Mathematical Modelling and Analysis
Volume 18 Number 1, February 2013, 1–21
http://dx.doi.org/10.3846/13926292.2013.756072

Publisher: Taylor&Francis and VGTU
http://www.tandfonline.com/TMMA
Print ISSN: 1392-6292
Online ISSN: 1648-3510

Applicability of Spline Collocation to Cordial Volterra Equations

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Received January 24, 2012; revised November 27, 2012; published online February 1, 2013

Abstract. We study the applicability of the standard spline collocation method, on a uniform grid, to linear Volterra integral equations of the second kind with the so-called cordial operators; these operators are noncompact and the applicability of the collocation method becomes crucial in the convergence analysis. In particular, piecewise constant, piecewise linear and piecewise quadratic collocation methods are applicable under wide, quite acceptable conditions. For higher order spline collocation, it is more complicated to carry out an analytical study of the applicability of the method; however, a numerical check is rather simple and this is illustrated by some numerical examples.

Keywords: Volterra equations, cordial operators, spline collocation.

AMS Subject Classification: 45A05; 45D05; 45E99; 65R20.

1 Introduction

In the present article we study the applicability of spline collocation methods to the Volterra integral equation

$$\mu u(t) = \int_0^t t^{-1}\varphi(t^{-1}s)u(s)\,ds + f(t), \quad 0 \le t \le T, \tag{1.1}$$

where $\varphi \in L^1(0,1)$ is the core of the (cordial) Volterra integral operator

$$(V_\varphi u)(t) = \int_0^t t^{-1}\varphi(t^{-1}s)u(s)\,ds, \quad 0 \le t \le T.$$

Equations and systems of type (1.1) can appear e.g. when solving differential equations with certain singularities and some partial differential equations with symmetries. A special case of (1.1) is equation (4.4) which arises in connection with a heat conduction problem with mixed-type boundary conditions (see [3] for details).

As known from [10], $V_\varphi : C^m[0,1] \to C^m[0,1]$, $m \geq 0$, is bounded but noncompact (if $\varphi \neq 0$) and its spectrum $\sigma_m(V_\varphi)$ is non-discrete:

$$\sigma_0(V_\varphi) = \{0\} \cup \big\{\hat\varphi(\lambda)\colon \lambda \in \mathbb{C},\ \operatorname{Re}\lambda \geq 0\big\}, \quad \hat\varphi(\lambda) := \int_0^1 \varphi(x) x^\lambda\, dx, \tag{1.2}$$
$$\sigma_m(V_\varphi) = \{0\} \cup \big\{\hat\varphi(k)\colon k = 0,\dots,m-1\big\} \cup \big\{\hat\varphi(\lambda)\colon \operatorname{Re}\lambda \geq m\big\}, \quad m \geq 1.$$

By $\varrho_0(V_\varphi) = \mathbb{C} \setminus \sigma_0(V_\varphi)$ we denote the resolvent set of $V_\varphi : C[0,1] \to C[0,1]$. Let us also recall that for $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda \geq 0$, the following relation holds

$$V_\varphi u_\lambda = \hat\varphi(\lambda) u_\lambda, \quad \text{where } u_\lambda(t) = t^\lambda,\ 0 < t \leq T. \tag{1.3}$$

In particular, V_φ maps polynomials into polynomials and, due to this property, it is easy to solve (1.1) approximating f by a polynomial. The polynomial algorithms become more complicated [11] if we have to solve a more general equation

$$\mu u(t) = \int_0^t t^{-1}\varphi\big(t^{-1}s\big) a(t,s) u(s)\, ds + f(t), \quad 0 \leq t \leq T; \tag{1.4}$$

here spline collocation methods are preferable. Since the applicability and convergence conditions of spline collocation methods for equations (1.1) and (1.4), and even for a class of related nonlinear equations, are the same in their essence [12, 13], we confine ourselves to the case of the model equation (1.1).

Let us describe the method. For $N \in \mathbb{N}$, denote $h = T/N$. For $m \in \mathbb{N}$, introduce the space $\mathcal{S}_N^m[0,T]$ of splines u_N such that

$$u_N|_{[ih,(i+1)h]} \in \mathcal{P}_{m-1}, \quad i = 0,\dots,N-1,$$

where $\mathcal{P}_{m-1}$ is the set of polynomials of degree not exceeding $m-1$. We accept that a spline $u_N \in \mathcal{S}_N^m[0,T]$ may have at the interior spline knots ih, $1 \leq i \leq N-1$, two values, so that $u_N|_{[ih,(i+1)h]} \in C[ih,(i+1)h]$, $i = 0,\dots,N-1$. For $u_N \in \mathcal{S}_N^m[0,T]$, we denote by $u_{N,i} \in \mathcal{S}_N^m[0,T]$ the spline coinciding with u_N on $[ih,(i+1)h]$, being zero on other subintervals $[lh,(l+1)h]$, $l \neq i$. The functions $u_{N,i}$ compose an m-dimensional subspace $S_{N,i} = S_{N,i}^m \subset \mathcal{S}_N^m[0,T]$.

Let us introduce the interpolation (collocation) parameters τ_k, $k = 1,\dots,m$,

$$0 \leq \tau_1 < \tau_2 < \cdots < \tau_m \leq 1.$$

For $v \in C[0,T]$, denote by $\Pi_{N,i} v \in S_{N,i}$, $0 \leq i \leq N-1$, the interpolation polynomial defined by

$$(\Pi_{N,i} v)(ih + \tau_k h) = v(ih + \tau_k h), \quad k = 1,\dots,m.$$

Let $L_k \in \mathcal{P}_{m-1}$, $k = 1, \dots, m$, be the Lagrange fundamental polynomials associated with the interpolation knots τ_k, $k = 1, \dots, m$, that is,

$$L_k(\tau_j) = \delta_{j,k} \quad \text{(Kronecker symbol)}, \quad j, k = 1, \dots, m.$$

Then the $L_{i,k}(t) := L_k(h^{-1}t - i)$, $k = 1, \dots, m$, are the fundamental polynomials associated with the interpolation knots $ih + \tau_k h \in [ih, (i+1)h]$, $k = 1, \dots, m$, that is,

$$L_{i,k}(ih + \tau_j h) = L_k\bigl(h^{-1}(ih + \tau_j h) - i\bigr) = L_k(\tau_j) = \delta_{j,k}, \quad j, k = 1, \dots, m.$$

For $v \in C[0, T]$ we have the representation

$$(\varPi_{N,i} v)(t) = \sum_{k=1}^{m} v(ih + \tau_k h) L_{i,k}(t), \quad t \in \bigl[ih, (i+1)h\bigr],\ i = 0, \dots N-1.$$

Define the interpolation operator $P_N : C[0, T] \to \mathcal{S}_N^m[0, T]$ by

$$(P_N v)(t) = (\varPi_{N,i} v)(t) \quad \text{for } t \in \bigl[ih, (i+1)h\bigr],\ i = 0, \dots, N-1,$$

and consider the spline collocation method

$$\mu u_N = P_N V_\varphi u_N + P_N f \tag{1.5}$$

for the approximate solution to equation (1.1).

We say that the collocation method (1.5) is **applicable** to equation (1.1) if the homogeneous equation $\mu u_N = P_N V_\varphi u_N$ has in $\mathcal{S}_N^m[0, T]$ only the trivial solution $u_N = 0$, that is, if there exist the inverses to the m-dimensional operators $\mu I - \varPi_{N,i} V_\varphi : S_{N,i} \to S_{N,i}$, $i = 0, 1, \dots, N-1$; then we can recurrently find $u_{N,i}$ from the equations

$$\mu u_{N,i} = \varPi_{N,i} V_\varphi u_{N,i} + \varPi_{N,i} f + \sum_{l=0}^{i-1} \varPi_{N,i} V_\varphi u_{N,l}, \quad i = 0, 1, \dots, N-1, \tag{1.6}$$

where for $i = 0$ we have adopted the agreement that $\sum_{l=0}^{-1} \dots = 0$.

