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Pseudo-Differential Operators and Equations in a Discrete Half-Space

Alexander V. Vasilyev[a], Vladimir B. Vasilyev[a]

[a] *Belgorod National Research University*
Studencheskaya 14/1, Belgorod 308007, Russia
E-mail: alexvassel@gmail.com
E-mail(*corresp.*): vladimir.b.vasilyev@gmail.com

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Abstract. We introduce a digital pseudo-differential operator acting in discrete Sobolev–Slobodetskii spaces and consider pseudo-differential equations with such operators in a discrete half-space. The theorem on a general solution of such equations is proved for a special case.

Keywords: discrete functional space, digital distribution, digital pseudo-differential operator, discrete pseudo-differential equation, general solution.

AMS Subject Classification: 35S15; 47G30.

1 Introduction

A certain theory of pseudo-differential operators and corresponding equations was constructed in the second half of the last century [3, 11, 16, 17], and it includes as usual boundedness theorems in different functional spaces and a certain variant of symbolic calculus. But for discrete situation there is no any variant of such a theory although there are a lot of approximate constructions for solving simplest kinds of pseudo-differential equations, for example singular integral and similar equations [1, 6, 7, 8, 9, 10, 13, 15, 18]. Moreover, there are some recent studies for these discrete situations from algebraic or symbolic calculus point of view on the whole m-dimensional lattice $\mathbb{Z}^m$ [2, 14]. But there are principal difficulties to transfer this approach to another discrete domains which are not $\mathbb{Z}^m$, for example a discrete half-space or a discrete cone.

We think to exclude this lacuna and to start studying these discrete analogues of pseudo-differential operators and equations. We also believe that such a discrete theory will help us to justify approximate solving schemes for these equations.

Earlier the authors obtained some initial results for special discrete pseudo-differential operators and equations, namely Calderon–Zygmund operators [19, 22, 23] including some comparison between discrete and continuous situations [21]. Moreover, we have some initial results for studying pseudo-differential equations and related boundary value problems for discrete domains in m-dimensional space which are different from $\mathbb{Z}^m$ [24, 26, 27]. Using the small parameter $h > 0$ we hope to obtain existing theory of pseudo-differential operators and boundary value problems on manifolds with a boundary passing to the limit $h \to 0$, to justify constructing approximate solutions, and to get error estimates between continuous and discrete solutions in appropriate discrete functional spaces.

The main goal of this paper is to prove a theorem on a structure of a general solution for a model discrete elliptic pseudo-differential equation in a discrete half-space.

2 Discrete Sobolev–Slobodetskii spaces

2.1 Discrete Fourier transform

We will use the following notations. Let $\mathbb{T}^m$ be the m-dimensional cube $[-\pi, \pi]^m, h > 0, \hbar = h^{-1}$. We will consider all functions defined on a cube as periodic functions in $\mathbb{R}^m$ with the same cube of periods.

If $u_d(\tilde{x}), \tilde{x} \in h\mathbb{Z}^m$, is a function of a discrete variable, then we call it "discrete function". For such discrete functions one can define the discrete Fourier transform

$$(F_d\, u_d)(\xi) \equiv \tilde{u}_d(\xi) = \sum_{\tilde{x} \in h\mathbb{Z}^m} e^{-i\tilde{x}\cdot\xi} u_d(\tilde{x}) h^m, \quad \xi \in \hbar\mathbb{T}^m,$$

if the latter series converges, and the function $\tilde{u}_d(\xi)$ is a periodic function on $\mathbb{R}^m$ with the basic cube of periods $\hbar\mathbb{T}^m$. This discrete Fourier transform preserves basic properties of the integral Fourier transform, particularly the inverse discrete Fourier transform is given by the formula

$$(F_d^{-1}\tilde{u}_d)(\tilde{x}) = \frac{1}{(2\pi)^m} \int_{\hbar\mathbb{T}^m} e^{i\tilde{x}\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in h\mathbb{Z}^m.$$

The discrete Fourier transform is a one-to-one correspondence between the spaces $L_2(h\mathbb{Z}^m)$ and $L_2(\hbar\mathbb{T}^m)$ with norms

$$||u_d||_2 = \left(\sum_{\tilde{x} \in h\mathbb{Z}^m} |u_d(\tilde{x})|^2 h^m \right)^{1/2}, \quad ||\tilde{u}_d||_2 = \left(\int_{\xi \in \hbar\mathbb{T}^m} |\tilde{u}_d(\xi)|^2 d\xi \right)^{1/2}.$$

