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Weighted Lipschitz Continuity, Schwarz–Pick's Lemma and Landau–Bloch's Theorem for Hyperbolic-Harmonic Mappings in $\mathbb{C}^n$

Shaolin Chen[a,c], **Saminathan Ponnusamy**[b,e] **and Xiantao Wang**[c,d]

[a] *Hengyang Normal University*
Hengyang, 421008 Hunan, China
[b] *Indian Institute of Technology Madras*
600 036 Chennai, India
[c] *Hunan Normal University*
Changsha, 410081 Hunan, China
[d] *Ministry of Education of China*
Changsha, Hunan, China
[e] *Indian Statistical Institute (ISI), Chennai Center*
CIT Campus, Taramani, 600 113 Chennai, India
E-mail: shlchen1982@yahoo.com.cn
E-mail: samy@iitm.ac.in
E-mail(*corresp.*): xtwang@hunnu.edu.cn

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Abstract. In this paper, we discuss some properties on hyperbolic-harmonic functions in the unit ball of $\mathbb{C}^n$. First, we investigate the relationship between the weighted Lipschitz functions and the hyperbolic-harmonic Bloch spaces. Then we establish the Schwarz–Pick type theorem for hyperbolic-harmonic functions and apply it to prove the existence of Landau-Bloch constant for functions in α-Bloch spaces.

Keywords: hyperbolic-harmonic function, Bloch space, Landau–Bloch's theorem, Schwarz–Pick's lemma.

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1 Introduction and Preliminaries

Let $\mathbb{C}^n$ denote the complex Euclidean n-space. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, the conjugate of z, denoted by $\overline{z}$, is defined by $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_n)$. For z and $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, the standard Hermitian scalar product on $\mathbb{C}^n$ and the

Euclidean norm of z are given by

$$\langle z, w\rangle := \sum_{k=1}^{n} z_k \overline{w}_k \quad \text{and} \quad |z| := \langle z, z\rangle^{1/2} = \left(|z_1|^2 + \cdots + |z_n|^2\right)^{1/2},$$

respectively. For $a \in \mathbb{C}^n$, $\mathbb{B}^n(a, r) = \{z \in \mathbb{C}^n \colon |z-a| < r\}$ is the (open) ball of radius r with center a. Also, we let $\mathbb{B}^n(r) := \mathbb{B}^n(0, r)$ and use $\mathbb{B}^n$ to denote the unit ball $\mathbb{B}^n(1)$, and $\mathbb{D} = \mathbb{B}^1$. We can interpret $\mathbb{C}^n$ as the real $2n$-space $\mathbb{R}^{2n}$ so that a ball in $\mathbb{C}^n$ is also a ball in $\mathbb{R}^{2n}$. We use the following standard notations. For $a \in \mathbb{R}^n$, we may let $\mathbb{B}^n_{\mathbb{R}}(a, r) = \{x \in \mathbb{R}^n \colon |x-a| < r\}$ so that $\mathbb{B}^n_{\mathbb{R}}(r) := \mathbb{B}^n_{\mathbb{R}}(0, r)$ and $\mathbb{B}^n_{\mathbb{R}} = \mathbb{B}^n_{\mathbb{R}}(1)$ denotes the open unit ball in $\mathbb{R}^n$ centered at the origin.

DEFINITION 1. A twice continuously differentiable complex-valued function $f = u + iv$ on $\mathbb{B}^n$ is called a *hyperbolic-harmonic* (briefly, h-harmonic, in the following) if and only if the real-valued functions u and v satisfy $\Delta_h u = \Delta_h v = 0$ on $\mathbb{B}^n$, where

$$\Delta_h := \left(1 - |z|^2\right)^2 \sum_{k=1}^{n} \left(\frac{\partial}{\partial x_k^2} + \frac{\partial}{\partial y_k^2}\right) + 4(n-1)\left(1 - |z|^2\right) \sum_{k=1}^{n} \left(x_k \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial y_k}\right)$$

denotes the *Laplace–Beltrami operator* and $z_k = x_k + iy_k$ for $k = 1, \ldots, n$.

Obviously, when $n = 1$, all h-harmonic functions are planar complex-valued harmonic functions (see [12]). We refer to [5, 13, 14, 25] for more details of h-harmonic functions.

By [5, P_{284}], it turns out that if $\psi \in C(\partial\mathbb{B}^n)$, then the Dirichlet problem

$$\begin{cases} \Delta_h f = 0 & \text{in } \mathbb{B}^n, \\ f = \psi & \text{on } \partial\mathbb{B}^n \end{cases}$$

has unique solution in $C(\overline{\mathbb{B}}^n)$ and can be represented by

$$f(z) = \int_{\partial\mathbb{B}^n} \mathrm{P}_h(z, \zeta)\psi(\zeta)\, d\sigma(\zeta),$$

where $d\sigma$ is the unique normalized surface measure on $\partial\mathbb{B}^n$ and $\mathrm{P}_h(z, \zeta)$ is the *hyperbolic Poisson kernel* defined by

$$\mathrm{P}_h(z, \zeta) = \left(\frac{1 - |z|^2}{|z - \zeta|^2}\right)^{2n-1} \quad \left(z \in \mathbb{B}^n,\ \zeta \in \partial\mathbb{B}^n\right).$$

Here $C(\Omega)$ stands for the set of all continuous functions on Ω. A planar complex-valued harmonic function f in $\mathbb{D}$ is called a *harmonic Bloch function* if and only if

$$\beta_f = \sup_{z, w \in \mathbb{D},\, z \neq w} \frac{|f(z) - f(w)|}{\rho(z, w)} < \infty.$$

Here β_f is the *Lipschitz number* of f and

$$\rho(z,w)=\frac{1}{2}\log\left(\frac{1+|\frac{z-w}{1-\overline{z}w}|}{1-|\frac{z-w}{1-\overline{z}w}|}\right)=\operatorname{arctanh}\left|\frac{z-w}{1-\overline{z}w}\right|$$

denotes the hyperbolic distance between z and w in $\mathbb{D}$. It can be proved that

$$\beta_f=\sup_{z\in\mathbb{D}}\left\{\left(1-|z|^2\right)\left[\left|f_z(z)\right|+\left|f_{\overline{z}}(z)\right|\right]\right\}.$$

We refer to [11, Theorem 2] (see also [8, Theorem 1] and [9, Theorem A]) for a proof of the last fact.