For $\mu \in \varrho_0(V_\varphi)$, the operator $\mu I - \varPi_{N,0} V_\varphi : S_{N,0} \to S_{N,0}$ is invertible since due to (1.3) V_φ maps $S_{N,0}$ into $S_{N,0}$. It is less obvious that $\mu I - \varPi_{N,i} V_\varphi : S_{N,i} \to S_{N,i}$ is invertible also for all sufficiently large $i \ge i_0$, with i_0 independent of N. This claim is more clear from the matrix form of equations (1.6). The ith equation of (1.6) is equivalent to the $m \times m$-system of linear equations to determine the knot values $u_{N,i}(ih + \tau_j h)$ of $u_{N,i}$ (see [12]):

$$\begin{aligned} \mu u_{N,i}(ih + \tau_j h) &= \sum_{k=1}^{m} d_{j,k}^{i,i} u_{N,i}(ih + \tau_k h) + f(ih + \tau_j h) \\ &\quad + \sum_{l=0}^{i-1} \sum_{k=1}^{m} d_{j,k}^{i,l} u_{N,l}(lh + \tau_k h), \quad j = 1, \dots, m, \end{aligned} \tag{1.7}$$

where for $j, k = 1, \dots, m$,

$$d_{j,k}^{i,i} = \int_{i/(i+\tau_j)}^{1} \varphi(x) L_k\big((i+\tau_j)x - i\big)\,dx, \quad i = 0,\dots,N-1,$$

$$\text{with } d_{1,k}^{0,0} = \delta_{k,1} \int_0^1 \varphi(x)\,dx, \quad \text{for } i=0,\ \tau_1 = 0,$$

$$d_{j,k}^{i,l} = \int_{l/(i+\tau_j)}^{(l+1)/(i+\tau_j)} \varphi(x) L_k\big((i+\tau_j)x - l\big)\,dx, \ \ l = 0,\dots,i-1,\ i = 1,\dots,N-1.$$

Hence the collocation method (1.5) is **applicable** to equation (1.1) iff

$$\det(\mu I_m - D_i) \neq 0, \quad i = 1,\dots,N-1,$$

where I_m is the $m\times m$ identity matrix and $D_i = (d_{j,k}^{i,i})_{j,k=1}^m$. For sufficiently large i this condition is fulfilled since

$$\|D_i\| := \max_{1\le j\le m} \sum_{k=1}^m \big|d_{j,k}^{i,i}\big| \le \int_{i/(i+\tau_m)}^1 \big|\varphi(x)\big|\,dx \max_{0\le t\le \tau_m} \sum_{k=1}^m \big|L_k(t)\big|$$

implies that $\|D_i\| \to 0$ as $i\to\infty$. Thus the **applicability condition** of the collocation method (1.5) reduces to

$$\det(\mu I_m - D_i) \neq 0 \quad \text{for } i = 1,\dots,i_0 - 1, \tag{1.8}$$

where $i_0 \ge 1$ is such that $\|D_{i_0}\| < |\mu|$.

We further note that the matrices D_i are independent of h (of N), so (1.8) implies the applicability of the collocation method (1.5) for all $N\ge 1$. In particular, if the eigenvalues $\mu_j^{(i)}$, $j = 1,\dots,m$, of D_i are such that

$$\mu_j^{(i)} \in \sigma_0(V_\varphi), \quad j = 1,\dots,m,\ i = 1,\dots,i_0-1, \tag{1.9}$$

then the applicability condition (1.8) is fulfilled for any $\mu\in\varrho_0(V_\varphi)$.

The following theorem [12] tells us that, roughly speaking, if equation (1.1) is uniquely solvable and the collocation method (1.5) is applicable to (1.1) then the method converges and the convergence is of optimal accuracy order.

Theorem 1. *Let $\varphi\in L^1(0,1)$, $f\in C[0,T]$ and $\mu\in\varrho_0(V_\varphi)$ in* (1.1)*, and let the applicability condition* (1.8) *be fulfilled. Then the collocation equation* (1.5) *has, for $N\ge 1$, a unique solution $u_N\in\mathcal{S}_N^m[0,T]$, and*

$$\|u - u_N\|_\infty \le c_0 \|u - P_N u\|_\infty \to 0 \quad \text{as } N\to\infty,$$

where $u\in C[0,T]$ is the unique solution of equation (1.1)*. If $f\in C^m[0,T]$, then also $u\in C^m[0,T]$ and*

$$\|u-u_N\|_\infty \le c_m h^m \big\|u^{(m)}\big\|_\infty.$$

The constants c_0 and c_m are independent of N and f.

We note that the cordial integral equation (1.1) is a special case of the Mellin convolution equations [6, 8]. In general, the solution of those equations (or perhaps its higher derivatives) will have a singularity at $t = 0$. A survey of numerical methods for Mellin-type integral equations based on piecewise polynomial approximation is presented in [7]. The use of graded meshes in conjunction with collocation methods is one approach to deal with the loss of optimal convergence order; the convergence analysis of these procedures has been treated in [1, 6, 8, 9].

Due to (1.2), we see that the solution u of a cordial integral equation (1.1) has no boundary singularities: if $\mu \in \varrho_0(V_\varphi)$ and $f \in C^m[0,T]$ then $u \in C^m[0,T]$. Thus the possibility of using a uniform grid to achieve optimal convergence orders is a privilege of the cordial integral equations. As easily seen [12], for $m = 1$ (piecewise constant approximation of the solution u), the applicability condition (1.8) is fulfilled for any non-negative $\varphi \in L^1(0,1)$ and any $\mu \in \varrho_0(V_\varphi)$. The main purpose of the current article is to analyze the case $m = 2$ (piecewise linear approximation of u); some theoretical results are obtained also in the case $m = 3$ (piecewise quadratic approximation of u). An analytical examination of condition (1.8) for $m \geq 3$ in the general case is a complicated task and remains as an open problem. On the other hand, in the course of the computation process (1.7) it is easy to check whether (1.8) is fulfilled or not; and there are different possibilities [12] to modify the method if (1.8) is violated or if μ, the parameter of equation (1.1), happens to be too close to the set of eigenvalues of some D_i, $1 \leq i \leq i_0 - 1$.

In Section 2 we reformulate the applicability condition (1.8) for $m = 2$, using the basis $\{1,\ t\}$ of $\mathcal{P}_1$ instead of the Lagrange basis. The main theoretical results of the article concern the case $m = 2$ and are established in Section 3. In Section 4 we apply these results to some concrete cordial equations including the Abel equation. In Section 5 we partly extend the results to the case $m = 3$. In Section 6 we comment on the numerical check of the applicability condition for different values of m and present some illustrative numerical examples.

2 Reformulation of the Applicability Condition

In theoretical considerations concerning the applicability of the collocation method (1.5), the Newton basis $\{1, t, \ldots, t^{m-1}\}$ of $\mathcal{P}_{m-1}$ is sometimes preferable to the Lagrange basis used in Section 1. Below we reformulate the applicability condition (1.8) for $m = 2$ using the Newton basis $\{1, t\}$ of $\mathcal{P}_1$. Let the collocation parameters τ_j, $0 \leq \tau_1 < \tau_2 \leq 1$, be given. Representing

$$u_{N,i}(t) = \gamma_{0,i} + \gamma_{1,i} t \quad \text{for } ih \leq t \leq (i+1)h,\ 0 \leq i \leq N-1,$$

we have, for $ih \leq t \leq (i+1)h$,

$$\begin{aligned}(V_\varphi u_{N,i})(t) &= \int_{ih}^{t} t^{-1}\varphi\big(t^{-1}s\big)\big(\gamma_{0,i} + \gamma_{1,i}s\big)\,ds \\ &= \gamma_{0,i}\int_{ih/t}^{1}\varphi(x)\,dx + \gamma_{1,i}t\int_{ih/t}^{1}x\varphi(x)\,dx.\end{aligned}$$

The ith equation (1.6) is uniquely solvable iff the corresponding homogeneous equation $\mu u_{N,i} = \Pi_{N,i} V_\varphi u_{N,i}$ has only the trivial solution $u_{N,i} = 0$, that is, iff

$$\mu u_{N,i}(ih+\tau_j h) - (V_\varphi u_{N,i})(ih+\tau_j h) = 0, \quad j = 1,2 \quad \Longrightarrow \quad \gamma_{0,i} = \gamma_{1,i} = 0,$$

or, equivalently, iff the 2×2 homogeneous system

$$\Big(\mu - \int_{i/(i+\tau_j)}^1 \varphi(x)\,dx\Big)\gamma_{0,i} + (i+\tau_j)\Big(\mu - \int_{i/(i+\tau_j)}^1 x\varphi(x)\,dx\Big)\gamma_{1,i}h = 0, \quad j = 1,2,$$

has only the trivial solution $\gamma_{0,i} = \gamma_{1,i}h = 0$. Denoting

$$\begin{aligned} a_{11}^{(i)} &= \int_{i/(i+\tau_1)}^1 \varphi(x)\,dx, \quad a_{12}^{(i)} = \int_{i/(i+\tau_1)}^1 x\varphi(x)\,dx, \\ a_{21}^{(i)} &= \int_{i/(i+\tau_2)}^1 \varphi(x)\,dx, \quad a_{22}^{(i)} = \int_{i/(i+\tau_2)}^1 x\varphi(x)\,dx, \end{aligned} \tag{2.1}$$

we obtain the following formulation: for $m = 2$, the collocation method (1.5) is applicable to equation (1.1) iff the parameter μ of equation (1.1) is such that, for $i = 1, \ldots, N-1$,

$$\det \begin{pmatrix} \mu - a_{11}^{(i)} & (i+\tau_1)(\mu - a_{12}^{(i)}) \\ \mu - a_{21}^{(i)} & (i+\tau_2)(\mu - a_{22}^{(i)}) \end{pmatrix} = (\tau_2 - \tau_1)\mu^2 - b_i\mu + c_i \neq 0,$$

where

$$b_i = (i+\tau_2)\big(a_{11}^{(i)} + a_{22}^{(i)}\big) - (i+\tau_1)\big(a_{12}^{(i)} + a_{21}^{(i)}\big), \tag{2.2}$$

$$c_i = (i+\tau_2)a_{11}^{(i)}a_{22}^{(i)} - (i+\tau_1)a_{12}^{(i)}a_{21}^{(i)}. \tag{2.3}$$

If for some i, $1 \le i \le N-1$, it happens that

$$(\tau_2 - \tau_1)\mu^2 - b_i\mu + c_i = 0, \tag{2.4}$$

then the collocation method (1.5) is inapplicable to equation (1.1).

