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A Numerical Scheme for Singularly Perturbed Delay Differential Equations of Convection-Diffusion Type on an Adaptive Grid

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Abstract. In this paper, an adaptive mesh strategy is presented for solving singularly perturbed delay differential equation of convection-diffusion type using second order central finite difference scheme. Layer adaptive meshes are generated via an entropy production operator. The details of the location and width of the layer are not required in the proposed method unlike the popular layer adaptive meshes mainly by Bakhvalov and Shishkin. An extensive amount of computational work has been carried out to demonstrate the applicability of the proposed method.

Keywords: singular perturbation, entropy like variable, delay differential equation, layer adaptive meshes, central finite difference scheme, convection-diffusion problem.

AMS Subject Classification: 65L10; 65L11.

1 Introduction

A singularly perturbed delay differential equation is a differential equation in which the highest order derivative is multiplied by a small parameter and involving at least one delay term. Singular perturbation problems are generally the first approximation of the considered physical model. Hence in such cases, more realistic model should include some of the past and the future states of the system and hence, a real system should be modelled by differential equations with delay or advance. Such type of equation arises frequently in the mathematical modelling of various practical phenomena, for example, in the modelling of the human pupil-light reflex [26], model of HIV infection [10, 33], the study of bistable devices in digital electronics [11], variational problem in control theory [17, 18], first exist time problems in modelling of activation of neuronal variability [37], immune response [30], evolutionary biology [32], dynamics of networks of two identical amplifiers [9], mathematical ecology [19], population dynamics [20] and in a variety of models for physiological process [27, 28, 29]. The theory and numerical treatment of delay differential equations can be found in [7, 12].

The numerical solution of singularly perturbed delay differential equations with large delays can be found in Lange and Miura [22], Amiraliyev and Erdogan [2], Amiraliyev and Cimen [1], Amiraliyeva et al. [3], Erdogan and Amiraliyev [13], Subburayan and Ramanujam [34], Bansal et. al. [5, 6].

It is well-known that numerical methods for singularly perturbed boundary value problems have to be very carefully created, due to boundary or interior layers in the solution. Finite difference methods are always a convenient choice for solving boundary value problems because of their simplicity. Whenever central finite difference schemes are applied to solve the singularly perturbed differential equations numerically on uniform meshes, oscillations are observed in the numerical solution and their magnitude increases in layers regions. The presence of oscillations in the approximated solution shows that central finite difference operators are unstable. To eliminate these oscillations while retaining the order of accuracy, one needs a fine mesh at the layer regions. This may be done either via uniformly fine meshing or via adaptive mesh strategy. The former strategy increases significantly in computational cost as the perturbation parameter ε decreases. The use of adaptive mesh refinement techniques is now-a-days a standard component in numerical computation. In fact, Bakhvalov [4] was the first to use layer adapted meshes in the context of reaction-diffusion problems. In the late 1970s and early 1980s, the special meshes for convection-diffusion problems have been investigated by Gartland [16], Liseikin [25], Vulanovic [35, 36] and others in order to achieve uniform convergence. In early 1990s, special piecewise-uniform meshes have been proposed by Shishkin [31]. Because of their simple structure, they have attracted much attention and are widely used for numerical approximation of SPPs. The limitation with Shishkin meshes is the requirement of a priori knowledge of the location of the layer regions. However, the performance of Shishkin meshes is inferior to that of Bakhvalov meshes, which has prompted efforts to improve them while retaining some of their simplicity. A detailed survey about layer adapted meshes for convection diffusion problems can be found in [24]. The aim of this paper is to improve the performance of the adaptive mesh such that there will be more mesh points in the layer region. In this paper, we generate the layer adaptive meshes to suppress the oscillations, via an entropy production operator which is positive automatically whenever the solution is unphysical, which effectively corresponds to insufficient resolution. We restrict ourselves to the class of delay differential equations which correspond to second order boundary value problems with a non-zero coefficient of the convection term.