Example 1. Since the definition for Sobolev–Slobodetskii spaces includes partial derivatives, we use their discrete analogue, i.e. divided difference of first order

$$(\Delta_k^{(1)} u_d)(\tilde{x}) = h^{-1}(u_d(x_1, \ldots, x_k + h, \ldots, x_m) - u_d(x_1, \ldots, x_k, \ldots, x_m)),$$

for which its discrete Fourier transform looks as follows

$$\widetilde{(\Delta_k^{(1)} u_d)}(\xi) = h^{-1}(e^{-ih\cdot\xi_k} - 1)\tilde{u}_d(\xi).$$

Further for the divided difference of second order we have

$$(\Delta_k^{(2)} u_d)(\tilde{x}) = h^{-2}(u_d(x_1, \dots, x_k + 2h, \dots, x_m) \\ - 2u_d(x_1, \dots, x_k + h, \dots, x_m) + u_d(x_1, \dots, x_k, \dots, x_m))$$

and its discrete Fourier transform

$$\widetilde{(\Delta_k^{(2)} u_d)}(\xi) = h^{-2}(e^{-ih\cdot\xi_k} - 1)^2\tilde{u}_d(\xi).$$

Thus, for the discrete Laplacian we have

$$(\Delta_d u_d)(\tilde{x}) = \sum_{k=1}^{m} (\Delta_k^{(2)} u_d)(\tilde{x}),$$

so that

$$\widetilde{(\Delta_d u_d)}(\xi) = h^{-2}\sum_{k=1}^{m}(e^{-ih\cdot\xi_k} - 1)^2\tilde{u}_d(\xi).$$

We will use the discrete Fourier transform to introduce special discrete Sobolev–Slobodetskii spaces which are very convenient for studying discrete pseudo-differential operators and related equations.

2.2 Definitions and notations

2.2.1 Discrete spaces and digital distributions

Now we will introduce the basic space $S(h\mathbb{Z}^m)$ which consists of discrete functions with finite semi-norms

$$|u_d| = \sup_{\tilde{x}\in h\mathbb{Z}^m} (1 + |\tilde{x}|)^l |\Delta^{(\mathbf{k})} u_d(\tilde{x})|$$

for arbitrary $l \in \mathbb{N}, \mathbf{k} = (k_1, \dots, k_m), k_r \in \mathbb{N}, r = 1, \dots, m$, where

$$\Delta^{(\mathbf{k})} u_d(\tilde{x}) = \Delta_1^{k_1} \dots, \Delta_m^{k_m} u_d(\tilde{x}).$$

In other words, the space $S(h\mathbb{Z}^m)$ is a discrete analogue of the Schwartz space $S(\mathbb{R}^m)$ of infinitely differentiable rapidly decreasing at infinity functions. Usually the space of distributions over the basic space $S(\mathbb{R}^m)$ is denoted by $S'(\mathbb{R}^m)$.

Digital distribution we call an arbitrary linear continuous functional defined on $S(h\mathbb{Z}^m)$. A set of such digital distributions we will denote by $S'(h\mathbb{Z}^m)$, and a value of the functional f_d on the basic function u_d will be denoted by (f_d, u_d).

Together with the space $S(h\mathbb{Z}^m)$ we consider the space $D(h\mathbb{Z}^m)$ consisting of discrete functions with a compact (finite) support. We say that $f_d = 0$ in the discrete domain $M_d \equiv M \cap h\mathbb{Z}^m, M \subset \mathbb{R}^m$, if $(f_d, u_d) = 0, \forall u_d \in D(M_d)$, where

$D(M_d) \subset D(h\mathbb{Z}^m)$ consists of discrete functions whose supports belong to M_d. If we will denote $\widetilde{M_d}$ a union of such M_d, where $f_d = 0$ then by definition *supp* $f_d = h\mathbb{Z}^m \setminus \widetilde{M_d}$.

As usual [28] we can define some simplest operations in the space $S'(h\mathbb{Z}^m)$ excluding the differentiation (see below), and a convergence is defined as a weak convergence in the space of functionals $S'(h\mathbb{Z}^m)$.