So we are interested in sufficient conditions onto the core $\varphi \in L^1(0,1)$, the collocation parameters τ_j $(0 \le \tau_1 < \tau_2 \le 1)$ and the parameter $\mu \in \varrho_0(V_\varphi)$ which guarantee that μ is different from the roots

$$\mu_1^{(i)} = \frac{b_i - (b_i^2 - 4(\tau_2-\tau_1)c_i)^{1/2}}{2(\tau_2-\tau_1)}, \qquad \mu_2^{(i)} = \frac{b_i + (b_i^2 - 4(\tau_2-\tau_1)c_i)^{1/2}}{2(\tau_2-\tau_1)} \tag{2.5}$$

of the quadratic equations (2.4), for $i = 1, \ldots, N-1$. Some results in this connection are presented in Section 3.

3 Applicability of Piecewise Linear Collocation

The main results of the present article are contained in Theorems 2 and 3.

Theorem 2. (i) *Let $\varphi \in L^1(0,1)$ be non-negative and $\mu \in \varrho_0(V_\varphi)$ in* (1.1)*, and let the collocation parameters be such that* $0 = \tau_1 < \tau_2 \leq 1$*. Then the piecewise linear collocation method is applicable to equation* (1.1) *and hence convergent.*

(ii) *Let $\varphi \in L^1(0,1)$ be non-negative and $\mu \in \mathbb{R}$ be such that $\mu > \int_0^1 \varphi(x)\,dx$ ($= \|V_\varphi\|$) in* (1.1)*, and let the collocation parameters satisfy* $0 < \tau_1 < \tau_2 \leq 1$*. Then the piecewise linear collocation method is applicable to equation* (1.1) *and hence convergent.*

(iii) *Let $\tau_1 \in (0,1)$ and $\mu < 0$ be given. Then, for any $\tau_2 > \tau_1$, sufficiently close to τ_1, there exists a non-negative core $\varphi \in L^\infty(0,1)$ (explicitly presented in the proof) such that $\mu < -\int_0^1 \varphi(x)\,dx$ and μ coincides with $\mu_1^{(1)}$ (defined by* (2.5)*, with $i = 1$). Thus the piecewise linear collocation method is inapplicable to equation* (1.1) *with this particular φ and the above μ, τ_1, τ_2, although $\mu \in \varrho_0(V_\varphi)$.*

Proof of claim (i). Assume that $\varphi \in L^1(0,1)$ is non-negative and let $0 = \tau_1 < \tau_2 \leq 1$. From (2.1)–(2.5) we observe that $a_{11}^{(i)} = a_{12}^{(i)} = 0$,

$$b_i = (i+\tau_2)a_{22}^{(i)} - i a_{21}^{(i)} = \int_{i/(i+\tau_2)}^1 \big((i+\tau_2)x - i\big)\varphi(x)\,dx \geq 0, \quad c_i = 0,$$

$$\mu_1^{(i)} = 0, \quad \mu_2^{(i)} = \frac{b_i}{\tau_2} = \int_{i/(i+\tau_2)}^1 \frac{(i+\tau_2)x - i}{\tau_2}\varphi(x)\,dx \leq \int_{i/(i+\tau_2)}^1 \varphi(x)\,dx,$$

where we have taken into account that

$$0 \leq \frac{(i+\tau_2)x - i}{\tau_2} \leq 1 \quad \text{for } \frac{i}{i+\tau_2} \leq x \leq 1,\ i \geq 1.$$

Due to (1.2) and (1.3), we have $[0, \int_0^1 \varphi(x)\,dx] \subset \sigma_0(V_\varphi)$, thus $\mu_1^{(i)}, \mu_2^{(i)} \in \sigma_0(V_\varphi)$; therefore, for any $\mu \in \varrho_0(V_\varphi)$ ($i \geq 1$) it is true that $\mu \neq \mu_1^{(i)}$ and $\mu \neq \mu_2^{(i)}$, that is, the applicability condition of the piecewise linear spline collocation method is fulfilled.

Proof of claim (ii). Assume that $\varphi \in L^1(0,1)$ is non-negative and let $0 < \tau_1 < \tau_2 \leq 1$. If $\mu_1^{(i)}$ and $\mu_2^{(i)}$ are complex, the claim (ii) is trivial. Assuming that $\mu_1^{(i)}$ and $\mu_2^{(i)}$ are real, we shall prove that $\mu_1^{(i)} \leq \mu_2^{(i)} \leq \|V_\varphi\|$, $i \geq 1$. Note that

$$a_{11}^{(i)} - a_{12}^{(i)} + a_{22}^{(i)} - a_{21}^{(i)} = -\int_{i/(i+\tau_2)}^{i/(i+\tau_1)} (1-x)\varphi(x)\,dx \leq 0, \tag{3.1}$$
$$a_{11}^{(i)} \leq a_{21}^{(i)}, \quad a_{22}^{(i)} \leq a_{21}^{(i)}.$$

Then

$$\begin{aligned} b_i &= (i+\tau_2)\big(a_{11}^{(i)} + a_{22}^{(i)}\big) - (i+\tau_1)\big(a_{12}^{(i)} + a_{21}^{(i)}\big) \\ &= \big((i+\tau_2) - (i+\tau_1)\big)\big(a_{11}^{(i)} + a_{22}^{(i)}\big) + (i+\tau_1)\big(a_{11}^{(i)} + a_{22}^{(i)} - a_{12}^{(i)} - a_{21}^{(i)}\big) \\ &\leq (\tau_2 - \tau_1)\big(a_{11}^{(i)} + a_{22}^{(i)}\big) \leq 2(\tau_2 - \tau_1)a_{21}^{(i)}, \quad i \geq 1. \end{aligned}$$

Therefore, denoting

$$d_i(\mu) := \det \begin{pmatrix} \mu - a_{11}^{(i)} & (i+\tau_1)(\mu - a_{12}^{(i)}) \\ \mu - a_{21}^{(i)} & (i+\tau_2)(\mu - a_{22}^{(i)}) \end{pmatrix} = (\tau_2 - \tau_1)\mu^2 - b_i\mu + c_i,$$

we have

$$\begin{aligned} d_i\big(a_{21}^{(i)}\big) &= \det \begin{pmatrix} a_{21}^{(i)} - a_{11}^{(i)} & (i+\tau_1)(a_{21}^{(i)} - a_{12}^{(i)}) \\ 0 & (i+\tau_2)(a_{21}^{(i)} - a_{22}^{(i)}) \end{pmatrix} \\ &= (i+\tau_2)\big(a_{21}^{(i)} - a_{11}^{(i)}\big)\big(a_{21}^{(i)} - a_{22}^{(i)}\big) \geq 0. \end{aligned}$$

On the other hand, $d_i'(\mu) = 2(\tau_2 - \tau_1)\mu - b_i$ and, since

$$d_i'\big(a_{21}^{(i)}\big) = 2(\tau_2 - \tau_1)a_{21}^{(i)} - b_i \geq 0, \quad \text{then} \quad d_i'(\mu) \geq 0 \quad \text{for } \mu \geq a_{21}^{(i)}.$$

Therefore, we may conclude that $d_i(\mu) > 0$ for $\mu > a_{21}^{(i)}$, hence

$$\mu_1^{(i)} \leq \mu_2^{(i)} \leq a_{21}^{(i)} \leq \int_0^1 \varphi(x)\,dx, \quad i \geq 1.$$

We see that the condition $\mu > \int_0^1 \varphi(x)\,dx$ implies $\mu > \mu_2^{(i)} \geq \mu_1^{(i)}$, $i \geq 1$, and the applicability condition is thus fulfilled.

Proof of claim (iii). Let $\mu < 0$ and $0 < \tau_1 < \tau_2 \leq 1$, $\tau_2 < 3\tau_1 + 2\tau_1^2$. Denote

$$x_j = 1/(1+\tau_j) \quad (j = 1, 2), \qquad \gamma = \frac{\mu}{x_1(x_1+x_2)/2 - x_2}.$$

Clearly, we have $1/2 \leq x_2 < x_1 < 1$. An elementary check confirms that $x_1(x_1+x_2)/2 - x_2 < 0$ and hence $\gamma > 0$ under certain conditions on μ and τ_1, τ_2. Consider the core

$$\varphi(x) = \gamma\,\chi_{[x_2,x_1]},$$

where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$. In accordance with (2.1)–(2.5), for $i = 1$ we have

$$\begin{gathered} a_{11}^{(1)} = a_{12}^{(1)} = 0, \qquad c_1 = 0, \\ a_{21}^{(1)} = \gamma \int_{x_2}^{x_1} dx = \gamma(x_1 - x_2), \quad a_{22}^{(1)} = \gamma \int_{x_2}^{x_1} x\,dx = \frac{\gamma}{2}(x_1^2 - x_2^2), \\ b_1 = x_2^{-1}a_{22}^{(1)} - x_1^{-1}a_{21}^{(1)} = \gamma x_1^{-1}x_2^{-1}(x_1 - x_2)\big\{x_1(x_1+x_2)/2 - x_2\big\} < 0, \\ \mu_1^{(1)} = \frac{b_1}{x_2^{-1} - x_1^{-1}} = \gamma\big(x_1(x_1+x_2)/2 - x_2\big), \qquad \mu_2^{(1)} = 0, \end{gathered}$$

and $\mu_1^{(1)} = \mu$ by the definition of γ. Clearly, if $\tau_2 > \tau_1$ is sufficiently close to τ_1 (or, if $x_2 < x_1$ is sufficiently close to x_1) then

$$\frac{x_1(x_1+x_2)}{2} - x_2 < -(x_1 - x_2)$$

(for $x_2 = x_1$ this inequality results in $x_1^2 - x_1 < 0$). Since

$$\|V_\varphi\| := \|V_\varphi\|_{C[0,T]\to C[0,T]} = \int_0^1 \varphi(x)\,dx = \gamma(x_1 - x_2),$$

we obtain $\mu_1^{(1)} < -\|V_\varphi\|$, hence $\mu = \mu_1^{(1)} \in \varrho_0(V_\varphi)$ for $\tau_2 > \tau_1$ close to τ_1. This completes the proof of the present theorem. □