The outline of this paper is as follows: In Section 2, we state singularly perturbed delay differential equation of convection-diffusion type. In Section 3, we introduce an adaptive mesh and use classical finite difference scheme to solve singularly perturbed delay differential equation of convection-diffusion type. In Section 4, we provide the error analysis for the proposed method. In Section 5, three numerical examples have been solved to demonstrate the applicability and efficiency of the proposed method and confirms the theoretical estimates. The paper ends with Section 6 with a brief conclusion.

2 Statement of the problem

In this paper, we shall study the following singularly perturbed delay differential equation of convection-diffusion type:

$$Lu(x) = -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x-1) = f(x), \ \ 0 < x < 2 \tag{2.1}$$

with the interval and boundary conditions,

$$u(x) = \phi(x), \ \ x \in [-1, 0], \ \ u(2) = \beta, \tag{2.2}$$

where $0 < \varepsilon \ll 1$ and $a(x) \geq \alpha > 0$, $\beta_0 \leq b(x) < 0$, $2\alpha + 5\beta_0 \geq \gamma > 0$, $a(x), b(x), f(x)$ are given sufficiently smooth functions on $\bar{\Omega}$, $\Omega = (0, 2)$, $\bar{\Omega} = [0, 2]$, $\Omega^- = (0, 1)$, $\Omega^+ = (1, 2)$, $\phi(x)$ is a smooth function on $[-1, 0]$ and β is a given constant which is independent of ε.

3 Adaptive mesh strategy

To generate the adaptive mesh, we followed the steps of Kumar and Srinivasan [21]. We now define an auxiliary equation i.e., the entropy production equation by multiplying with an appropriate test function. From the case of scaler hyperbolic conservation laws in Leveque [23], we know that for scalar conservation laws, u^2 is always an appropriate entropy variable and, therefore, $2u(x)$ is a suitable multiplying test function. On multiplying equation (2.1) (componentwise), we obtain

$$Lu * 2u = f * 2u, \tag{3.1}$$

after simplifying, we can write the above equation (3.1) as

$$-\varepsilon(\xi'' - 2(u')^2) + a\xi' + 2buu(x-1) - 2uf = 0, \ \text{where} \ \ \xi = u^2.$$

Rewriting the above equation, we get

$$-\varepsilon\xi'' + a\xi' - 2uf + 2buu(x-1) = -2\varepsilon(u')^2. \tag{3.2}$$

We label the linear operator on the left hand side (LHS) in the above equation (3.2), as the entropy production operator with analogy to similar operators in the hyperbolic conservation laws in Leveque [23]. The continuous operator should obviously be negative for all $x \in [0, 2]$ (as the right hand side (RHS) is always negative for all $x \in [0, 2]$).

3.1 Discretization of entropy production operator

We know that if we solve the discrete problem of equation (2.1)–(2.2) computationally using central difference method, we get non-physical oscillations inside and near the boundary layer region. Similarly, if we calculate the discrete problem of the left hand side equation in (3.2) using the same central difference operator by taking $\xi_i = u_i^2$, where u_i is the central difference computed solution for equation (2.1), we observe that LHS is negative wherever the solution is smooth enough and positive where we have boundary layers (or oscillation in the computed solution of equation (2.1)). After investigation, we have found that if we write the RHS term $-2\varepsilon(u')^2$ of equation (3.2) in the difference operator at the i^{th} mesh point, we get

$$-2\varepsilon\left(\frac{u_i - u_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{u_{i+1} - u_i}{x_{i+1} - x_i}\right) \tag{3.3}$$

by applying a combination of forward and backward difference operator for $u^{'}$ which is of opposite sign wherever the oscillations occur and hence results in positive value of the difference operator as in equation (3.2) for the oscillatory numerical solution u_i. We exploit this property of the entropy production operator to precisely determine regions of insufficient resolution.

3.2 Mesh selection strategy

The following algorithm is proposed to generate an adaptive mesh for solving second order singularly perturbed delay differential equations:

Step 1: Choose an initial grid with few uniform mesh points.

Step 2: Calculate the entropy on the mesh points using (3.3).

Step 3: Identify mesh points where entropy is positive and then choose the mesh point with the maximum entropy.