For a complex-valued h-harmonic function f on $\mathbb{B}^n$, we introduce

$$D_f=\left(\frac{\partial f}{\partial z_1},\ldots,\frac{\partial f}{\partial z_n}\right)\quad\text{and}\quad \overline{D}_f=\left(\frac{\partial f}{\partial \overline{z}_1},\ldots,\frac{\partial f}{\partial \overline{z}_n}\right).$$

Definition 2. The *h-harmonic Bloch space* $\mathcal{HB}$ consists of complex-valued h-harmonic functions f defined on $\mathbb{B}^n$ such that

$$\|f\|_{\mathcal{HB}}=\sup_{z\in\mathbb{B}^n}\left\{\left(1-|z|^2\right)\left[\left|D_f(z)\right|+\left|\overline{D}_f(z)\right|\right]\right\}<\infty.$$

Obviously, when $n=1$, $\|f\|_{\mathcal{HB}}=\beta_f$. For a pair of distinct points z and w in $\mathbb{B}^n$, let

$$\mathcal{L}_f(z,w)=\frac{(1-|z|^2)^{\frac{1}{2}}\left(1-|w|^2\right)^{\frac{1}{2}}|f(z)-f(w)|}{|z-w|}$$

denote the *weighted Lipschitz function* of a given h-harmonic function $f:\mathbb{B}^n\to\mathbb{C}$. The relationship between weighted Lipschitz functions and (analytic) Bloch spaces has attracted much attention (cf. [1, 2, 11, 15, 16, 21]). Our first aim is to characterize the functions in h-harmonic Bloch spaces in terms of their corresponding weighted Lipschitz functions. This is done in Theorem 1 which is indeed a generalization of [11, Theorem 1] and [15, Theorem 3].

Throughout, $\mathcal{H}(\mathbb{B}^n,\mathbb{C}^n)$ denotes the set of all continuously differentiable functions f from $\mathbb{B}^n$ into $\mathbb{C}^n$ with $f=(f_1,\ldots,f_n)$ and $f_j(z)=u_j(z)+iv_j(z)$ $(1\leq j\leq n)$, where u_j and v_j are real-valued functions on $\mathbb{B}^n$. For $f\in\mathcal{H}(\mathbb{B}^n,\mathbb{C}^n)$, the real Jacobian matrix of f is given by

$$J_f=\begin{pmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial y_2} & \cdots & \frac{\partial u_1}{\partial x_n} & \frac{\partial u_1}{\partial y_n}\\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial y_2} & \cdots & \frac{\partial v_1}{\partial x_n} & \frac{\partial v_1}{\partial y_n}\\ & & \vdots & \vdots & & & \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial y_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial y_2} & \cdots & \frac{\partial u_n}{\partial x_n} & \frac{\partial u_n}{\partial y_n}\\ \frac{\partial v_n}{\partial x_1} & \frac{\partial v_n}{\partial y_1} & \frac{\partial v_n}{\partial x_2} & \frac{\partial v_n}{\partial y_2} & \cdots & \frac{\partial v_n}{\partial x_n} & \frac{\partial v_n}{\partial y_n}\end{pmatrix}.$$

A vector-valued function $f\in\mathcal{H}(\mathbb{B}^n,\mathbb{C}^n)$ is said to be *h-harmonic*, if each component f_j $(1\leq j\leq n)$ is a h-harmonic function from $\mathbb{B}^n$ into $\mathbb{C}$. We denote by $\mathcal{H}_h(\mathbb{B}^n, \mathbb{C}^n)$ the set of all vector-valued h-harmonic functions from $\mathbb{B}^n$ into $\mathbb{C}^n$.

For each $f = (f_1, \dots, f_n) \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$, denote by $f_z = (D_{f_1}, \dots, D_{f_n})^T$ the matrix formed by the complex gradients $D_{f_1}, \dots, D_{f_n}$, and let denote by $f_{\overline{z}} = (\overline{D}_{f_1}, \dots, \overline{D}_{f_n})^T$, where T means the matrix transpose.

For an $n \times n$ matrix $A = (a_{ij})_{n\times n}$, the operator norm of A is given by

$$|A| = \sup_{z\neq 0} \frac{|Az|}{|z|} = \max\bigl\{|A\theta| \colon \theta \in \partial\mathbb{B}^n\bigr\}.$$

Then for $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$, we use the standard notations:

$$\Lambda_f(z) = \max_{\theta\in\partial\mathbb{B}^n} \bigl|f_z(z)\theta + f_{\overline{z}}(z)\overline{\theta}\bigr| \quad \text{and} \quad \lambda_f(z) = \min_{\theta\in\partial\mathbb{B}^n} \bigl|f_z(z)\theta + f_{\overline{z}}(z)\overline{\theta}\bigr|. \tag{1.1}$$

We see that (see for instance [6])

$$\Lambda_f = \max_{\theta\in\partial\mathbb{B}^{2n}_{\mathbb{R}}} |J_f\theta| \quad \text{and} \quad \lambda_f = \min_{\theta\in\partial\mathbb{B}^{2n}_{\mathbb{R}}} |J_f\theta|. \tag{1.2}$$

Let $\mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ denote the set of all $f = (f_1, \dots, f_n) \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$ such that all partial derivatives $\partial f_j/\partial z_k$ and $\partial f_j/\partial \overline{z}_k$ $(1 \le j, k \le n)$ are h-harmonic in $\mathbb{B}^n$.

