it, \underline{\alpha}, \underline{\omega}_3; \underline{\mathbf{a}}) \in A\}.$$

Theorem 6. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ satisfy the hypotheses of Theorem 3, and that $\min_{1 \leq j \leq r} \sigma_j > \frac{1}{2}$. Then, on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$, there exists a probability measure P such that the measures P_T and \hat{P}_T both converge weakly to P as $T \rightarrow \infty$.*

Proof. Let θ be a random variable defined on a certain probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and uniformly distributed on the interval $[0, 1]$. Define

$$\underline{X}_{T,n} = \underline{X}_{T,n}(\underline{\sigma}) = (X_{T,n,1}(\sigma_1), \dots, X_{T,n,r}(\sigma_r)) = \underline{\zeta}(\underline{\sigma} + iT\theta, \underline{\alpha}; \underline{\mathbf{a}})$$

which is a \mathbb{C}^r -valued random element on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then we have, in view of Theorem 5, that

$$\underline{X}_{T,n} \xrightarrow[T \rightarrow \infty]{\mathcal{D}} \underline{X}_n, \tag{5.1}$$

where $\underline{X}_n = \underline{X}_n(\underline{\sigma}) = (X_{n,1}(\sigma_1), \dots, X_{n,r}(\sigma_r))$ is the \mathbb{C}^r -valued random element which distribution is the limit measure P_n in Theorem 5, and, as usual, $\xrightarrow{\mathcal{D}}$ means the convergence in distribution.

Since the series for $\zeta_n(\sigma_j + it, \alpha_j; \mathbf{a}_j)$ converges absolutely, the properties of Dirichlet series imply

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\zeta_n(\sigma_j + it, \alpha_j; \mathbf{a}_j)|^2 dt = \sum_{m=0}^{\infty} \frac{|a_{mj}|^2 v^2(m, n, \alpha_j)}{(m + \alpha_j)^{2\sigma_j}} \leq \sum_{m=0}^{\infty} \frac{|a_{mj}|^2}{(m + \alpha_j)^{2\sigma_j}},$$

for all $n \in \mathbb{N}$, $j = 1, \dots, r$. Hence, for $j = 1, \dots, r$,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\zeta_n(\sigma_j + it, \alpha_j; \mathbf{a}_j)| dt \leq R_j, \tag{5.2}$$

where $R_j = \left(\sum_{m=0}^{\infty} \frac{|a_{mj}|^2}{(m+\alpha_j)^{2\sigma_j}} \right)^{\frac{1}{2}} < \infty$. Now let $\varepsilon > 0$ be arbitrary number, and $M_j = M_j(\varepsilon) = R_j r \varepsilon^{-1}$, $j = 1, \dots, r$. Then, in view of (5.2),

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \mathbb{P}(|X_{T,n,j}(\sigma_j)| > M_j \text{ for at least one } j = 1, \dots, r) \\ & \leq \sum_{j=1}^r \limsup_{T \rightarrow \infty} \mathbb{P}(|X_{T,n,j}(\sigma_j)| > M_j) \\ & \leq \sum_{j=1}^r \limsup_{T \rightarrow \infty} \frac{1}{T} \text{meas}\{t \in [0, T]: |\zeta_n(\sigma_j + it, \alpha_j; \mathbf{a}_j)| \geq M_j\} \\ & \leq \sum_{j=1}^r \frac{1}{M_j} \sup_{n \in \mathbb{N}} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\zeta_n(\sigma_j + it, \alpha_j; \mathbf{a}_j)| dt \leq \sum_{j=1}^r R_j / M_j = \varepsilon. \end{aligned}$$