Step 4: Add one mesh point on the left and the right side of the location found in Step 3 and then calculate the entropy using (3.3) at each point on newly generated non-uniform mesh.

Step 5: If the entropy is positive for at least one mesh point, go to the Step 3. Otherwise stop the iterative process. The resulting mesh is our adapted mesh which is represented by N_g.

3.3 Finite difference scheme

Initially, we discretize the interval [0,2] into few equal parts with mesh spacing h. In this case, we discretize in such a way that the delay $x = 1$ is a mesh point. As mentioned in the previous section, keep on adding the points on both sides of the mesh point where the entropy is maximum and positive in each iteration, we get a stage in which the entropy is negative at each mesh point through out the interval. We assume the final non-uniform (or adaptive mesh N_g) mesh as $\bar{\Omega}^N = \{0 = x_0 < x_1 < .... < x_{2N} = 2\}$. In each iteration, we find the index m such that $x_m = m.h_i = 1$. We discretize the delay differential equation (2.1)–(2.2), using central finite difference scheme on a non-uniform mesh as follows:

$$L^N U_i = -\varepsilon\delta^2 U(x_i) + a(x_i)D^0 U(x_i) + b(x_i)U(x_{i-m}) = f(x_i), \tag{3.4}$$
$$U(x_0) = \phi(0), \quad U(x_{2N}) = \beta,$$

where,

$$D^+V(x_i) = \frac{V(x_{i+1}) - V(x_i)}{x_{i+1} - x_i}, \quad D^-V(x_i) = \frac{V(x_i) - V(x_{i-1})}{x_i - x_{i-1}},$$
$$D^0V(x_i) = \frac{V(x_{i+1}) - V(x_{i-1})}{x_{i+1} - x_{i-1}},$$

and the second order centered difference operator δ^2 is defined by

$$\delta^2 V(x_i) = \frac{(D^+V(x_i) - D^-V(x_i))}{(x_{i+1} - x_{i-1})/2}.$$

Since, $h_i = x_{i+1} - x_i$, $h_{i-1} = x_i - x_{i-1}$ and $h_i + h_{i-1} = x_{i+1} - x_{i-1}$, the equation (3.4), can be written in the form

$$\left(\frac{-2\varepsilon}{h_{i-1}(h_i + h_{i-1})} - \frac{a_i}{h_i + h_{i-1}}\right)U_{i-1} + \left(\frac{2\varepsilon}{h_i h_{i-1}}\right)U_i$$
$$+ \left(\frac{-2\varepsilon}{h_i(h_i + h_{i-1})} + \frac{a_i}{h_i + h_{i-1}}\right)U_{i+1} + b_i U_{i-m} = f_i.$$

Multiplying both sides by $(h_i + h_{i-1})h_i h_{i-1}$, we get

$$(-2\varepsilon h_i - a_i h_i h_{i-1})U_{i-1} + 2\varepsilon(h_i + h_{i-1})U_i + (-2\varepsilon h_{i-1} + a_i h_i h_{i-1})U_{i+1}$$
$$+ b_i(h_i + h_{i-1})h_i h_{i-1} U_{i-m} = f_i(h_i + h_{i-1})h_i h_{i-1}. \tag{3.5}$$

The equation (3.5) can be rewritten in the form

$$\begin{cases} E_i U_{i-1} - F_i U_i + G_i U_{i+1} = Q_i, & \text{for } i = 1, 2, ..., m, \\ E_i U_{i-1} - F_i U_i + G_i U_{i+1} + H_i U_{i-m} = Q_i, & \text{for } i = m+1, m+2, ..., 2N-1, \end{cases} \tag{3.6}$$

with the boundary conditions,

$$U_0 = \phi(0), \quad U_{2N} = \beta, \tag{3.7}$$

where

$$E_i = -2\varepsilon h_i - a_i h_i h_{i-1}, \quad F_i = -2\varepsilon(h_i + h_{i-1}),$$
$$G_i = -2\varepsilon h_{i-1} + a_i h_i h_{i-1}, \quad H_i = b_i(h_i + h_{i-1})h_i h_{i-1},$$

and

$$Q_i = \begin{cases} (h_i + h_{i-1})h_i h_{i-1}(f_i - b_i\phi_{i-m}), & \text{for } i = 1, 2, \ldots, m, \\ f_i(h_i + h_{i-1})h_i h_{i-1}, & \text{for } i = m+1, m+2, \ldots, 2N-1. \end{cases}$$

We have solved the system of equations (3.6) with the boundary conditions (3.7) by Gauss elimination method with partial pivoting.