We remark that when $n = 1$, every complex-valued harmonic function from $\mathbb{D}$ to $\mathbb{C}$ belongs to $\mathcal{PH}(\mathbb{D}, \mathbb{C})$. The converse is not true as the function $f(z) = |z|^2$ shows.

Definition 3. For $\alpha > 0$, the *vector-valued h-harmonic α-Bloch space* $\mathcal{HB}_n(\alpha)$ consists of all functions in $\mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ such that

$$\|f\|_{\mathcal{HB}_n(\alpha)} = \sup_{z\in\mathbb{B}^n} \bigl\{\bigl(1 - |z|^2\bigr)^\alpha \bigl[\bigl|f_z(z)\bigr| + \bigl|f_{\overline{z}}(z)\bigr|\bigr]\bigr\} < \infty.$$

Obviously, $\mathcal{HB}_1(\alpha)$ contains the harmonic α-Bloch space as a proper subset (see [9]). One of the long standing open problems in function theory is to determine the precise value of the univalent Landau-Bloch constant for analytic functions of $\mathbb{D}$. In recent years, this problem has attracted much attention, see [4, 18, 20] and references therein. For general holomorphic functions of more than one complex variable, no Landau-Bloch constant exists (cf. [26]). In order to obtain some analogs of Landau-Bloch's theorem for functions with several complex variables, it became necessary to restrict the class of functions considered (cf. [3, 6, 10, 17, 22, 24, 26]).

Based on Heinz's Lemma and Colonna's Distortion Theorem ([11, Theorem 3]) for planar complex-valued harmonic functions, in [6], the authors established the Schwarz–Pick type theorem for bounded pluriharmonic mappings and pluriharmonic K-mappings. As a consequence of it, the authors in [6] obtained Landau-Bloch theorem as generalizations of the main results [7, Theorems 1–7]. It is known that every pluriharmonic mapping f defined in $\mathbb{B}^n$ admits a decomposition $f = h + \overline{g}$, where h and g are holomorphic in $\mathbb{B}^n$. This decomposition property is no longer valid for functions in $\mathcal{HB}_n(\alpha)$. Hence the methods of proof used in [6, Theorem 4] and [6, Theorem 5] are no longer applicable for functions in $\mathcal{H}_h(\mathbb{B}^n, \mathbb{C}^n)$ and $\mathcal{HB}_n(\alpha)$. In view of this reasoning, in this article, we use entirely a different approach and prove Schwarz–Pick type theorem for functions in $\mathcal{H}_h(\mathbb{B}^n, \mathbb{C}^n)$ and then establish the Landau-Bloch theorem for functions in $\mathcal{HB}_n(\alpha)$ (see Theorems 2 and 3). It is worth pointing out that Theorems 2 and 3 are indeed generalizations of [11, Theorem 1] and [9, Theorem 2.4], respectively.

2 Characterization of Mappings in h-Harmonic Bloch Spaces

Consider the group $\mathrm{Aut}(\mathbb{B}^n)$ consisting of all biholomorphic mappings of $\mathbb{B}^n$ onto itself. Then for each $a \in \mathbb{B}^n$, ϕ_a defined by [23]:

$$\phi_a(z) = \frac{a - P_a z - (1-|a|^2)^{\frac{1}{2}}(z - P_a z)}{1 - \langle z, a\rangle}$$

belongs to $\mathrm{Aut}(\mathbb{B}^n)$, where $P_a z = a\langle z, a\rangle / \langle a, a\rangle$. Moreover, we find that

$$1 - \left|\phi_a(z)\right|^2 = \frac{(1-|z|^2)(1-|a|^2)}{|1-\langle z, a\rangle|^2}. \tag{2.1}$$

Using arguments similar to those in the proof of [19, Lemma 2.5], we have

Lemma 1. *Suppose $f : \overline{\mathbb{B}}^n_{\mathbb{R}}(a,r) \to \mathbb{R}$ is a continuous, and h-harmonic in $\mathbb{B}^n_{\mathbb{R}}(a,r)$. Then*

$$\left|\nabla f(a)\right| \le \frac{2(n-1)\sqrt{n}}{nV(n)r^n} \int_{\partial \mathbb{B}^n_{\mathbb{R}}(a,r)} \left|f(a) - f(t)\right| d\sigma(t),$$

where $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$, $d\sigma$ denotes the surface measure on $\partial\mathbb{B}^n_{\mathbb{R}}(a,r)$ and $V(n)$, the volume of the unit ball in $\mathbb{R}^n$.

Proof. Without loss of generality, we may assume that $a = 0$ and $f(0) = 0$. Let

$$K(x,t) = \frac{1}{nr^{n-1}V(n)}\left(\frac{r^2 - |x|^2}{|x-t|^2}\right)^{n-1}.$$

Then by the assumption on f, we see that [5]

$$f(x) = \int_{\partial \mathbb{B}^n_{\mathbb{R}}(r)} K(x,t) f(t)\, d\sigma(t), \quad x \in \mathbb{B}^n_{\mathbb{R}}(r).$$