If $f_d(\tilde{x})$ is a local summable function then one can define the digital distribution f_d by the formula

$$(f_d, u_d) = \sum_{\tilde{x} \in h\mathbb{Z}^m} f_d(\tilde{x}) u_d(\tilde{x}) h^m, \quad \forall u_d \in S(h\mathbb{Z}^m). \tag{2.1}$$

Such distributions we call *regular* digital distributions. But there are so-called *singular* digital distributions like the Dirac mass-function $(\delta_d, u_d) = u_d(0)$, which can not be represented by the above formula (2.1).

2.2.2 Digital distributions and the Liouville theorem

Here we will consider our discrete functions from distribution point of view [28]. For simplicity we consider one-dimensional case because a multidimensional situation will be almost the same.

A multiplication by basic function. If $\varphi(\tilde{x})$ is a discrete function such that for some l

$$|\varphi(\tilde{x})| \leq c|\tilde{x}|^l, \quad \forall \tilde{x} \in h\mathbb{Z}^m,$$

one can define the discrete distribution φf_d for arbitrary $f_d \in S'(h\mathbb{Z}^m)$ by the formula

$$(\varphi f_d, u_d) = (f_d, \varphi u_d), \quad \forall u_d \in S(h\mathbb{Z}^m).$$

A shift. If $f_d, u_d \in S(h\mathbb{Z})$, we can define the shift $(T_h f_d)(\tilde{x}) \equiv f_d(\tilde{x} + h)$ by the following formula

$$\begin{aligned}(T_h f_d, u_d) &= \sum_{\tilde{x} \in h\mathbb{Z}^m} (T_h f_d)(\tilde{x}) u_d(\tilde{x}) h = \sum_{\tilde{x} \in h\mathbb{Z}^m} f_d(\tilde{x} + h) u_d(\tilde{x}) h \\ &= \sum_{\tilde{x} \in h\mathbb{Z}^m} f_d(\tilde{x}) u_d(\tilde{x} - h) h = \sum_{\tilde{x} \in h\mathbb{Z}^m} f_d(\tilde{x}) (T_{-h} u_d)(\tilde{x}) h = (f_d, T_{-h} u_d),\end{aligned}$$

so we can take the following definition for a shift of digital distribution

$$(T_h f_d, u_d) = (f_d, T_{-h} u_d), \quad \forall u_d \in S(h\mathbb{Z}^m). \tag{2.2}$$

A difference operator. For $u_d \in S(h\mathbb{Z}^m)$ the difference operators of first order are defined

$$(\Delta_+^{(1)} u_d)(\tilde{x}) = \frac{1}{h}(u_d(\tilde{x} + h) - u_d(\tilde{x})), \quad (\Delta_-^{(1)} u_d)(\tilde{x}) = \frac{1}{h}(u_d(\tilde{x} - h) - u_d(\tilde{x})),$$

and thus according to (2.2) we can write for $f_d \in S(h\mathbb{Z})$

$$(\Delta_+^{(1)} f_d, u_d) = \sum_{\tilde{x} \in h\mathbb{Z}} (\Delta^{(1)} f_d)(\tilde{x}) u_d(\tilde{x}) h = \frac{1}{h} \sum_{\tilde{x} \in h\mathbb{Z}} f_d(\tilde{x} + h) u_d(\tilde{x}) h$$

$$-\frac{1}{h}\sum_{\tilde{x}\in h\mathbb{Z}} f_d(\tilde{x})u_d(\tilde{x})h = \frac{1}{h}\sum_{\tilde{y}\in h\mathbb{Z}} f_d(\tilde{y})u_d(\tilde{y}-h)h$$
$$-\frac{1}{h}\sum_{\tilde{y}\in h\mathbb{Z}} f_d(\tilde{y})u_d(\tilde{y})h = \sum_{\tilde{y}\in h\mathbb{Z}} f_d(\tilde{y})(\varDelta^{(1)}u_d)(\tilde{y})h = (f_d, \varDelta_-^{(1)}u_d).$$

It implies the following

Definition 1. For digital distribution $f_d \in S'(h\mathbb{Z})$ the digital distribution $\varDelta^{(1)}f_d$ is defined by the formula

$$(\varDelta_+^{(1)}f_d, u_d) = (f_d, \varDelta_-^{(1)}u_d), \quad \forall u_d \in S(h\mathbb{Z}).$$

Below we will not distinguish $\varDelta_\pm$. One can define the divided difference of k-th order $\varDelta^{(k)}f_d$ for a digital distribution f_d by induction

$$\varDelta^{(k)}f_d = \varDelta^{(1)}(\varDelta^{(k-1)}f_d).$$

We need some difference analogue for a digital distribution supported at the origin. To obtain these properties we need some preliminary results, these are discrete analogues of Schwartz's theorems [28].