4 Error analysis

4.1 Truncation error

By Taylor series expansion we have,

$$u(x_i + h_i) = u(x_i) + h_i u'(x_i) + \frac{h_i^2}{2} u''(x_i) + \frac{h_i^3}{6} u'''(x_i) + \dots,$$
$$u(x_i - h_{i-1}) = u(x_i) - h_{i-1} u'(x_i) + \frac{h_{i-1}^2}{2} u''(x_i) - \frac{h_{i-1}^3}{6} u'''(x_i) + \dots.$$

The truncation error at each nodal point is given by

$$\tau_i = \left(\frac{-2\varepsilon}{h_{i-1}(h_i + h_{i-1})} - \frac{a_i}{h_i + h_{i-1}}\right) u(x_{i-1}) + \left(\frac{2\varepsilon}{h_i h_{i-1}}\right) u(x_i)$$
$$+ \left(\frac{-2\varepsilon}{h_i(h_i + h_{i-1})} + \frac{a_i}{h_i + h_{i-1}}\right) u(x_{i+1}) + b_i u(x_{i-m}) - f_i.$$

After simplification, we get the truncation error as

$$\tau_i = (h_i - h_{i-1}) \left(-\frac{\varepsilon}{3} u'''(\zeta_i) + \frac{a_i}{2} u''(\eta_i)\right).$$

As $h_i \to 0$, the truncation error tends to zero, which shows that scheme is consistent. The order of the truncation error is given by $O(h_i - h_{i-1})$.

4.2 Discretization error

Our aim is to express, if possible, the discretization error $e_i = u_i - U_i$ at the i^{th} mesh point in terms of h_i. The difference equations approximating the problem are given by

$$-\varepsilon \frac{2(D^+U_i - D^-U_i)}{h_i + h_{i-1}} + a_i \frac{U_{i+1} - U_{i-1}}{h_i + h_{i-1}} + b_i U_{i-m} - f_i = 0, \tag{4.1}$$
$$\tau_i = -\varepsilon \frac{2(D^+u_i - D^-u_i)}{h_i + h_{i-1}} + a_i \frac{u_{i+1} - u_{i-1}}{h_i + h_{i-1}} + b_i u_{i-m} - f_i. \tag{4.2}$$

Subtracting equation (4.1) from equation (4.2), we get

$$\left(\frac{-2\varepsilon}{h_{i-1}(h_i + h_{i-1})} - \frac{a_i}{h_i + h_{i-1}}\right) e_{i-1} + \left(\frac{2\varepsilon}{h_i h_{i-1}}\right) e_i$$
$$+ \left(\frac{-2\varepsilon}{h_i(h_i + h_{i-1})} + \frac{a_i}{h_i + h_{i-1}}\right) e_{i+1} + b_i e_{i-m} = \tau_i.$$

Multiplying both sides by $(h_i + h_{i-1})h_i h_{i-1}$, we get

$$(-2\varepsilon h_i - a_i h_i h_{i-1}) e_{i-1} + 2\varepsilon(h_i + h_{i-1}) e_i + (-2\varepsilon h_{i-1} + a_i h_i h_{i-1}) e_{i+1}$$
$$+ b_i (h_i + h_{i-1}) h_i h_{i-1} e_{i-m} = \tau_i (h_i + h_{i-1}) h_i h_{i-1}, \tag{4.3}$$