Further, a computation shows that

$$\frac{\partial}{\partial x_i} K(x,t) = \frac{-2(n-1)(r^2-|x|^2)^{n-2}}{nr^{n-1}V(n)} \cdot \frac{[|x-t|^2 x_i + (r^2-|x|^2)(x_i - t_i)]}{|x-t|^{2n}},$$

which yields

$$\frac{\partial}{\partial x_i} K(0,t) = \frac{2(n-1)t_i}{nV(n)r^{n+1}},$$

whence

$$\begin{aligned}\big|\nabla f(0)\big| &= \Bigg[\sum_{i=1}^{n}\Bigg|\int_{\partial\mathbb{B}^n_{\mathbb{R}}(r)} \frac{\partial}{\partial x_i}K(0,t)f(t)\,d\sigma(t)\Bigg|^2\Bigg]^{\frac{1}{2}}\\ &\le \sum_{i=1}^{n}\Bigg|\int_{\partial\mathbb{B}^n_{\mathbb{R}}(r)} \frac{\partial}{\partial x_i}K(0,t)f(t)\,d\sigma(t)\Bigg| \le \int_{\partial\mathbb{B}^n_{\mathbb{R}}(r)} \big|f(t)\big|\sum_{i=1}^{n}\Bigg|\frac{\partial}{\partial x_i}K(0,t)\Bigg|\,d\sigma(t)\\ &\le \sqrt{n}\int_{\partial\mathbb{B}^n_{\mathbb{R}}(r)}\big|f(t)\big|\Bigg(\sum_{i=1}^{n}\Bigg|\frac{\partial}{\partial x_i}K(0,t)\Bigg|^2\Bigg)^{\frac{1}{2}}\,d\sigma(t)\\ &= \frac{2(n-1)\sqrt{n}}{nV(n)r^n}\int_{\partial\mathbb{B}^n_{\mathbb{R}}(r)}\big|f(t)\big|\,d\sigma(t),\end{aligned}$$

from which the lemma follows. □

Lemma 2. *Let $f = u + iv$ be a continuously differentiable function from $\mathbb{B}^n$ into $\mathbb{C}$, where u and v are real-valued functions. Then for $z \in \mathbb{B}^n$,*

$$\big|D_f(z)\big| + \big|\overline{D}_f(z)\big| \le \big|\nabla u(z)\big| + \big|\nabla v(z)\big|, \tag{2.2}$$

where $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial y_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial y_n})$ *and* $\nabla v = (\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial y_1}, \dots, \frac{\partial v}{\partial x_n}, \frac{\partial v}{\partial y_n})$.

Proof. By a basic change of variables, for each $k = 1, 2, \dots, n$, we have

$$f_{z_k}(z) = \frac{1}{2}\big(f_{x_k}(z) - if_{y_k}(z)\big) \quad\text{and}\quad f_{\overline{z}_k}(z) = \frac{1}{2}\big(f_{x_k}(z) + if_{y_k}(z)\big),$$

which implies

$$\begin{aligned} f_{z_k}(z) &= \frac{1}{2}\big[u_{x_k}(z) + v_{y_k}(z) + i(v_{x_k}(z) - u_{y_k}(z))\big],\\ f_{\overline{z}_k}(z) &= \frac{1}{2}\big[u_{x_k}(z) - v_{y_k}(z) + i\big(v_{x_k}(z) + u_{y_k}(z)\big)\big].\end{aligned}$$

The classical Cauchy–Schwarz inequality gives

$$\begin{aligned}\big|D_f(z)\big| &= \frac{1}{2}\sqrt{\sum_{k=1}^{n}\Big[\big(u_{x_k}(z) + v_{y_k}(z)\big)^2 + \big(v_{x_k}(z) - u_{y_k}(z)\big)^2\Big]}\\ &\le \frac{1}{2}\big(\big|\nabla u(z)\big| + \big|\nabla v(z)\big|\big)\end{aligned}$$

and similarly,

$$\begin{aligned}\big|\overline{D}_f(z)\big| &= \frac{1}{2}\sqrt{\sum_{k=1}^{n}\Big[\big(u_{x_k}(z) - v_{y_k}(z)\big)^2 + \big(v_{x_k}(z) + u_{y_k}(z)\big)^2\Big]}\\ &\le \frac{1}{2}\big(\big|\nabla u(z)\big| + \big|\nabla v(z)\big|\big),\end{aligned}$$

from which we obtain the desired inequality (2.2). □

Example 1. Consider $f(z)=z^2+\overline{z}=u(x,y)+iv(x,y)$ so that $u(x,y)=x^2+x-y^2$ and $v(x,y)=2xy-y$. It is easy to see that

$$\big|f_z(0)\big|+\big|f_{\overline{z}}(0)\big|=1 \quad \text{and} \quad \big|\nabla u(0)\big|+\big|\nabla v(0)\big|=2,$$

showing that strict inequality in (2.2) is possible.

Theorem 1. *$f\in\mathcal{HB}$ if and only if* $\sup_{z,w\in\mathbb{B}^n, z\neq w}\mathcal{L}_f(z,w)<\infty$.

Proof. First we prove the necessity. For each pair of distinct points z and w in $\mathbb{B}^n$, we have

$$\begin{aligned}|f(z)-f(w)| &= \left|\int_0^1 \frac{df}{dt}\big(zt+(1-t)w\big)\,dt\right|\\ &= \left|\sum_{k=1}^n (z_k-w_k)\int_0^1 \frac{df}{d\varsigma_k(t)}\big(zt+(1-t)w\big)\,dt\right.\\ &\quad \left.+\sum_{k=1}^n(\overline{z}_k-\overline{w}_k)\int_0^1\frac{df}{d\overline{\varsigma}_k(t)}\big(zt+(1-t)w\big)\,dt\right|\\ &\leq \sum_{k=1}^n|z_k-w_k|\cdot\left|\int_0^1\frac{df}{d\varsigma_k(t)}\big(zt+(1-t)w\big)\,dt\right|\\ &\quad +\sum_{k=1}^n|\overline{z}_k-\overline{w}_k|\cdot\left|\int_0^1\frac{df}{d\overline{\varsigma}_k(t)}\big(zt+(1-t)w\big)\,dt\right|,\end{aligned}$$