Proposition 1. *$f_d \in S'(h\mathbb{Z}^m)$ iff there exist a positive number C and integer $p \geq 0$ such that for arbitrary $u_d \in S(h\mathbb{Z}^m)$ the following inequality*

$$|(f_d, u_d)| \leq C|u_d|_p$$

holds, where

$$|u_d|_p = \sup_{k\leq p,\ \tilde{x}\in h\mathbb{Z}^m} (1+|\tilde{x}|)^p |(\varDelta^{(k)}u_d)(\tilde{x})|.$$

Proof. We will prove the necessity only because one can prove the immediately. Let $f_d \in S'(h\mathbb{Z}^m)$. We will prove this property by contradiction and suppose that there are no such numbers C and p. Then there is a sequence $\{u_{d,k}\}_{k=1}^{\infty}, u_{d,k} \in S(h\mathbb{Z}^m)$, such that

$$|(f_d, u_{d,k})| \geq k|u_{d,k}|_k. \tag{2.3}$$

The following sequence

$$v_{d,k}(\tilde{x}) = \frac{u_{d,k}(\tilde{x})}{\sqrt{k}|u_{d,k}|_k}, \quad k = 1, 2, \ldots$$

tends to zero in $S(h\mathbb{Z}^m)$ since for $k \geq s, k \geq r$ we have

$$|\tilde{x}^s \varDelta^{(r)} v_{d,k}(\tilde{x})| = \frac{|\tilde{x}^s \varDelta^{(s)} u_{d,k}(\tilde{x})|}{\sqrt{k}|u_{d,k}|_k} \leq \frac{1}{\sqrt{k}}.$$

Since the functional f_d is continuous in $S(h\mathbb{Z}^m)$, we obtain

$$\lim_{k\to\infty} (f_d, v_{d,k}) = 0.$$

On the other hand, we obtain from (2.3)

$$|(f_d, v_{d,k})| = \frac{|(f_d, u_{d,k})|}{\sqrt{k}|u_{d,k}|_k} \geq \sqrt{k}.$$

This contradiction proves the Proposition 1. □

Lemma 1. *If a digital distribution $f_d \in S'(h\mathbb{Z})$ is supported at zero then it is a finite span of divided differences of f up to n-th order. In other words*

$$f_d(\tilde{x}) = \sum_{k=0}^{n} c_k (\Delta^{(k)}\delta_d)(\tilde{x}).$$

Proof. Since *supp* $f_d = \{0\}$, then for arbitrary $k > 0$

$$f_d = \varphi(k\tilde{x}) f_d, \tag{2.4}$$

where $\varphi(\tilde{x}) \in S(h\mathbb{Z}^m)$ is equal to 1 in some neighbourhood of 0, and equals to 0 for $|\tilde{x}| > 1$. According to the Proposition 1, we have

$$|(f_d, u_d)| \leq C|u_d|_n, \quad \forall u_d \in S(h\mathbb{Z}^m) \tag{2.5}$$

for some $C > 0, n \geq 0$, non-depending on u_d.

For arbitrary $u_d \in S(h\mathbb{Z}^m)$ we set

$$u_{d,n}(\tilde{x}) = u_{d,n}(\tilde{x}) - \sum_{l=0}^{n} \frac{(\Delta^{(l)} u_d)(0)}{l!}\tilde{x}^l, \quad v_k(\tilde{x}) = u_{d,n}(\tilde{x})\varphi(k\tilde{x}).$$

Taking into account that

$$(\Delta^{(r)} u_{d,n})(\tilde{x}) = O(|\tilde{x}|^{n+1-r}), \quad \tilde{x} \to \infty \ (r \leq n),$$
$$(\Delta^{(s)}\varphi)(k\tilde{x}) = O(k^s), \quad k \to \infty,$$

and applying (2.5) to $v_k(\tilde{x})$ we obtain

$$\begin{aligned}
|(f_d, v_k)| \leq C|v_k|_n = C \sup_{l \leq n, |\tilde{x}| \leq \frac{1}{k}} &(1+|\tilde{x}|)^n |\Delta^{(l)}\left(u_{d,n}(\tilde{x})\varphi(k\tilde{x})\right) \\
&\leq C_1 \max_{l \leq n, |\tilde{x}| \leq \frac{1}{k}} \sum_{s=0}^{l} |\Delta^{(s)} u_{d,n}(\tilde{x})||\Delta^{(l-s)}\varphi(k\tilde{x})| \\
&\leq C_2 \max_{l \leq n} \sum_{s=0}^{l} k^{-n-1+s} k^{l-s} = \frac{C_3}{k} \to 0, \quad k \to \infty.
\end{aligned}$$