Lemma 1. *Let $\varphi \in L^1(0,1)$ be non-negative and the collocation parameters satisfy $0 \le \tau_1 < \tau_2 \le 1$. If the b_i, c_i, defined in* (2.2) *and* (2.3), *are such that $b_i \ge 0$, $c_i \ge 0$ $(1 \le i \le N-1)$, then the piecewise linear collocation method is applicable to equation* (1.1) *for any $\mu \in \mathbb{R} \cap \varrho_0(V_\varphi)$, and hence convergent.*

Proof. Again, the claim is trivial if $\mu_1^{(i)}$ and $\mu_2^{(i)}$ are complex. If the roots $\mu_1^{(i)}$ and $\mu_2^{(i)}$ are real then it follows from the conditions $b_i \ge 0$, $c_i \ge 0$ that $0 \le (b_i^2 - 4(\tau_2 - \tau_1)c_i)^{1/2} \le b_i$ (see (2.5)); this implies that $0 \le \mu_1^{(i)} \le \mu_2^{(i)}$ and the applicability condition is fulfilled for $\mu < 0$, $\mu \in \varrho_0(V_\varphi)$. For $\mu > \|V_\varphi\|$, the applicability condition is fulfilled by the statements (i), (ii) of Theorem 2, where $[0, \|V_\varphi\|] \subset \sigma_0(V_\varphi)$. □

Below we examine conditions for $\varphi \ge 0$ which guarantee that $b_i \ge 0$, $c_i \ge 0$ for all $i \ge 1$ and any choice of the collocation parameters.

Lemma 2. *Let $\varphi \in L^1(0,1) \cap C(0,1)$ be non-negative. If the function $\Phi(x) = x\varphi(x)/(1-x)$ is non-decreasing on $[1/2, 1)$ then $b_i \ge 0$ for all $i \ge 1$ and all choices of τ_1, τ_2 such that $0 \le \tau_1 < \tau_2 \le 1$.*

Proof. For a fixed $i \ge 1$, denote $x_j := x_j^{(i)} = i/(i + \tau_j)$, $j = 1, 2$, and let

$$g(x_1, x_2) := g_i(x_1, x_2) = x_1\big(a_{11}^{(i)} + a_{22}^{(i)}\big) - x_2\big(a_{12}^{(i)} + a_{21}^{(i)}\big),$$

where $i/(i+1) \le x_2 < x_1 \le 1$. Note that $b_i = i x_1^{-1} x_2^{-1} g(x_1, x_2) \ge 0$ iff $g \ge 0$. So it is sufficient to prove that $g(x_1, x_2) \ge 0$ for $i/(i+1) \le x_2 < x_1 \le 1$. Using (2.1), we have the expressions

$$a_{11}^{(i)} = \int_{x_1}^1 \varphi(x)\,dx, \quad a_{12}^{(i)} = \int_{x_1}^1 x\varphi(x)\,dx,$$
$$a_{21}^{(i)} = \int_{x_2}^1 \varphi(x)\,dx, \quad a_{22}^{(i)} = \int_{x_2}^1 x\varphi(x)\,dx,$$

to be substituted into the derivatives

$$\frac{\partial g}{\partial x_1} = a_{11}^{(i)} + a_{22}^{(i)} - x_1\varphi(x_1) + x_1 x_2 \varphi(x_1),$$
$$\frac{\partial g}{\partial x_2} = -x_1 x_2 \varphi(x_2) - \big(a_{12}^{(i)} + a_{21}^{(i)}\big) + x_2 \varphi(x_2).$$

With the help of (3.1) and making use of the non-decreasing property of Φ, we obtain

$$\begin{aligned}\frac{\partial g}{\partial x_1}+\frac{\partial g}{\partial x_2}&=a_{11}^{(i)}+a_{22}^{(i)}-\left(a_{12}^{(i)}+a_{21}^{(i)}\right)+x_2\varphi(x_2)(1-x_1)-x_1\varphi(x_1)(1-x_2)\\&\le x_2\varphi(x_2)(1-x_1)-x_1\varphi(x_1)(1-x_2)\\&=(1-x_1)(1-x_2)\left(\frac{x_2\varphi(x_2)}{1-x_2}-\frac{x_1\varphi(x_1)}{1-x_1}\right)\le 0.\end{aligned}$$

Thus

$$\frac{\partial g}{\partial x_1}+\frac{\partial g}{\partial x_2}\le 0 \quad \text{for } i/(i+1)\le x_2<x_1<1,$$

which means that g is non-increasing along the straight lines $x_2=x_1-\theta$, $\theta\in(0,1)$. Since

$$g(1,x_2)=\int_{x_2}^1(x-x_2)\varphi(x)\,dx\ge 0 \quad \text{for } i/(i+1)\le x_2<1,$$

it then $g(x_1,x_2)\ge 0$ for $i/(i+1)\le x_2<x_1\le 1$, thus completing the proof of this lemma. □

Remark 1. The non-decreasing property of Φ for a non-negative $\varphi\in L^1(0,1)\cap C^1(0,1)$ in Lemma 2 can be expressed in the form

$$\varphi(x)+x(1-x)\varphi'(x)\ge 0 \quad \text{for } \frac{1}{2}\le x<1; \tag{3.2}$$

a simplified sufficient condition for (3.2) and for the non-decreasing behaviour of Φ is given by

$$\varphi(x)+x\varphi'(x)\ge 0 \quad \text{for } \frac{1}{2}\le x<1. \tag{3.3}$$

Condition (3.3) will appear in another context when we examine conditions for $c_i\ge 0$.

Lemma 3. *Let $\varphi\in L^1(0,1)\cap C(0,1)$ be non-negative. Then the inequality $c_i\ge 0$ holds for all $i\ge 1$ and all choices of the collocation parameters $0\le\tau_1<\tau_2\le 1$ if and only if φ satisfies the inequality*

$$x\varphi(x)\int_x^1(s-x)\varphi(s)\,ds-\int_x^1\varphi(s)\,ds\int_x^1 s\varphi(s)\,ds\le 0 \quad \textit{for } \frac{1}{2}\le x<1. \tag{3.4}$$

Proof. We examine the inequality $c_i\ge 0$ for a fixed $i\ge 1$ and omit the upper index i in the $x_j^{(i)}$:

$$x_1=x_1^{(i)}:=i/(i+\tau_1),\qquad x_2=x_2^{(i)}:=i/(i+\tau_2),\quad i/(i+1)\le x_2<x_1\le 1.$$

With these notations formula (2.3) takes the form

$$c_i=ix_2^{-1}\int_{x_1}^1\varphi(s)\,ds\int_{x_2}^1 s\varphi(s)\,ds-ix_1^{-1}\int_{x_1}^1 s\varphi(s)\,ds\int_{x_2}^1\varphi(s)\,ds.$$

Since φ is non-negative, then $\int_{x_1}^1 \varphi(s)\,ds = 0$ implies $\varphi(x) = 0$ on $(x_1, 1)$ yielding $\int_{x_1}^1 s\varphi(s)\,ds = 0$ and $c_i = 0$. Therefore, we may assume that $\int_{x_1}^1 \varphi(s)\,ds > 0$ which implies $\int_{x_2}^1 \varphi(s)\,ds > 0$, and we can write

$$c_i = i \int_{x_1}^1 \varphi(s)\,ds \int_{x_2}^1 \varphi(s)\,ds \left(\frac{\int_{x_2}^1 s\varphi(s)\,ds}{x_2 \int_{x_2}^1 \varphi(s)\,ds} - \frac{\int_{x_1}^1 s\varphi(s)\,ds}{x_1 \int_{x_1}^1 \varphi(s)\,ds} \right).$$

We observe that $c_i \ge 0$ for any x_1, x_2, and $i/(i+1) \le x_2 < x_1 \le 1$ (for any $\tau_1 < \tau_2$ satisfying $0 \le \tau_1 < \tau_2 \le 1$) iff the function

$$g(x) = \frac{\int_x^1 s\varphi(s)\,ds}{x \int_x^1 \varphi(s)\,ds}$$

is non-increasing, i.e., if $g'(x) \le 0$ for $x \in [\frac{i}{i+1}, 1)$. Since

$$g'(x) = \frac{x\varphi(x) \int_x^1 (s-x)\varphi(s)\,ds - \int_x^1 \varphi(s)\,ds \int_x^1 s\varphi(s)\,ds}{(x \int_x^1 \varphi(s)\,ds)^2}$$

the condition takes the form (3.4); for $i = 1$ the interval is $[\frac{1}{2}, 1)$. $\square$