where $\varsigma(t)=(\varsigma_1(t),\ldots,\varsigma_n(t))=zt+(1-t)w$. Hence we see that

$$\begin{aligned}\big|f(z)-f(w)\big| &\leq \left(\sum_{k=1}^n|z_k-w_k|^2\right)^{\frac12}\left\{\left[\sum_{k=1}^n\left(\int_0^1\left|\frac{\partial f}{\partial\varsigma_k(t)}\big(zt+(1-t)w\big)\right|dt\right)^2\right]^{\frac12}\right.\\ &\quad\left.+\left[\sum_{k=1}^n\left(\int_0^1\left|\frac{\partial f}{\partial\overline{\varsigma}_k(t)}\big(zt+(1-t)w\big)\right|dt\right)^2\right]^{\frac12}\right\}\\ &\leq\sqrt{n}|z-w|\int_0^1\Big[\big|D_f\big(tz+(1-t)w\big)\big|+\big|\overline{D}_f\big(tz+(1-t)w\big)\big|\Big]\,dt.\end{aligned}$$

This gives

$$\begin{aligned}\frac{|f(z)-f(w)|}{|z-w|} &\leq\sqrt{n}\int_0^1\frac{[|D_f(\varsigma(t))|+|\overline{D}_f(\varsigma(t))|](1-|\varsigma(t)|^2)}{1-|\varsigma(t)|^2}\,dt\\ &\leq\sqrt{n}\|f\|_{\mathcal{HB}}\int_0^1\frac{dt}{1-|\varsigma(t)|^2}\leq\sqrt{n}\|f\|_{\mathcal{HB}}\int_0^1\frac{dt}{[(1-t)(1-|z|)]^{\frac12}[t(1-|w|)]^{\frac12}}\\ &=\frac{\pi\sqrt{n}\|f\|_{\mathcal{HB}}}{(1-|z|)^{\frac12}(1-|w|)^{\frac12}}.\end{aligned}$$

Thus,

$$\sup_{z,w\in\mathbb{B}^n,\, z\neq w}\mathcal{L}_f(z,w)\leq\pi\sqrt{n}\|f\|_{\mathcal{HB}}.$$

Next we prove the sufficiency part. Let $f = u + iv$, where u and v are real h-harmonic functions. Fix $r \in (0,1)$. In view of (2.1) and the fact that $|\langle z, a\rangle| \le |z|\,|a|$, we easily have

$$\big|\phi_a(z)\big| \le \frac{|z-a|}{|1-\langle z,a\rangle|} \le \frac{|z-a|}{1-|a|}, \tag{2.3}$$

whence for $a \in \mathbb{B}^n$,

$$\mathbb{B}^n\left(a, \frac{r(1-|a|^2)}{2}\right) \subset E(a,r),$$

where $E(a,r) = \{z \in \mathbb{B}^n \colon |\phi_a(z)| < r\}$. By Lemma 1, we have

$$\begin{aligned}\big(1-|z|^2\big)\big|\nabla u(z)\big| &\le \frac{(2n-1)\sqrt{2n}(1-|z|^2)}{nV(2n)[\frac{r(1-|z|^2)}{2}]^{2n}} \int\limits_{\partial\mathbb{B}^n(z,\frac{r(1-|z|^2)}{2})} \big|u(\zeta)-u(z)\big|\,d\sigma(\zeta)\\ &= M\big(|z|,r\big) \int\limits_{\partial\mathbb{B}^n(z,\frac{r(1-|z|^2)}{2})} \big|u(\zeta)-u(z)\big|\,d\sigma(\zeta),\end{aligned}$$

where $V(2n)$ denotes the volume of the unit ball in $\mathbb{R}^{2n}$ (or $\mathbb{C}^n$) and

$$M\big(|z|,r\big) = \frac{2^{2n}(2n-1)\sqrt{2n}}{nV(2n)(1-|z|^2)^{2n-1}r^{2n}}.$$

Similarly, we obtain

$$\big(1-|z|^2\big)\big|\nabla v(z)\big| \le M\big(|z|,r\big) \int\limits_{\partial\mathbb{B}^n(z,\frac{r(1-|z|^2)}{2})} \big|v(\zeta)-v(z)\big|\,d\sigma(\zeta).$$

By Lemma 2, we have

$$\begin{aligned}&\big(1-|z|^2\big)\big(\big|D_f(z)\big| + \big|\overline{D}_f(z)\big|\big)\\ &\quad\le \big(1-|z|^2\big)\big(\big|\nabla u(z)\big| + \big|\nabla v(z)\big|\big)\\ &\quad\le M\big(|z|,r\big) \int\limits_{\partial\mathbb{B}^n(z,\frac{r(1-|z|^2)}{2})} \big(\big|u(\zeta)-u(z)\big| + \big|v(\zeta)-v(z)\big|\big)\,d\sigma(\zeta)\\ &\quad\le \sqrt{2}M\big(|z|,r\big)M_1 \int\limits_{\partial\mathbb{B}^n(z,\frac{r(1-|z|^2)}{2})} d\sigma(\zeta) = \frac{4\sqrt{n}(2n-1)}{r}M_1,\end{aligned}$$

where $M_1 = \sup\{\big|f(z)-f(w)\big| \colon w \in E(z,r)\}$.