This together with (5.1) implies

$$\mathbb{P}(|X_{n,j}(\sigma_j)| > M_j \text{ for at least one } j = 1, \dots, r) \leq \varepsilon. \tag{5.3}$$

for all $n \in \mathbb{N}$. Now define $K_\varepsilon^r = \{z \in \mathbb{C}^r: |z_j| \leq M_j, j = 1, \dots, r\}$. Then K_ε is a bounded closed set, thus it is a compact set on \mathbb{C}^r . Moreover, by (5.3),

$$\mathbb{P}(\underline{X}_n \in K_\varepsilon^r) \geq 1 - \varepsilon,$$

or, by the definition of \underline{X}_n ,

$$P_n(K_\varepsilon^r) \geq 1 - \varepsilon$$

for all $n \in \mathbb{N}$. This means that the family of probability measures $\{P_n: n \in \mathbb{N}\}$ is tight, and, by the Prokhorov theorem, see, for example, [1], it is relatively compact. Therefore, there exists a subsequence $\{P_{n_k}\} \subset \{P_n\}$ such that the measure P_{n_k} converges weakly to a certain probability measure P on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ as $k \rightarrow \infty$. In other words,

$$\underline{X}_{n_k} \xrightarrow[k \rightarrow \infty]{\mathcal{D}} P. \tag{5.4}$$

Now define

$$\underline{X}_T = \underline{X}_T(\sigma) = \zeta(\sigma + iT\theta, \alpha; \mathbf{a}).$$

Then \underline{X}_T is a \mathbb{C}^r -valued random element on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Using Lemma 1.1, we find that, for every $\varepsilon > 0$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \text{meas}\{t \in [0, T]: \rho(\underline{\zeta}(\sigma + it, \alpha; \mathbf{a}), \underline{\zeta}_n(\sigma + it, \alpha; \mathbf{a})) \geq \varepsilon\} \\ & \leq \lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{\varepsilon T} \int_0^T \rho(\underline{\zeta}(\sigma + it, \alpha; \mathbf{a}), \underline{\zeta}_n(\sigma + it, \alpha; \mathbf{a})) dt = 0. \end{aligned}$$

This and the definitions of the random elements $\underline{X}_{T,n}$ and \underline{X}_T imply

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \mathbb{P}(\rho(\underline{X}_T, \underline{X}_{T,n}) \geq \varepsilon) = 0. \tag{5.5}$$

The relations (5.1), (5.4) and (5.5) show that the hypotheses of Theorem 4.2 from [1] are satisfied. Therefore,

$$\underline{X}_T \xrightarrow[T \rightarrow \infty]{\mathcal{D}} P, \tag{5.6}$$

and this shows that the measure P_T converges weakly to P as $T \rightarrow \infty$. Moreover, in virtue of (5.6) we have that the measure P is independent of the choice of the subsequence P_{n_k} . Thus, the relative compactness of the family $\{P_n\}$ yields the relation

$$\underline{X}_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} P. \tag{5.7}$$

Reasoning similarly to the case of the measure P_T and using Theorem 5, Lemma 2 and (5.7), we obtain without difficulty that the measure \hat{P}_T also converges weakly to P as $T \rightarrow \infty$. \square

In view of Theorem 6, for proving Theorem 3, it remains to identify the measure P in (5.6).

6 Proof of Theorem 3

We start with one statement from ergodic theory. Let $\underline{a}_t = \{((m + \alpha_1)^{-it} : m \in \mathcal{M}(\alpha_1)), \dots, ((m + \alpha_r)^{-it} : m \in \mathcal{M}(\alpha_r))\}$, $t \in \mathbb{R}$. Define the one-parameter family $\{\Phi_t : t \in \mathbb{R}\}$ of transformation of $\underline{\Omega}_3$ by the formula $\Phi_t(\underline{\omega}_3) = \underline{a}_t \underline{\omega}_3$, $\underline{\omega}_3 \in \underline{\Omega}_3$. Then $\{\Phi_t : t \in \mathbb{R}\}$ is a one-parameter group of measurable measure preserving transformations on the group $\underline{\Omega}_3$.