Theorem 3. *Assume $\varphi \in L^1(0,1) \cap C^1(0,1)$ is non-negative and satisfies* (3.3), *and*

$$(1-x)\varphi(x) \int_x^1 \varphi(s)\,ds \to 0 \quad \text{as } x \to 1. \tag{3.5}$$

Then the piecewise linear collocation method is, for any $\mu \in \mathbb{R} \cap \varrho_0(V_\varphi)$ and all choices of the collocation parameters τ_1, τ_2 $(0 \le \tau_1 < \tau_2 \le 1)$, applicable to equation (1.1) *and hence convergent.*

Proof. In view of Lemmas 1–3 and Remark 1, we only have to show that condition (3.4) is fulfilled. Denote

$$\gamma(x) = x\varphi(x) \int_x^1 (s-x)\varphi(s)\,ds - \int_x^1 \varphi(s)\,ds \int_x^1 s\varphi(s)\,ds, \quad \frac{1}{2} \le x < 1.$$

Due to (3.3), for $\frac{1}{2} \le x < 1$,

$$\gamma'(x) = \big(\varphi(x) + x\varphi'(x)\big) \int_x^1 (s-x)\varphi(s)\,ds + \varphi(x) \int_x^1 s\varphi(s)\,ds \ge 0. \tag{3.6}$$

Secondly, we have $\gamma(x) \to 0$ as $x \to 1$. In order to prove this, let us denote $\psi(s) = \int_s^1 \varphi(t)dt$. Integrating by parts and using (3.5) we obtain

$$\int_x^1 (s-x)\varphi(s)\,ds = \int_x^1 \psi(s)\,ds, \quad 0 \le \int_x^1 (s-x)\varphi(s)\,ds \le (1-x)\psi(x),$$

$$0 \le \varphi(x) \int_x^1 (s-x)\varphi(s)\,ds \le (1-x)\varphi(x) \int_x^1 \varphi(s)\,ds \to 0 \text{ as } x \to 1,$$

so that $\gamma(x) \to 0$ as $x \to 1$. Finally, (3.6) implies that $\gamma(x) \le 0$ for $\frac{1}{2} \le x < 1$, i.e., (3.4) is fulfilled. $\square$

4 Application to Some Concrete Cordial Equations

Consider the Abel type equation

$$\mu u(t) = \int_0^t t^{-\alpha} s^{-\beta} \big(t^\gamma - s^\gamma\big)^{-\nu} u(s)\, ds + f(t), \quad 0 \le t \le T, \tag{4.1}$$

in the role of (1.1). The integral operator

$$(V_{\varphi_{\beta,\gamma,\nu}} u)(t) = \int_0^t t^{-\alpha} s^{-\beta} \big(t^\gamma - s^\gamma\big)^{-\nu} u(s)\, ds, \quad 0 \le t \le T,$$

is cordial with the core

$$\varphi_{\beta,\gamma,\nu}(x) = x^{-\beta}(1 - x^\gamma)^{-\nu}, \quad 0 < x < 1, \tag{4.2}$$

provided that the real parameters α, β, γ, ν satisfy the conditions

$$\gamma > 0, \qquad 0 \le \nu < 1, \qquad \beta < 1, \qquad \alpha + \beta + \gamma\nu = 1. \tag{4.3}$$

The first three conditions in (4.3) imply that $\varphi_{\beta,\gamma,\nu} \in L^1(0,1)$ whereas the equality $\alpha + \beta + \gamma\nu = 1$ determines α as a function of the other parameters β, γ, ν and is responsible for the cordiality, i.e., for the equality $(V_{\varphi_{\beta,\gamma,\nu}} u)(t) = \int_0^t t^{-1}\varphi_{\beta,\gamma,\nu}(t^{-1}s)u(s)\, ds$. It holds [10, 11]

$$\sigma_0(V_{\varphi_{\beta,\gamma,\nu}}) \cap \mathbb{R} = \big[0, \|V_{\varphi_{\beta,\gamma,\nu}}\|\big], \quad \|V_{\varphi_{\beta,\gamma,\nu}}\| = \frac{1}{\gamma}\,\frac{\Gamma(1-\nu)\Gamma(\frac{1-\beta}{\gamma})}{\Gamma(1-\nu+\frac{1-\beta}{\gamma})}.$$

In the case $\nu = 0$ we have

$$\alpha + \beta = 1, \quad \frac{1}{\gamma}\,\frac{\Gamma(1-\nu)\Gamma(\frac{1-\beta}{\gamma})}{\Gamma(1-\nu+\frac{1-\beta}{\gamma})} = \frac{1}{1-\beta} = \frac{1}{\alpha}.$$

Theorem 4. *For all choices of the collocation parameters τ_1, τ_2, satisfying $0 \le \tau_1 < \tau_2 \le 1$, and all values of $\mu \in \varrho_0(V_{\varphi_{\beta,\gamma,\nu}}) \cap \mathbb{R}$ (i.e., for $\mu < 0$ as well as for $\mu > \|V_{\varphi_{\beta,\gamma,\nu}}\|$), the piecewise linear collocation method is applicable to equation* (4.1)*, with conditions* (4.3)*, and hence convergent.*

Proof. The function $\varphi_{\beta,\gamma,\nu}(x) > 0$ is smooth for $0 < x < 1$ and satisfies conditions (3.3) and (3.5):

$$\varphi_{\beta,\gamma,\nu}(x) + x\varphi'_{\beta,\gamma,\nu}(x) = (1-\beta)\varphi_{\beta,\gamma,\nu}(x) + \gamma\nu x^{\gamma-\beta}(1-x^\gamma)^{-\nu-1} > 0,$$
$$(1-x)\varphi_{\beta,\gamma,\nu}(x) \sim \gamma^{-\nu}(1-x)^{1-\nu} \to 0 \quad \text{as } x \to 1.$$

An application of Theorem 3 completes the proof. □

Consider now the following equation

$$\mu u(t) = \int_0^t t^{-\alpha} s^{\alpha-1} u(s)\, ds + f(t), \quad 0 \le t \le T, \tag{4.4}$$

which was first introduced in [5]; there a (fourth order) Hermite type collocation method based on cubic splines in the continuity class C^1 was considered, where the approximate solution was required to satisfy both the original and the differentiated form of the equation at the mesh points. Further studies for (4.4) include [2, 4].

We see that (4.4) is a special case of the cordial equation (4.1), with conditions (4.3) being satisfied by the parameter values $\nu = 0$, $\alpha + \beta = 1$, and with core $\varphi_\alpha(x) = x^{\alpha-1}$ $(0 < x < 1)$, $\alpha > 0$. For the spectrum of the operator

$$(V_{\varphi_\alpha} u)(t) = \int_0^t t^{-\alpha} s^{\alpha-1} u(s)\, ds,$$

formula (1.2) yields (see [10])

$$\sigma_0(V_{\varphi_\alpha}) = \left\{ \lambda \in \mathbb{C}\colon \left(\operatorname{Re}\lambda - \frac{1}{2\alpha} \right)^2 + (\operatorname{Im}\lambda)^2 \le \frac{1}{4\alpha^2} \right\},$$
$$\sigma_0(V_{\varphi_\alpha}) \cap \mathbb{R} = \big[0, \|V_{\varphi_\alpha}\|\big], \quad \|V_{\varphi_\alpha}\| = 1/\alpha. \tag{4.5}$$

By Theorem 4, the piecewise linear collocation method is applicable to equation (4.4) and hence convergent, for any real μ satisfying either $\mu < 0$ or $\mu > 1/\alpha$. Below we extend this result to complex $\mu \in \varrho_0(V_{\varphi_\alpha})$.