Hence for all $w \in \mathbb{B}^n(z, \frac{r(1-|z|^2)}{2}) \subset E(z,r)$, it follows from (2.1) and (2.3) that

$$\begin{aligned}\frac{(1-|z|^2)^{\frac12}(1-|w|^2)^{\frac12}}{|z-w|} &= \frac{(1-|z|^2)^{\frac12}(1-|w|^2)^{\frac12}}{|1-\langle z,w\rangle|}\cdot\frac{|1-\langle z,w\rangle|}{|z-w|}\\ &= \sqrt{1-\big|\phi_z(w)\big|^2}\cdot\frac{|1-\langle z,w\rangle|}{|z-w|}\end{aligned}$$

$$\geq \sqrt{1-r^2} \cdot \frac{|1-\langle z,w\rangle|}{|z-w|} \geq \frac{\sqrt{1-r^2}}{r}.$$

Therefore, there exists a positive constant $M_2(n,r)$ such that

$$\big(1-|z|^2\big)\big(\big|D_f(z)\big| + \big|\overline{D}_f(z)\big|\big) \leq M_2(n,r) \sup_{w\in E(z,r),\, w\neq z} \mathcal{L}_f(z,w),$$

from which we see that $f \in \mathcal{HB}$. □

3 Schwarz–Pick Type Theorem and Landau–Bloch Theorem

The following result is a Schwarz–Pick type theorem for h-harmonic functions in $\mathcal{H}_h(\mathbb{B}^n,\mathbb{C}^n)$.

Theorem 2. *Let $f \in \mathcal{H}_h(\mathbb{B}^n,\mathbb{C}^n)$ with $|f(z)| \leq M$ for $z \in \mathbb{B}^n$, where M is a positive constant. Then*

$$\left|f(z) - \frac{(1-|z|)^{2n-1}}{(1+|z|)^{2n-1}} f(0)\right| \leq M\left[1 - \frac{(1-|z|)^{2n-1}}{(1+|z|)^{2n-1}}\right] \tag{3.1}$$

and

$$\Lambda_f \leq \frac{2(2n-1)M}{(1-|z|)^2}. \tag{3.2}$$

Proof. We first prove (3.1). Without loss of generality, we assume that f is also h-harmonic on $\partial\mathbb{B}^n$. The hyperbolic Poisson integral formula states that

$$f(z) = \int_{\partial\mathbb{B}^n} \mathrm{P}_h(z,\zeta) f(\zeta)\, d\sigma(\zeta), \qquad \int_{\partial\mathbb{B}^n} \mathrm{P}_h(z,\zeta)\, d\sigma(\zeta) = 1. \tag{3.3}$$

As $\mathrm{P}_h(0,\zeta) = 1$ and $|\mathrm{P}_h(z,\zeta)| \leq 1$ for $\zeta \in \partial\mathbb{B}^n$ and all $z \in \mathbb{B}^n$, the representation (3.3) immediately yields

$$\begin{aligned}\left|f(z) - \frac{(1-|z|)^{2n-1}}{(1+|z|)^{2n-1}} f(0)\right| &= \left|\int\limits_{\partial\mathbb{B}^n} \left[\frac{(1-|z|^2)^{2n-1}}{|z-\zeta|^{2(2n-1)}} - \frac{(1-|z|)^{2n-1}}{(1+|z|)^{2n-1}}\right] f(\zeta)\, d\sigma(\zeta)\right| \\ &\leq \int\limits_{\partial\mathbb{B}^n} \left[\frac{(1-|z|^2)^{2n-1}}{|z-\zeta|^{2(2n-1)}} - \frac{(1-|z|)^{2n-1}}{(1+|z|)^{2n-1}}\right] \big|f(\zeta)\big|\, d\sigma(\zeta) \\ &\leq M\left[1 - \frac{(1-|z|)^{2n-1}}{(1+|z|)^{2n-1}}\right]\end{aligned}$$

and the proof of (3.1) follows.

Next, we prove (3.2). Let $f = (f_1, \dots, f_n)$ and $\theta = (\theta_1, \dots, \theta_n)^T \in \partial\mathbb{B}^n$. Without loss of generality, we assume that f is also h-harmonic on $\partial\mathbb{B}^n$. If we consider the formula (3.3) for f componentwise and then the partial derivatives with respect to the variables z_k and $\overline{z}_k$, we see that

$$\begin{aligned}&\big(f_j(z)\big)_{z_k} \\ &\quad = \int\limits_{\partial\mathbb{B}^n} \frac{-(2n-1)(1-|z|^2)^{2n-2}[\overline{z}_k|\zeta-z|^2 + (1-|z|^2)(\overline{z}_k - \overline{\zeta}_k)]}{|z-\zeta|^{4n}} f_j(\zeta)\, d\sigma(\zeta)\end{aligned}$$

and

$$\begin{aligned}&\big(f_j(z)\big)_{\overline{z}_k}\\ &\quad= \int\limits_{\partial\mathbb{B}^n} \frac{-(2n-1)(1-|z|^2)^{2n-2}[z_k|\zeta-z|^2+(1-|z|^2)(z_k-\zeta_k)]}{|z-\zeta|^{4n}} f_j(\zeta)\,d\sigma(\zeta),\end{aligned}$$

which hold clearly for each $j,k\in\{1,\dots,n\}$. Now, we introduce

$$\Gamma_{f_j}=\sum_{k=1}^{n}\big(f_j(z)\big)_{z_k}\cdot\theta_k+\sum_{k=1}^{n}\big(f_j(z)\big)_{\overline{z}_k}\cdot\overline{\theta}_k.$$