Lemma 3. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ satisfy the hypotheses of Theorem 3. Then the one-parameter group $\{\Phi_t : t \in \mathbb{R}\}$ is ergodic.*

Proof. Let χ be a character of the group $\underline{\Omega}_3$. We have seen in the proof of Theorem 4 that

$$\chi(\underline{\omega}_3) = \prod_{m \in \mathcal{M}(\alpha_1)} \omega_{31}^{k_{1m}}(m) \dots \prod_{m \in \mathcal{M}(\alpha_r)} \omega_{3r}^{k_{rm}}(m), \quad \underline{\omega}_3 = (\omega_{31}, \dots, \omega_{3r}) \in \underline{\Omega}_3,$$

where only a finite number of integers k_{jm} , $j = 1, \dots, r$, are distinct from zero. First suppose that χ is a non-trivial character. Then we have that

$$\begin{aligned} \chi(\underline{a}_t) &= \prod_{m \in \mathcal{M}(\alpha_1)} (m + \alpha_1)^{-itk_{1m}} \dots \prod_{m \in \mathcal{M}(\alpha_r)} (m + \alpha_r)^{-itk_{rm}} \\ &= \exp \left\{ -it \left(\sum_{m \in \mathcal{M}(\alpha_1)} k_{1m} \log(m + \alpha_1) + \dots + \sum_{m \in \mathcal{M}(\alpha_r)} k_{rm} \log(m + \alpha_r) \right) \right\}. \end{aligned}$$

Using the linear independence of the set $\bigcup_{j=1}^r I(\alpha_j)$, hence, we find that there exists $\tau_0 \in \mathbb{R} \setminus \{0\}$ such that $\chi(\underline{a}_{\tau_0}) = 1$. The further proof is standard, see, for example, [5]. \square

Proof of Theorem 3. We take a fixed continuity set A of the limit measure P in Theorem 6. Then, using the equivalent of the weak convergence of probability measures in terms of continuity sets, see Theorem 2.1 from [1], we have that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \text{meas}\{t \in [0, T]: \zeta_3(\sigma + it, \underline{\alpha}, \underline{\omega}_3; \underline{\mathbf{a}}) \in A\} = P(A). \tag{6.1}$$

Let $\hat{\theta}$ be a random variable on $(\underline{\Omega}_3, \mathcal{B}(\underline{\Omega}_3), \underline{m}_{3H})$ given by

$$\hat{\theta}(\underline{\omega}_3) = \begin{cases} 1 & \text{if } \zeta_3(\sigma, \underline{\alpha}, \underline{\omega}_3; \underline{\mathbf{a}}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then, obviously, the expectation $\mathbb{E}(\hat{\theta})$ of $\hat{\theta}$ equals

$$\int_{\underline{\Omega}_3} \hat{\theta} \, d\underline{m}_{3H} = \underline{m}_{3H}(\underline{\omega}_3 \in \underline{\Omega}_3: \zeta_3(\sigma, \underline{\alpha}, \underline{\omega}_3; \underline{\mathbf{a}}) \in A) = P_{\zeta_3}(A). \tag{6.2}$$

From Lemma 3, the ergodicity of the random process $\hat{\theta}(\Phi_t(\underline{\omega}_3))$ follows. Therefore, an application of the classical Birkhoff–Khinchine theorem, see, for example, [2], shows that, for almost all $\underline{\omega}_3 \in \underline{\Omega}_3$,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \hat{\theta}(\Phi_t(\underline{\omega}_3)) \, dt = \mathbb{E}(\hat{\theta}). \tag{6.3}$$

On the other hand, by the definitions of $\hat{\theta}$ and Φ_t , we have that

$$\frac{1}{T} \int_0^T \hat{\theta}(\Phi_t(\underline{\omega}_3)) \, dt = \frac{1}{T} \text{meas}\{t \in [0, T]: \zeta_3(\sigma + it, \underline{\alpha}, \underline{\omega}_3; \underline{\mathbf{a}}) \in A\}.$$

This, (6.2) and (6.3) imply, for almost all $\underline{\omega}_3 \in \underline{\Omega}_3$, the equality

$$\lim_{T \rightarrow \infty} \frac{1}{T} \text{meas}\{t \in [0, T]: \zeta_3(\sigma + it, \underline{\alpha}, \underline{\omega}_3; \underline{\mathbf{a}}) \in A\} = P_{\zeta_3}(A).$$

Hence, taking into account (6.1), we obtain that $P(A) = P_{\zeta_3}(A)$. Since A was an arbitrary continuity set of P , the latter relation is true for all continuity sets of P , and this shows that $P(A) = P_{\zeta_3}(A)$ for all $A \in \mathcal{B}(\mathbb{C}^r)$. \square

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