Theorem 5. *For all choices of the collocation parameters τ_1, τ_2, satisfying $0 \le \tau_1 < \tau_2 \le 1$, and any $\mu \in \varrho_0(V_{\varphi_\alpha})$, the piecewise linear collocation method is applicable to equation* (4.4) *and hence convergent.*

Proof. We have to show that, for all choices of the collocation parameters τ_1, τ_2, $0 \le \tau_1 < \tau_2 \le 1$, and all i, $1 \le i \le N-1$, it holds $\mu_1^{(i)}, \mu_2^{(i)} \in \sigma_0(V_{\varphi_\alpha})$. For real $\mu_1^{(i)}$, $\mu_2^{(i)}$ we know this from previous considerations, so let us consider the case of complex

$$\mu_j^{(i)} = \frac{b_i \pm (b_i^2 - 4(\tau_2 - \tau_1)c_i)^{1/2}}{2(\tau_2 - \tau_1)}, \qquad b_i^2 < 4(\tau_2 - \tau_1)c_i,$$

with

$$\operatorname{Re}\mu_j^{(i)} = \frac{b_i}{2(\tau_2 - \tau_1)}, \qquad \big(\operatorname{Im}\mu_j^{(i)}\big)^2 = \frac{4(\tau_2 - \tau_1)c_i - b_i^2}{4(\tau_2 - \tau_1)^2}, \quad j = 1, 2.$$

According to (4.5), the inclusions $\mu_j^{(i)} \in \sigma_0(V_{\varphi_\alpha})$, $j = 1, 2$, hold iff

$$\left(\frac{b_i}{2(\tau_2 - \tau_1)} - \frac{1}{2\alpha} \right)^2 + \frac{4(\tau_2 - \tau_1)c_i - b_i^2}{4(\tau_2 - \tau_1)^2} \le \frac{1}{4\alpha^2},$$

i.e., $b_i - 2\alpha c_i \ge 0$. Now denote $x_j := x_j^{(i)} = i/(i + \tau_j)$, $j = 1, 2$, and $g(x_1, x_2) := i^{-1} x_1 x_2 (b_i - 2\alpha c_i)$. Then the last inequality is equivalent to

$$g(x_1, x_2) = x_1(a_{11} + a_{22} - 2\alpha a_{11} a_{22}) - x_2(a_{12} + a_{21} - 2\alpha a_{12} a_{21}) \ge 0 \tag{4.6}$$

for $1/2 \le 1/(i+1) \le x_2 < x_1 \le 1$, $1 \le i \le N-1$; arguing for a fixed i, we omit the upper index also in the $a^{(i)}_{jk}$ to obtain

$$a_{11} = \int_{x_1}^1 \varphi_\alpha(x)\,dx = \frac{1-x_1^\alpha}{\alpha}, \qquad a_{12} = \int_{x_1}^1 x\varphi_\alpha(x)\,dx = \frac{1-x_1^{\alpha+1}}{\alpha+1},$$
$$a_{21} = \int_{x_2}^1 \varphi_\alpha(x)\,dx = \frac{1-x_2^\alpha}{\alpha}, \qquad a_{22} = \int_{x_2}^1 x\varphi_\alpha(x)\,dx = \frac{1-x_2^{\alpha+1}}{\alpha+1}.$$

We see that a_{11} and a_{12} are functions of x_1 whereas a_{21} and a_{22} are functions of x_2. For $1/2 \le x_2 < 1$, we have

$$g(x_1,x_2)|_{x_1=x_2} = 0, \quad g(1,x_2) = a_{22} - x_2 a_{21} = \int_{x_2}^1 (x-x_2)\varphi_\alpha(x)\,dx > 0. \tag{4.7}$$

Further, elementary evaluations yield

$$(\partial/\partial x_1)g = a_{11} + a_{22} - 2\alpha a_{11}a_{22} + \big\{x_2(1-2\alpha a_{21}) - (1-2\alpha a_{22})\big\},$$
$$(\partial/\partial x_1)g|_{x_1=x_2} = \frac{1-x_2^2+\alpha^2(1-x_2)x_2^\alpha}{\alpha(\alpha+1)} > 0 \quad \text{for } 1/2 \le x_2 < 1, \tag{4.8}$$
$$(\partial/\partial x_1)^2 g = x_1^{\alpha-1}\big(\alpha(1-x_2)-1\big) \quad \text{for } 1/2 \le x_2 < x_1 \le 1. \tag{4.9}$$

According to (4.9), $(\partial/\partial x_1)^2 g$ does not change its sign for $x_1 \in (x_2, 1]$ and fixed x_2. Together with (4.7) and (4.8) this implies (4.6). □

5 Applicability of Piecewise Quadratic Collocation

Some of the results of Sections 3 and 4 can be extended to piecewise quadratic collocation, by using the following lemma.

Lemma 4. *Let $\varphi \in L^1(0,1)$, $\mu \in \varrho_0(V_\varphi)$, $m \ge 1$. Assume that, for any choice of the collocation parameters $0 \le \tau_1 < \cdots < \tau_m \le 1$, the homogeneous equation $\mu v_N = P_N V_{\varphi_1} v_N$, with $\varphi_1(x) = x\varphi(x)$, has only the trivial solution $v_N = 0$, i.e., the spline collocation method of degree $m-1$, with arbitrary collocation parameters $\tau_1, \dots, \tau_m$ $(0 \le \tau_1 < \cdots < \tau_m \le 1)$, is applicable to equation $\mu u = V_{\varphi_1}u + f$. Denote by $P'_N : C[0,T] \to \mathcal{S}_N^{m+1}[0,T]$ the interpolation operator corresponding to the collocation parameters $\tau'_1, \dots, \tau'_{m+1}$, with $\tau'_1 = 0$ and any $0 < \tau'_2 < \cdots < \tau'_{m+1} \le 1$. Then the homogeneous equation $\mu u_N = P'_N V_\varphi u_N$ has only the trivial solution $u_N = 0$, i.e., the spline collocation method of degree m, with $\tau'_1 = 0$ and arbitrary $\tau'_2, \dots, \tau'_{m+1}$ $(0 < \tau'_2 < \cdots < \tau'_{m+1} \le 1)$ is applicable to equation* (1.1)*.*

Proof. Let $u_N^0 \in \mathcal{S}_N^{m+1}[0,T]$ be a solution to equation $\mu u_N = P'_N V_\varphi u_N$. We have to show that $u_N^0 = 0$. Since $\mu \in \varrho_0(V_\varphi)$, $u_N^0|_{[0,h]} \in \mathcal{P}_m$, and V_φ maps $\mathcal{P}_m$ into $\mathcal{P}_m$, we have $u_N^0|_{[0,h]} = 0$. Let $\ell \ge 1$ be the greatest integer such that $u_N^0|_{[0,\ell h]} = 0$, $\ell \le N$. Below we demonstrate that $\ell < N$ leads to a contradiction with the definition of ℓ, hence $\ell = N$ and $u_N^0 = 0$.

Due to the equality $\mu u_N^0 = P_N' V_\varphi u_N^0$, we have

$$\mu u_N^0(\ell h + \tau_k' h) - (V_\varphi u_N^0)(\ell h + \tau_k' h) = 0, \quad k = 1, \dots, m+1. \tag{5.1}$$

Since $\tau_1' = 0$, we have, in particular, $\mu u_{N,\ell+1}^0(\ell h) = \int_0^{\ell h} \frac{s}{\ell h} \varphi\big(\frac{s}{\ell h}\big) u_N^0(s)\, ds = 0$, where $u_{N,\ell+1}^0 := u_N^0|_{[\ell h,(\ell+1)h]}$; since $\mu \neq 0$ for $\mu \in \varrho_0(V_\varphi)$, we conclude that $u_{N,\ell+1}^0(\ell h) = 0$. Thus the left and right hand side limits of u_N^0 at the knot ℓh coincide (are equal to 0), and u_N^0 is continuous and piecewise continuously differentiable on $[0,(\ell+1)h)$. Furthermore, from (5.1) we conclude that the derivative of $\mu u_N^0 - V_\varphi u_N^0$ vanishes at some m intermediate points $\ell h + \tau_k h \in (\ell h, (\ell+1)h)$ with $\tau_k' < \tau_k < \tau_{k+1}'$, $k = 1, \dots, m$. Denoting $v_N^0(t) = \frac{d}{dt} u_N^0(t)$, we observe that

$$\begin{aligned} \frac{d}{dt}\big(V_\varphi u_N^0\big)(t) &= \frac{d}{dt} \int_0^t t^{-1} \varphi\big(t^{-1}s\big) u_N^0(s)\, ds = \frac{d}{dt} \int_0^1 \varphi(x) u_N^0(tx)\, dx \\ &= \int_0^1 x\varphi(x) v_N^0(tx)\, dx = \int_0^1 \varphi_1(x) v_N^0(tx)\, dx = \big(V_{\varphi_1} v_N^0\big)(t) \end{aligned}$$

for $0 \le t \le (\ell+1)h$. Thus

$$\mu v_N^0(\ell h + \tau_k h) - \big(V_{\varphi_1} v_N^0\big)(\ell h + \tau_k h) = 0, \quad k = 1, \dots, m,$$

hence $v_{N,\ell+1}^0 = \Pi_{N,\ell} V_{\varphi_1} v_{N,\ell+1}^0$, where $v_{N,\ell+1}^0 = v_N^0|_{[\ell h,(\ell+1)h]}$ is a polynomial of degree $m-1$, and the interpolation operator $\Pi_{N,\ell}$ corresponds to the polynomials of degree $m-1$ and the interpolation (collocation) points $\tau_1, \dots, \tau_m$, $0 < \tau_1 < \dots < \tau_m < 1$, specified above. Due to the assumption of the lemma, we get $v_{N,\ell+1}^0 = 0$ which implies $u_{N,\ell+1}^0 = 0$ and contradicts the definition of ℓ. Thus $\ell = N$ which completes the proof of Lemma 4. $\square$

With the help of Lemma 4 we extend Theorems 3, 4 and 5 for piecewise quadratic collocation as follows.