The error equation (4.3) can be represented as

$$E_ie_{i-1}-F_ie_i+G_ie_{i+1}+H_ie_{i-m}=(h_i^2-h_{i-1}^2)h_ih_{i-1}\Big(-\frac{\varepsilon}{3}u'''(\zeta_i)+\frac{a_i}{2}u''(\eta_i)\Big), \tag{4.4}$$

where

$$E_i=-2\varepsilon h_i-a_ih_ih_{i-1},\quad F_i=-2\varepsilon(h_i+h_{i-1}),$$
$$G_i=-2\varepsilon h_{i-1}+a_ih_ih_{i-1},\quad H_i=b_i(h_i+h_{i-1})h_ih_{i-1}.$$

It can be easily verified that

$$|F_i-E_i-G_i-H_i|=|b_i|(h_i+h_{i-1})h_ih_{i-1}>0.$$

Now from equation (4.4)

$$|F_i||e_i|\le|(h_i^2-h_{i-1}^2)|(h_ih_{i-1})|g_i|+|E_i||e_{i-1}|+|G_i||e_{i+1}|+|H_i||e_{i-m}|, \tag{4.5}$$

where

$$g_i=-\frac{\varepsilon}{3}u'''(\zeta_i)+\frac{a_i}{2}u''(\eta_i).$$

Let $\lambda=\|e\|_\infty$ and select the index i for which $|e_i|=\|e\|_\infty=\lambda$. Now from equation (4.5), we have

$$|F_i|\lambda\le|(h_i^2-h_{i-1}^2)|(h_ih_{i-1})|g_i|+|E_i|\lambda+|G_i|\lambda+|H_i|\lambda,$$
$$\lambda\{|F_i|-|E_i|-|G_i|-|H_i|\}\le|h_i^2-h_{i-1}^2|(h_ih_{i-1})|g_i|,$$
$$\lambda\le\frac{|(h_i^2-h_{i-1}^2)|(h_ih_{i-1})}{|b_i|(h_i+h_{i-1})(h_ih_{i-1})}\|g\|_\infty,\quad \lambda\le\frac{|(h_i-h_{i-1})|}{\inf|b_i|}\|g\|_\infty,$$

where

$$\|g\|_\infty\le\frac{\varepsilon}{3}\|u'''\|_\infty+|\frac{a(x)}{2}|\|u''\|_\infty,$$

which is bounded and independent of h_i. Thus $\|e\|_\infty=O(h_i-h_{i-1})$ as $h_i\to 0$. This shows the convergence of the proposed numerical scheme.

5 Numerical examples and results

In this section three illustrative examples will be studied. To demonstrate the applicability and efficiency of the proposed method, we have implemented the present method on three examples with right-end boundary layer in the interval $[0,2]$. These problems were widely discussed in the literature.

Since the exact solutions of the problems are not known, the maximum absolute errors for the examples are calculated using the following double mesh principle

$$E_\varepsilon^N=\max|U_i^N-U_{2i}^{2N}|.$$

To use the double mesh principle, we incorporated the mesh points at the middle of each mesh and applied the central finite difference scheme on variable mesh. The numerical rate of convergence is estimated by the formula

$$R^N=\log|E_\varepsilon^N/E_\varepsilon^{2N}|/\log 2.$$

Example 1. We consider singularly perturbed delay differential equation of convection-diffusion type as [8]

$$-\varepsilon u''(x) + 3u'(x) - u(x-1) = 0, \quad u(x) = 1, \quad -1 \le x \le 0, \quad u(2) = 2.$$

The exact solution of this problem is given by

$$u(x) = \begin{cases} 1 + c_1\big[\exp\big(3x/\varepsilon\big) - 1\big] + x/3, \quad 0 \le x \le 1, \\ c_2 + \frac{x}{3} + \frac{(x-1)^2}{18} + \frac{\varepsilon x}{27} - \frac{c_1 x}{3} - \frac{c_1 x}{3}\exp\left(3(x-1)/\varepsilon\right) \\ \quad + \exp\left(\frac{3(x-2)}{\varepsilon}\right)\left[\frac{23}{18} - \frac{2\varepsilon}{27} - c_2 + \frac{2c_1}{3} + \frac{2c_1}{3}\exp\left(\frac{3}{\varepsilon}\right)\right], \quad 1 \le x \le 2, \end{cases}$$