Then the classical Cauchy–Schwarz inequality yields

$$\begin{aligned}&\frac{|\Gamma_{f_j}|^2}{(2n-1)^2(1-|z|^2)^{4n-4}}\\ &\quad=\bigg|\sum_{k=1}^{n}\int_{\partial\mathbb{B}^n}\frac{[\overline{z}_k|\zeta-z|^2+(1-|z|^2)(\overline{z}_k-\overline{\zeta}_k)]\theta_k}{|z-\zeta|^{4n}}f_j(\zeta)\,d\sigma(\zeta)\\ &\qquad+\sum_{k=1}^{n}\int_{\partial\mathbb{B}^n}\frac{[z_k|\zeta-z|^2+(1-|z|^2)(z_k-\zeta_k)]\overline{\theta}_k}{|z-\zeta|^{4n}}f_j(\zeta)\,d\sigma(\zeta)\bigg|^2\\ &\quad\le 4\bigg[\int_{\partial\mathbb{B}^n}\frac{[|z||\zeta-z|^2+(1-|z|^2)|\zeta-z|]|f_j(\zeta)|}{|z-\zeta|^{4n}}\,d\sigma(\zeta)\bigg]^2\\ &\quad\le 4\bigg[\int_{\partial\mathbb{B}^n}\frac{[|z||\zeta-z|+(1-|z|^2)]^2}{|z-\zeta|^{4n-2}}\,d\sigma(\zeta)\bigg]\bigg[\int_{\partial\mathbb{B}^n}\frac{|f_j(\zeta)|^2}{|z-\zeta|^{4n}}\,d\sigma(\zeta)\bigg],\end{aligned}$$

whence

$$\begin{aligned}&\frac{|\Lambda_f|^2}{(2n-1)^2(1-|z|^2)^{4n-4}}=\frac{\max_{\theta\in\partial\mathbb{B}^n}\big(\sum_{j=1}^{n}|\Gamma_{f_j}|^2\big)}{(2n-1)^2(1-|z|^2)^{4n-4}}\\ &\quad\le 4\bigg[\int\limits_{\partial\mathbb{B}^n}\frac{[|z||\zeta-z|+(1-|z|^2)]^2}{|z-\zeta|^{4n-2}}\,d\sigma(\zeta)\bigg]\bigg[\int\limits_{\partial\mathbb{B}^n}\frac{\sum_{j=1}^{n}|f_j(\zeta)|^2}{|z-\zeta|^{4n}}\,d\sigma(\zeta)\bigg]\\ &\quad\le\frac{4M^2}{(1-|z|)^2(1-|z|^2)^{2n-1}}\bigg[\int_{\partial\mathbb{B}^n}\frac{(1+|z|)^2}{|z-\zeta|^{4n-2}}\,d\sigma(\zeta)\bigg]\\ &\quad\le\frac{4M^2(1+|z|)^2}{(1-|z|)^2(1-|z|^2)^{2n-1}}\bigg[\int_{\partial\mathbb{B}^n}\frac{1}{|z-\zeta|^{4n-2}}\,d\sigma(\zeta)\bigg]\\ &\quad\le\frac{4M^2(1+|z|)^2}{(1-|z|)^2(1-|z|^2)^{4n-2}}.\end{aligned}$$

Hence

$$|\Lambda_f|^2\le\frac{4(2n-1)^2M^2}{(1-|z|)^4},$$

from which the inequality (3.2) follows. □

Definition 4. A matrix-valued function $A(z)=\big(a_{i,j}(z)\big)_{n\times n}$ is called h-*harmonic* if each of its entries $a_{i,j}(z)$ is a h-harmonic function from an open subset $\Omega\subset\mathbb{C}^n$ into $\mathbb{C}$.

As an application of Theorem 2, we get

Lemma 3. *Suppose that* $A(z) = (a_{i,j}(z))_{n\times n}$ *is a matrix-valued h-harmonic function of* $\mathbb{B}^n(r)$ *such that* $A(0) = 0$ *and* $|A(z)| \leq M$ *in* $\mathbb{B}^n(r)$*. Then*

$$\big|A(z)\big| \leq M\left[1 - \frac{(r-|z|)^{2n-1}}{(r+|z|)^{2n-1}}\right].$$

Proof. For an arbitrary $\theta = (\theta_1, \ldots, \theta_n)^T \in \partial\mathbb{B}^n$, we introduce

$$P_\theta(z) = A(z)\theta = \big(p_1(z), \ldots, p_n(z)\big)$$

and let $F_\theta(\zeta) = P_\theta(r\zeta)$ for $\zeta \in \mathbb{B}^n$. By Theorem 2, we see that

$$\left|F_\theta(\zeta) - \frac{(1-|\zeta|)^{2n-1}}{(1+|\zeta|)^{2n-1}}F_\theta(0)\right| \leq M\left[1 - \frac{(1-|\zeta|)^{2n-1}}{(1+|\zeta|)^{2n-1}}\right], \quad \zeta \in \mathbb{B}^n,$$

which is equivalent to

$$\big|P_\theta(z)\big| \leq M\left[1 - \frac{(r-|z|)^{2n-1}}{(r+|z|)^{2n-1}}\right], \quad z \in \mathbb{B}^n(r).$$

The arbitrariness of θ yields the desired inequality. $\square$

We recall the following result which is crucial for the proof of our next theorem.

Lemma A. **[[6, Lemma 1] or [17, Lemma 4]]** *Let* A *be an* $n \times n$ *complex (real) matrix and* $|A| \neq 0$*. Then for* $\theta \in \partial\mathbb{B}^n$*, the inequality* $|A\theta| \geq |\det A|\,|A|^{1-n}$ *holds.*

Theorem 3. *Suppose that* $f \in \mathcal{HB}_n(\alpha)$*,* $f(0) = 0$*,* $\det J_f(0) = 1$ *and*

$$\|f\|_{\mathcal{HB}_n(\alpha)} \leq M,$$

where M *is a positive constant. Then* f *is univalent in* $\mathbb{B}^n(\rho/2)$*, where*

$$\rho = \frac{3^\alpha}{(2M)^{2n}(3^\alpha + 4^\alpha)}. \tag{3.4}$$

Moreover, the range $f(\mathbb{B}^n(\rho/2))$ *contains a univalent ball* $\mathbb{B}^n(R)$*, where*

$$R \geq \frac{\rho}{4M^{2n-1}}.$$

Proof. For $\zeta \in \mathbb{B}^n$, let $F(\zeta) = 2f(\frac{1}{2}\zeta)$. Then

$$\big|F_\zeta(\zeta)\big| + \big|F_{\overline{\zeta}}(\zeta)\big| \leq \frac{M}{(1-\frac{|\zeta|^2}{4})^\alpha} \leq \frac{4^\alpha}{3^\alpha}M,$$

which gives

$$\big|F_\zeta(\zeta) - F_\zeta(0)\big| \leq \big|F_\zeta(\zeta)\big| + \big|F_\zeta(0)\big| \leq \left(1 + \frac{4^\alpha}{3^\alpha}\right)M.$$