But according to (2.4) (f_d, v_k) does not depend on k. Thus, we have

$$(f_d, v_1) = \lim_{k \to \infty} (f_d, v_k) = 0.$$

Therefore, using (2.4) for $k = 1$ we obtain the following representation

$$(f_d, u_d) = (\varphi f_d, u_d) = (f_d, \varphi u_d) = (f_d, v_1 + \sum_{l=0}^{n} \frac{(\Delta^{(l)} u_d)(0)}{l!} \tilde{x}^l)$$
$$= (f_d, v_1) + \sum_{l=0}^{n} \frac{(\Delta^{(l)} u_d)(0)}{l!} (f_d, \tilde{x}^l \varphi(\tilde{x})) = \sum_{l=0}^{n} C_l (\Delta^{(l)} \delta_d, u_d),$$

where we set $C_l = (f_d, \tilde{x}^l \varphi)$. One can easily prove a uniqueness of such representation. $\square$

The Fourier transform. Let us note that every digital distribution $f_d \in S'(h\mathbb{Z})$ can be treated as a distribution $f_d \in S'(\mathbb{R})$ supported on $h\mathbb{Z}$. Since the Fourier transform for a distribution f_d is defined by the standard formula

$$(F f_d, u) = (f_d, F u), \quad \forall u \in S(\mathbb{R}),$$

then we have

$$(F \Delta^{(1)} f_d, u) = (f_d, \Delta^{(1)} F u).$$

Now we will calculate the last Fourier transform. For $u \in S(\mathbb{R})$ we have

$$(\Delta^{(1)} \tilde{u})(\xi) = \frac{1}{h}(\tilde{u}(\xi + h) - \tilde{u}(\xi)) = \frac{1}{h} \int_{-\infty}^{+\infty} (e^{-ihx} - 1) e^{-ix\xi} u(x) dx,$$

so that for $f_d \in S'(\mathbb{R})$

$$(f_d, \Delta^{(1)} F u) = (f_d, F(\frac{e^{-ihx} - 1}{h} u(x))$$
$$= (F f_d, \frac{e^{-ihx} - 1}{h} u) = (\frac{e^{-ih\xi} - 1}{h} F f_d, u).$$

If $f_d \in S(h\mathbb{Z})$, then

$$(F_d \Delta_+^{(1)} f_d)(\xi) = \sum_{\tilde{x} \in h\mathbb{Z}} e^{-i\tilde{x}\cdot\xi} \frac{f_d(\tilde{x} + h) - f_d(\tilde{x})}{h} h = \frac{e^{-ih\xi} - 1}{h} (F_d f_d)(\xi),$$

and the latter formula is agreed with above calculations.

Corollary 1. For the digital distribution

$$f_d(\tilde{x}) = \sum_{k=0}^{n} c_k (\Delta^{(k)} \delta_d)(\tilde{x})$$

we have the Fourier transform $\tilde{f}_d(\xi) = \sum\limits_{k=0}^{n} c_k \zeta^k$, where $\zeta = \hbar(e^{-ih\xi} - 1)$.

Remark 1. We use the term "Liouville theorem" because such functions are related with holomorphy properties their Fourier transforms (see, for example, [3, 25]).

2.2.3 Discrete $\mathbf{H^s}$-spaces

Let us denote $\zeta^2 = h^{-2}\sum\limits_{k=1}^{m}(e^{-ih\cdot\xi_k}-1)^2$ and introduce the following

Definition 2. The space $H^s(h\mathbb{Z}^m)$ is a closure of the space $S(h\mathbb{Z}^m)$ with respect to the norm

$$||u_d||_s = \left(\int_{\hbar\mathbb{T}^m}(1+|\zeta^2|)^s|\tilde{u}_d(\xi)|^2 d\xi\right)^{1/2}. \tag{2.6}$$

We would like to note that a lot of properties for such spaces were studied in [4].