where

$$c_1 = \exp\left(-6/\varepsilon\right)\left[\frac{4\varepsilon/9 - \varepsilon^2/27 - 3}{3 - 4\exp(-6/\varepsilon) + \frac{2\varepsilon}{3}[\exp(-3/\varepsilon) - \exp(-6/\varepsilon)]}\right],$$

$$c_2 = \frac{1 - \frac{23}{18}\exp(\frac{-3}{\varepsilon}) + \frac{2\varepsilon}{27}\exp(\frac{-3}{\varepsilon}) - \frac{\varepsilon}{27}}{1 - \exp(-3/\varepsilon)} + \frac{c_1\exp(\frac{3}{\varepsilon})[1 - \exp(\frac{-3}{\varepsilon}) - \frac{2}{3}\exp(\frac{-6}{\varepsilon})]}{1 - \exp(-3/\varepsilon)}.$$

The maximum absolute errors, number of adaptive mesh generated points and the rate of convergence are presented in Table 1 for different values of perturbation parameter ε.

Table 1. Maximum absolute error for Example 1 for different values of ε with $N = 23$ (initially)

ε	Max. error	Generated mesh (N_g)	Rate of Convergence (R^N)
2^{-5}	0.0089	29	2.1924
2^{-6}	0.0089	31	2.1953
2^{-7}	0.0089	33	2.2041
2^{-8}	0.0090	35	2.2195
2^{-9}	0.0091	37	2.2377
2^{-10}	0.0092	39	2.2496
2^{-11}	0.0092	41	2.2516
2^{-12}	0.0092	43	2.2486
2^{-13}	0.0092	47	2.2454
2^{-14}	0.0092	49	2.2435
2^{-15}	0.0092	51	2.2425
2^{-16}	0.0092	53	2.2421
2^{-17}	0.0092	55	2.2419
2^{-18}	0.0092	57	2.2419
2^{-19}	0.0092	59	2.2418
2^{-20}	0.0092	63	2.2418

Example 2. We consider singularly perturbed delay differential equation of convection-diffusion type as [8]

$$-\varepsilon u''(x) + (3 + x^2)u'(x) - u(x-1) = \exp x,$$
$$u(x) = \exp x, \quad -1 \le x \le 0, \quad u(2) = 2.$$

The maximum absolute errors, number of adaptive mesh generated points and the rate of convergence are presented in Table 2 for different values of perturbation parameter ε.

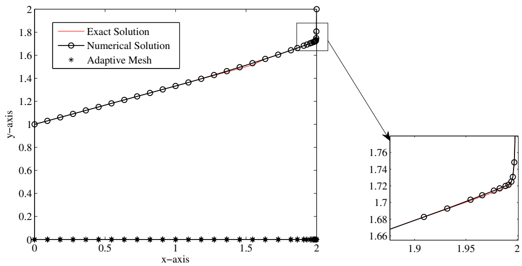

Figure 1. Numerical solution for Example 1 with $\varepsilon = 2^{-8}$, $N = 23$ (initially)

Table 2. Maximum absolute error for Example 2 for different values of ε with $N = 23$ (initially)

ε	Max. error	Generated mesh (N_g)	Rate of Convergence (R^N)
2^{-5}	0.0257	39	2.2519
2^{-6}	0.0256	41	2.2508
2^{-7}	0.0254	43	2.2431
2^{-8}	0.0252	45	2.2358
2^{-9}	0.0250	47	2.2453
2^{-10}	0.0250	47	2.2695
2^{-11}	0.0250	49	2.2917
2^{-12}	0.0250	49	2.3044
2^{-13}	0.0250	51	2.3100
2^{-14}	0.0250	53	2.3121
2^{-15}	0.0250	53	2.3129
2^{-16}	0.0250	55	2.3132
2^{-17}	0.0250	57	2.3133
2^{-18}	0.0250	59	2.3134
2^{-19}	0.0250	61	2.3134
2^{-20}	0.0250	63	2.3134

Example 3. We consider singularly perturbed delay differential equation of convection-diffusion type [8]

$$
\begin{aligned}
& -\varepsilon u''(x) + 5u'(x) - \frac{1}{2}u(x-1) = \begin{cases} 1, & 0 < x < 1, \\ -1, & 1 < x < 2, \end{cases} \\
& u(x) = 1, \quad -1 \le x \le 0, \quad u(2) = 2.
\end{aligned}
$$