Theorem 6. *Assume $\varphi \in L^1(0,1) \cap C^1(0,1)$ is non-negative and satisfies conditions* (3.3) *and* (3.5). *Then the piecewise quadratic collocation method with $\tau_1 = 0$ and arbitrary τ_2, τ_3, satisfying $0 < \tau_2 < \tau_3 \le 1$, is, for any $\mu \in \mathbb{R} \cap \varrho_0(V_\varphi)$, applicable to equation* (1.1) *and hence convergent.*

Proof. It is easily proved that $\varphi_1(x) = x\varphi(x)$ satisfies (3.3) and (3.5). By Theorem 3, the piecewise linear collocation method with any τ_1, τ_2 ($0 \le \tau_1 < \tau_2 \le 1$) is applicable to equation $\mu u = V_{\varphi_1} u + f$, $\mu \in \mathbb{R} \cap \varrho_0(V_\varphi)$. Then, by Lemma 4, the piecewise quadratic collocation method with $\tau_1 = 0$ and arbitrary τ_2, τ_3, $0 < \tau_2 < \tau_3 \le 1$, is for $\mu \in \mathbb{R} \cap \varrho_0(V_\varphi)$ applicable to equation (1.1). $\square$

Theorem 7. *For $\tau_1 = 0$, all choices of the collocation parameters τ_2, τ_3, satisfying $0 < \tau_2 < \tau_3 \le 1$, and any $\mu \in \varrho_0(V_{\varphi_{\beta,\gamma,\nu}}) \cap \mathbb{R}$ (i.e., for $\mu < 0$ as well as for $\mu > \|V_{\varphi_{\beta,\gamma,\nu}}\|$), the piecewise quadratic collocation method is applicable to equation* (4.1) *subject to* (4.3) *and hence convergent.*

Table 1. Eq. (6.1); cubic collocation with the Gauss points.
Maximum of errors at mesh points and convergence rates.

	$\mu = -0.4$			$\mu = 4$		
N	e^N	e^N/e^{2N}	p	e^N	e^N/e^{2N}	p
4	1.898×10^{-4}	–	–	1.235×10^{-4}	–	–
8	1.126×10^{-5}	16.86	4.08	7.140×10^{-5}	17.29	4.11
16	6.658×10^{-7}	16.91	4.08	4.127×10^{-7}	17.30	4.11
32	3.959×10^{-8}	16.82	4.07	2.407×10^{-8}	17.14	4.10
64	2.374×10^{-9}	16.67	4.06	2.191×10^{-9}	16.40	4.03
128	1.44×10^{-10}	16.51	4.04	1.350×10^{-10}	16.30	4.02

Table 2. Eq. (6.1); cubic collocation with the Chebyshev points.
Maximum of errors at mesh points and convergence rates.

	$\mu = -0.4$			$\mu = 4$		
N	e^N	e^N/e^{2N}	p	e^N	e^N/e^{2N}	p
4	1.74×10^{-4}	–	–	8.503×10^{-5}	–	–
8	1.008×10^{-5}	17.26	4.11	4.988×10^{-6}	17.04	4.09
16	5.970×10^{-7}	16.89	4.08	2.981×10^{-7}	16.73	4.06
32	3.595×10^{-8}	16.61	4.05	1.807×10^{-8}	16.49	4.04
64	1.422×10^{-9}	16.93	4.08	1.107×10^{-9}	16.33	4.03
128	1.118×10^{-10}	15.82	3.98	6.824×10^{-11}	16.22	4.02

Proof. We know that the core function $\varphi_{\beta,\gamma,\nu}(x) > 0$ is smooth for $0 < x < 1$ and satisfies conditions (3.3) and (3.5) (cf. the proof of Theorem 4). Applying Theorem 6 we obtain the statement of the present theorem. □

Theorem 8. *For $\tau_1 = 0$, all choices of the collocation parameters τ_2, τ_3, satisfying $0 < \tau_2 < \tau_3 \le 1$, and any $\mu \in \varrho_0(V_{\varphi_\alpha})$ (described in (4.5)), with $\alpha > 0$, the piecewise quadratic collocation method is applicable to equation (4.4) and hence convergent.*

Proof. Noting that $x\varphi_\alpha(x) = \varphi_{\alpha+1}(x)$, we obtain the claim of the present theorem by Lemma 4 and Theorem 5. □

6 Numerical Experiments

In the first part of this section we start by presenting some results which illustrate the performance of the collocation method. Then Subsection 6.2 is devoted to the numerical investigation of the applicability of the method.

6.1 Numerical results by cubic collocation

Example 1. We consider the integral equation

$$\mu u(t) = \int_0^t s^{-1/2}(t-s)^{-1/2} u(s)\, ds + f(t), \quad 0 \le t \le 1, \tag{6.1}$$

Table 3. Eq. (6.1); cubic collocation with $c_1 = 0,\ c_2 = 1/4,\ c_3 = 1/2,\ c_4 = 3/4$. Maximum of errors at collocation points and convergence rates.

	$\mu = -0.4$			$\mu = 4$		
N	E^N	E^N/E^{2N}	p	E^N	E^N/E^{2N}	p
4	2.727×10^{-2}	–	–	6.69×10^{-5}	–	–
8	1.538×10^{-3}	17.73	4.1	4.50×10^{-6}	14.85	3.9
16	6.800×10^{-6}	22.63	4.5	2.86×10^{-7}	15.72	4.0
32	3.005×10^{-7}	22.63	4.5	1.79×10^{-8}	16.01	4.0
64	1.328×10^{-8}	22.63	4.5	1.11×10^{-9}	16.11	4.0
128	5.869×10^{-10}	22.63	4.5	6.88×10^{-11}	16.12	4.0

which is obtained from (4.1) setting $\gamma = 1$, $\beta = \nu = 1/2$ and $\alpha = 1-\beta-\gamma\nu = 0$. In the case $f(t) = (\frac{256}{315} + \mu)t^{9/2}$ its exact solution is given by $u(t) = t^{9/2}$.

Associated with (6.1) is the integral operator

$$(V_{\varphi_{\beta,\gamma,\nu}} u) = \int_0^t t^{-1/2} s^{-1/2} (t-s)^{-1/2} u(s)\, ds,$$

which is cordial with core $\varphi(x) = x^{-1/2}(1-x)^{-1/2}$ and whose spectrum is such that

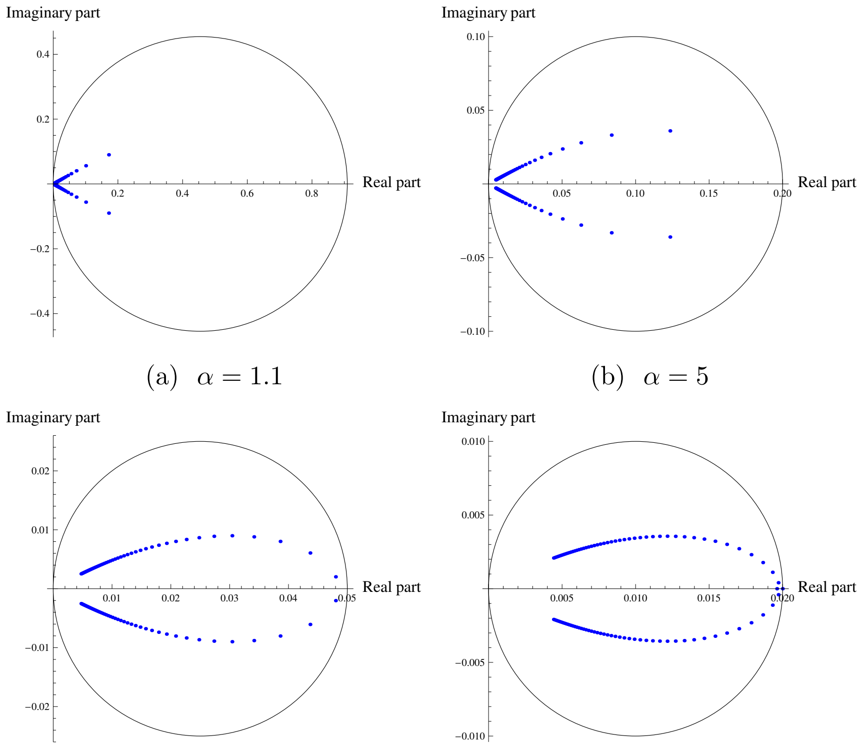
$$\sigma_0(V_\varphi) \cap \mathbb{R} = \big[0, \|V_\varphi\|\big] = \left[0, \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)}\right] = [0, \pi].$$

Let $u_N \in S_N^4$ denote the approximate solution of equation (6.1) obtained by the collocation method (1.7) based on cubic splines, that is, with $m = 4$. Denote the errors at the mesh points by $e_i^N = u_N(ih) - u(ih)$, $i = 0, 1, \ldots, N$, and $e^N = \max_{i=0,1,\ldots,N} |e_i^N|$. Similarly, we define the errors at the collocation points by $e_{il}^N = u_N((i+\tau_l)h) - u((i+\tau_l)h)$ and $E^N = \max_{i=0,1,\ldots,N} |e_{il}^N|$, for $i = 0, 1, \ldots, N,\ l = 1, 2, \ldots, m$. By p we denote the experimental convergence exponent such that $e^N \sim N^{-p}$ or $E^N \sim N^{-p}$.

In Tables 1–3 we have listed the maximum of the errors at the mesh points and at the collocation points, in the cases $\mu \in \{-0.4, 4\}$, using several choices of collocation parameters. Note that the values $\mu = -0.4$ and $\mu = 4$ are outside the interval $[0, \pi]$. The global convergence order on $[0, 1]$ is of course 4; in Table 3, for $\mu = -0.4$ we observe superconvergence behaviour at the collocation points.