Further, let $D \subset \mathbb{R}^m$ be a domain, and $D_d = D \cap h\mathbb{Z}^m$ be a discrete domain.

Definition 3. The space $H^s(D_d)$ consists of discrete functions from $H^s(h\mathbb{Z}^m)$ which supports belong to $\overline{D_d}$. A norm in the space $H^s(D_d)$ is induced by a norm of the space $H^s(h\mathbb{Z}^m)$. The space $H_0^s(D_d)$ consists of discrete functions u_d with a support in D_d, and these discrete functions should admit a continuation into the whole $H^s(h\mathbb{Z}^m)$. A norm in the $H_0^s(D_d)$ is given by the formula

$$||u_d||_s^+ = \inf||\ell u_d||_s,$$

where infimum is taken over all continuations ℓ.

The Fourier image of the space $H^s(D_d)$ will be denoted by $\widetilde{H}^s(D_d)$. Such spaces were studied in detail in the paper [4]. Of course, all norms (2.6) are equivalent to the L_2-norm but this equivalence depends on h. Let us note that all constants below in our considerations do not depend on h.

3 Digital pseudo-differential operators and discrete equations

3.1 Operators and equations

Let $\widetilde{A}_d(\xi)$ be a periodic function in $\mathbb{R}^m$ with the basic cube of periods $\hbar\mathbb{T}^m$. Such functions are called symbols. As usual, we will define a digital pseudo-differential operator by its symbol.

Definition 4. A digital pseudo-differential operator A_d in a discrete domain D_d is called an operator of the following kind

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in h\mathbb{Z}^m}\int_{\hbar\mathbb{T}^m}\widetilde{A}_d(\xi)e^{i(\tilde{x}-\tilde{y})\cdot\xi}\tilde{u}_d(\xi)d\xi, \quad \tilde{x}\in D_d.$$

An operator A_d is called an elliptic operator if

$$ess \inf_{\xi\in\hbar\mathbb{T}^m}|\widetilde{A}_d(\xi)| > 0.$$

First, as usual, we define the operator A_d on the dense set $S(h\mathbb{Z}^m)$ and then extend it on more general space.

Remark 2. One can introduce the symbol $\widetilde{A}_d(\tilde{x},\xi)$ depending on a spatial variable $\tilde{x}$ and define a general pseudo-differential operator by the formula

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in h\mathbb{Z}^m} \int_{\hbar\mathbb{T}^m} \widetilde{A}_d(\tilde{x},\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x}\in D_d.$$

For studying such operators and related equations one needs to use more fine and complicated technique.

DEFINITION 5. By definition the class E_α includes symbols satisfying the following condition

$$c_1(1+|\zeta^2|)^{\alpha/2} \le |A_d(\xi)| \le c_2(1+|\zeta^2|)^{\alpha/2} \tag{3.1}$$

with universal positive constants c_1, c_2 non-depending on h and the symbol $A_d(\xi)$. The number $\alpha\in\mathbb{R}$ is called an order of a digital pseudo-differential operator A_d.

Obviously, operator A_d satisfying (3.1) is an elliptic operator. Using the last definition one can easily get the following property.

Lemma 2. *A digital pseudo-differential operator $A_d\in E_\alpha$ is a linear bounded operator $H^s(h\mathbb{Z}^m)\to H^{s-\alpha}(h\mathbb{Z}^m)$.*

We study the equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x}\in D_d, \tag{3.2}$$

assuming that we are interested in a solution $u_d\in H^s(D_d)$, taking into account $v_d\in H_0^{s-\alpha}(D_d)$.

Main difficulty for this problem is related to a geometry of the domain D. Indeed, if $D=\mathbb{R}^m$ then the condition (3.1) guarantees the unique solvability for the equation (3.2). We will consider here only so-called canonical domains and simplest digital pseudo-differential operators with symbols non-depending on a spatial variable $\tilde{x}$. This fact is dictated by using in future the local principle. The last asserts that for a Fredholm solvability of the general equation (3.2) with symbol $A_d(\tilde{x},\xi)$ in an arbitrary discrete domain D_d, one needs to obtain invertibility conditions for so-called local representatives of the operator A_d, i.e. for an operator with symbol $A_d(\cdot,\xi)$ in a special canonical domain.