Lemma 3 implies that

$$\begin{aligned}
&\left|F_\zeta(\zeta) - F_\zeta(0)\right| \\
&\quad \le \left(1 + \frac{4^\alpha}{3^\alpha}\right) M \left[1 - \frac{(1-|\zeta|)^{2n-1}}{(1+|\zeta|)^{2n-1}}\right] \\
&\quad = \frac{2M(3^\alpha + 4^\alpha)}{3^\alpha} \frac{(C_{2n-1}^1|\zeta| + C_{2n-1}^3|\zeta|^3 + \cdots + C_{2n-1}^{2n-1}|\zeta|^{2n-1})}{(1+|\zeta|)^{2n-1}} \\
&\quad \le \frac{2^{2n-1}(3^\alpha + 4^\alpha)M}{3^\alpha(1+|\zeta|)^{2n-1}}|\zeta| \le \frac{2^{2n-1}(3^\alpha + 4^\alpha)M}{3^\alpha}|\zeta|, \qquad (3.5)
\end{aligned}$$

where $C_n^k = \binom{n}{k}$ $(k = 1, 2, \ldots, n)$ denote the binomial coefficients. Similarly,

$$\left|F_{\overline{\zeta}}(\zeta) - F_{\overline{\zeta}}(0)\right| \le \frac{2^{2n-1}(3^\alpha + 4^\alpha)M}{3^\alpha}|\zeta|. \qquad (3.6)$$

On the other hand, for $\theta \in \partial\mathbb{B}^n$, we infer from (1.1), (1.2) and Lemma A that

$$\lambda_F(0) \ge \frac{\det J_F(0)}{\Lambda_F^{2n-1}(0)} \ge \frac{1}{M^{2n-1}}. \qquad (3.7)$$

In order to prove the univalence of F in $\mathbb{B}^n(\rho)$, we choose two distinct points ζ' and ζ'' in $\mathbb{B}^n(\rho)$ with $\zeta' - \zeta'' = |\zeta' - \zeta''|\theta$, and let $[\zeta', \zeta'']$ denote the line segment with endpoints ζ' and ζ'', where

$$\rho = \frac{3^\alpha}{(2M)^{2n}(3^\alpha + 4^\alpha)}.$$

Set $d\zeta = (d\zeta_1, \ldots, d\zeta_n)^T$ and $d\overline{\zeta} = (d\overline{\zeta}_1, \ldots, d\overline{\zeta}_n)^T$. Then we infer from (3.5), (3.6) and (3.7) that

$$\begin{aligned}
\left|F(\zeta') - F(\zeta'')\right| &\ge \left| \int_{[\zeta',\zeta'']} F_\zeta(0)\, d\zeta + F_{\overline{\zeta}}(0)\, d\overline{\zeta} \right| \\
&- \left| \int_{[\zeta',\zeta'']} \big(F_\zeta(\zeta) - F_\zeta(0)\big)\, d\zeta + \big(F_{\overline{\zeta}}(\zeta) - F_{\overline{\zeta}}(0)\big)\, d\overline{\zeta} \right| \\
&\ge \left|F_\zeta(0)\theta + F_{\overline{\zeta}}(0)\overline{\theta}\right| \int_{[\zeta',\zeta'']} |d\zeta| - \frac{2^{2n}(3^\alpha + 4^\alpha)M}{3^\alpha} \int_{[\zeta',\zeta'']} |\zeta|\,|d\zeta| \\
&> |\zeta' - \zeta''| \left\{ \lambda_F(0) - \frac{2^{2n}(3^\alpha + 4^\alpha)M}{3^\alpha}\rho \right\} \\
&\ge |\zeta' - \zeta''| \left\{ \frac{1}{M^{2n-1}} - \frac{2^{2n}(3^\alpha + 4^\alpha)M}{3^\alpha}\rho \right\} = 0,
\end{aligned}$$

where $\theta = \frac{d\zeta}{|d\zeta|}$. Thus, F is univalent in $\mathbb{B}^n(\rho)$ which is equivalent to saying that f is univalent in $\mathbb{B}^n(\rho/2)$.

Furthermore, for each z with $|\zeta| = \rho$, we have

$$
\begin{aligned}
\big|F(\zeta) - F(0)\big| \geq & \left| \int\limits_{[0,\zeta]} F_\zeta(0)\, d\zeta + F_{\overline{\zeta}}(0)\, d\overline{\zeta} \right| \\
& - \left| \int\limits_{[0,\zeta]} \big(F_\zeta(\zeta) - F_\zeta(0)\big)\, d\zeta + \big(F_{\overline{\zeta}}(\zeta) - F_{\overline{\zeta}}(0)\big)\, d\overline{\zeta} \right| \\
\geq & \, \rho \left\{ \frac{1}{M^{2n-1}} - \frac{2^{2n-1}(3^\alpha + 4^\alpha) M\rho}{3^\alpha} \right\} \\
= & \, \frac{\rho}{2M^{2n-1}} \quad \text{(by (3.4))},
\end{aligned}
$$

showing the range $f(\mathbb{B}^n(\rho/2))$ contains a univalent ball $\mathbb{B}^n(R)$, where $R \geq \rho/(4M^{2n-1})$. The proof of this theorem is complete. □

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