6.2 Numerical check of the applicability condition

Here we are concerned with illustrating the applicability condition (1.9) of the collocation method (1.7) to equation (4.4). We have undertaken an exhaustive numerical study for different values of $m \geq 2$ and various standard collocation parameters (including equidistant, Gauss, Radau, Lobatto and Chebyshev points, among others). Some of the computational results are graphically illustrated in Figures 1–3. Recall that $\sigma_0(V_{\varphi_\alpha}) \cap \mathbb{R} = [0, 1/\alpha]$ and assume that



(a) $\alpha = 1.1$ (b) $\alpha = 5$

(c) $\alpha = 20$ (d) $\alpha = 50$

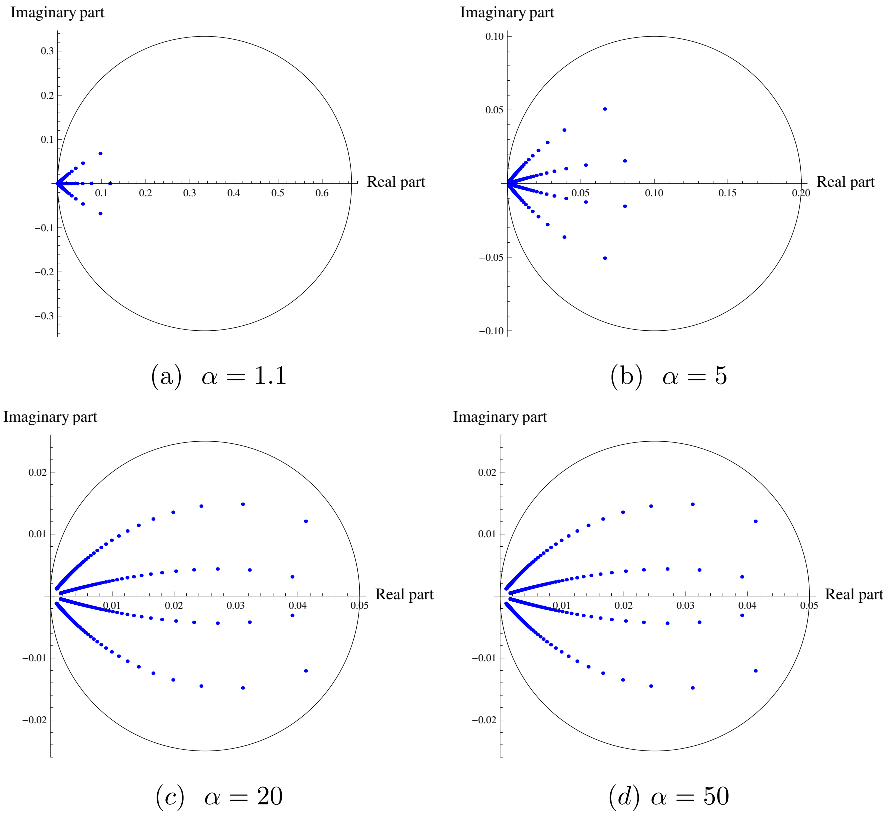
Figure 1. Eq. (4.4); eigenvalues of D_i, $i = 1, 2, \ldots$; linear spline collocation with Gauss points.

$\mu \in \rho_0(V_{\varphi_\alpha})$. The applicability condition concerns the invertibility of the operator $\mu I - \mathit{\Pi}_{N,i} V_{\varphi_\alpha}$, $i = 0, 1 \ldots, N-1$, which is equivalent to the condition $\det(\mu I_m - D_i) \neq 0$ being satisfied for all $i = 0, 1 \ldots, N-1$.

Let $c_* := \max_{0 \le x \le \tau_m} \sum_{k=1}^m |L_k(x)|$. For big i such that $c_* \int_{i/(i+\tau_m)}^1 x^{\alpha-1} dx < |\mu|$ the operator $\mu I - \mathit{\Pi}_{N,i} V_{\varphi_\alpha}$ is invertible. Let $i_{\alpha,\mu}$ be the first $i \in \mathbf{N}$ such that $c_* \int_{i/(i+\tau_m)}^1 x^{\alpha-1}\, dx < |\mu|$. Then we can conclude that the operator $\mu I - \mathit{\Pi}_{N,i} V_{\varphi_\alpha}$ is invertible for $i \ge i_{\alpha,\mu}$.

For $i = 1, \ldots, i_{\alpha,\mu} - 1$ we investigate the invertibility of of $\mu I_m - \mathit{\Pi}_{N,i} V_{\varphi_\alpha}$ by computing the eigenvalues $\mu^{(i)} \in \sigma_0(\mathit{\Pi}_{N,i} V_{\varphi_\alpha}) = \sigma(D_i)$, $i = 1, \ldots, m$. Namely, if $|\mu^{(i)} - \frac{1}{2\alpha}| < \frac{1}{2\alpha}$, that is, if $\mu^{(i)} \in \sigma(D_i)$, $i = 1, \ldots, m$, then $\mu \in \rho_0(V_{\varphi_\alpha})$ is different from $\mu^{(i)}$, $i = 1, \ldots, m$, hence the operator $\mu I_m - \mathit{\Pi}_{N,i} V_{\varphi_\alpha}$ is invertible and the collocation method is applicable to equation (4.4).

The graphics below show the distribution of the computed eigenvalues for $\alpha \in \{1.1, 5, 20, 50\}$, in the cases of linear and cubic spline collocation (Figures 1 and 2, respectively), using the Gauss points as collocation parameters; and also in the case of quintic spline collocation using the Chebyshev points (Figure 3). We can see that the $\mu^{(i)}$ belong to the circle with center in $(1/2\alpha, 0)$ and radius $1/(2\alpha)$. Moreover, in all the numerical experiments we have carried out, the



(a) $\alpha = 1.1$ (b) $\alpha = 5$

(c) $\alpha = 20$ (d) $\alpha = 50$

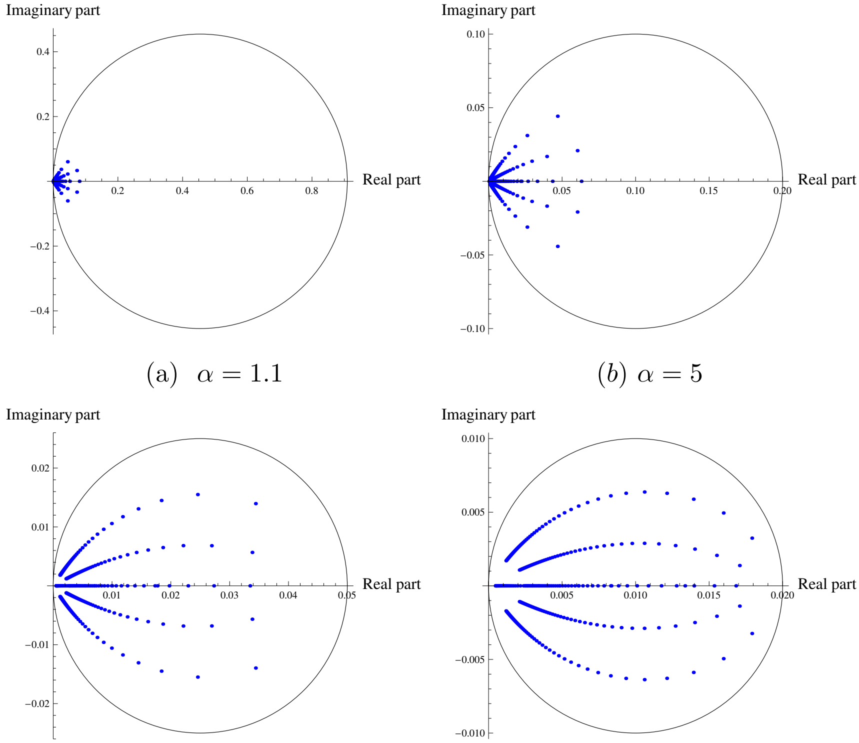
Figure 2. Eq. (4.4); eigenvalues of D_i, $i = 1, 2, \ldots$; cubic spline collocation with Gauss points.

condition (1.9) was fulfilled (in the frame of the computation accuracy).

The question of the applicability of the collocation method was not addressed in [4] and [2], where a conjecture was used that the method is always applicable to equation (4.4). This claim is trivially true for $m = 1$, $\mu \in \varrho_0(V_{\varphi_\alpha})$; Theorems 5 and 8 confirm it analytically for $m = 2$ and $m = 3$, $\mu \in \varrho_0(V_{\varphi_\alpha})$, with $\tau_1 = 0$ in the case $m = 3$. Our numerical check supports the conjecture that the spline collocation method is applicable to equation (4.4) for arbitrary $m \in \mathbb{N}$, $\mu \in \varrho_0(V_{\varphi_\alpha})$ and any choice of collocation parameters.

Acknowledgments

This work has been supported by FCT grant PTDC/MAT/101867/2008 (Portugal) and ETF grant 9104 (Estonia).



(a) $\alpha = 1.1$ (b) $\alpha = 5$

(c) $\alpha = 20$ (d) $\alpha = 50$

Figure 3. Eq. (4.4); Eigenvalues of D_i, $i = 1, 2, \ldots$; five degree spline collocation case with Chebyshev points.

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