In the previous two examples, it is assumed that $f(x)$ is continuous on $[0, 2]$. Motivated by the works of [14,15], in this example, we suppose that $f(x)$ has a simple discontinuity at $x = 1$, that is $f(1-) \neq f(1+)$, where $f(1-)$ and $f(1+)$ are left and right limits respectively. This boundary value problem exhibits a strong boundary layer at $x = 2$ and an interior week layer at $x = 1$.

The maximum absolute errors, number of adaptive mesh generated points and the rate of convergence are presented in Table 3 for different values of perturbation parameter ε. It is observed from the Table 3 that after $\varepsilon = 2^{-9}$ the rate of convergence has been reduced to almost 1, which is due to the effect

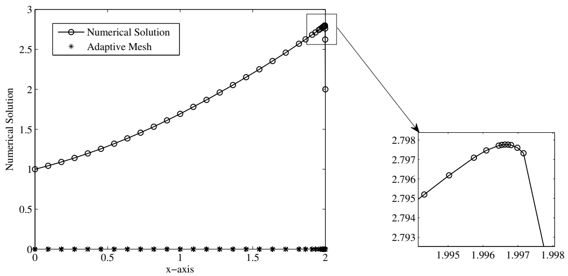

Figure 2. Numerical solution for Example 2 with $\varepsilon = 2^{-8}$, $N = 23$ (initially)

of weak interior layer at $x = 1$. From the Figure 3, it is clear that some mesh points are added around $x = 1$ to resolve the weak boundary layer and some mesh points are added around $x = 2$ to resolve the strong boundary layer.

Table 3. Maximum absolute error for Example 3 for different values of ε with $N = 23$ (initially)

ε	Max. error	Generated mesh (N_g)	Rate of Convergence (R^N)
2^{-5}	0.0720	43	2.1185
2^{-6}	0.0719	47	2.1180
2^{-7}	0.0718	49	2.1180
2^{-8}	0.0718	53	2.1177
2^{-9}	0.0717	53	2.1177
2^{-10}	0.0414	39	1.1016
2^{-11}	0.0392	41	0.9597
2^{-12}	0.0379	43	0.9034
2^{-13}	0.0376	45	0.8961
2^{-14}	0.0380	47	0.9167
2^{-15}	0.0382	49	0.9281
2^{-16}	0.0383	51	0.9341
2^{-17}	0.0384	53	0.9372
2^{-18}	0.0384	55	0.9387
2^{-19}	0.0384	57	0.9395
2^{-20}	0.0384	59	0.9399

6 Conclusions

In this paper, we have presented an adaptive mesh method to deal with oscillation produced by central finite difference method, when applied to convection dominated singularly perturbed delay differential equations. The method is analysed for convergence. The proposed method is an ε-uniform convergent. An extensive amount of computational work has been carried out to demonstrate the proposed method. The maximum absolute error is tabulated in the form of Tables 1–3 for the considered examples in support of the predicted theory. The graphs of the solution of the considered examples for certain values of

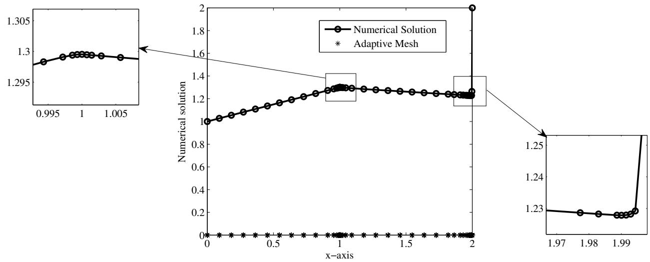

Figure 3. Numerical solution for Example 3 with $\varepsilon = 2^{-8}$, $N = 23$ (initially)

perturbation parameter are plotted in Figures 1–3. On the basis of the numerical results of a variety of examples, it is concluded that the present method offers significant advantage for the linear singularly perturbed delay differential equations.

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