Earlier authors have extracted some canonical domains, namely $D=\mathbb{R}^m$, $\mathbb{R}^m_+$, C^a_+, where $\mathbb{R}^m_+ = \{x\in\mathbb{R}^m : x=(x',x_m), x_m>0\}, C^a_+ = \{x\in\mathbb{R}^m : x_m > a|x'|, a>0\}$. Methods for studying two last cases are related to special boundary value problems for holomorphic functions [19, 20, 22, 24, 26, 27].

Everywhere below we study the case $D=\mathbb{R}^m_+$.

3.2 Periodic Riemann boundary value problem

For studying the discrete half-space case we need a special technique like continue case [3, 5, 12]. It was found for this case [20, 22] the periodic analogue of the Hilbert transform [3, 5, 11, 12] with the parameter ξ'

$$(H^{per}_{\xi'}\tilde{u}_d)(\xi',\xi_m) = \frac{1}{2\pi i} v.p. \int_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(\xi_m-\eta_m)}{2} \tilde{u}_d(\xi',\eta_m) d\eta_m,$$

where

$$v.p. \int_{-\hbar\pi}^{\hbar\pi} \cot \frac{h(\xi_m - \eta_m)}{2} \tilde{u}_d(\xi', \eta_m) d\eta_m$$
$$= \lim_{\varepsilon \to 0+} \left(\int_{-\hbar\pi}^{\xi_m - \varepsilon} + \int_{\xi_m + \varepsilon}^{\hbar\pi} \right) \cot \frac{h(\xi_m - \eta_m)}{2} \tilde{u}_d(\xi', \eta_m) d\eta_m.$$

This operator generates two projectors

$$P_{\xi'}^{per} = \frac{1}{2}(I + H_{\xi'}^{per}), \quad Q_{\xi'}^{per} = \frac{1}{2}(I - H_{\xi'}^{per}),$$

which permit to formulate and solve the following problem.

Let us denote $\Pi_\pm$ half-strips in the complex plane $\mathbb{C}$

$$\Pi_\pm = \{z \in \mathbb{C} : z = s + i\tau, s \in [-\pi, \pi], \pm\tau > 0\},$$

and let $H^\pm(\hbar\mathbb{T}^m) \subset L_2(\hbar\mathbb{T}^m)$ be subspaces of functions $u(\xi', \xi_m \pm i\tau)$ which admit holomorphic continuation in the strips $\hbar\Pi_\pm$ and satisfy the condition

$$\int_{-\hbar\pi}^{\hbar\pi} |u(\xi', \xi_m \pm i\tau)|^2 d\xi_m < +\infty, \quad \forall \tau > 0, \xi' \in \hbar\mathbb{T}^{m-1}.$$

A statement of the problem: find two functions $\Phi^\pm \in H^\pm(\hbar\mathbb{T}^m)$, which satisfy the linear relation

$$\Phi^+(\xi) = G(\xi)\Phi^-(\xi) + g(\xi), \tag{3.3}$$

where $G(\xi), g(\xi)$ are given functions defined on $\hbar\mathbb{T}^m$.

If $G(\xi) \equiv 1$ then the problem (3.3) is called a jump problem. For $g(\xi) \in L_2(\hbar\mathbb{T}^m)$ the jump problem has unique solution [19, 20, 22]

$$\Phi^+ = P_{\xi'}^{per} g, \quad \Phi^- = -Q_{\xi'}^{per} g.$$

The last assertion correspond to the unique representation as the direct sum

$$L_2(\hbar\mathbb{T}^m) = H^+(\hbar\mathbb{T}^m) \oplus H^-(\hbar\mathbb{T}^m).$$

This fact can be generalized for more wide spaces $H^s(\hbar\mathbb{T}^m)$ using a boundedness of the operator $H_{\xi'}^{per}$ in such spaces for small $|s| < 1/2$ (see also Theorem 1 below).

4 A general solution

4.1 Index of factorization

To study the general Riemann boundary value problem we will use the following concept.

DEFINITION 6. Periodic factorization of an elliptic symbol $A_d(\xi) \in E_\alpha$ is called its representation in the form

$$A_d(\xi) = A_{d,+}(\xi) A_{d,-}(\xi),$$

where the factors $A_{d,\pm}(\xi)$ admit an analytical continuation into half-strips $\hbar\Pi_\pm$ on the last variable ξ_m for almost all fixed $\xi' \in \hbar\mathbb{T}^{m-1}$ and satisfy the estimates

$$|A^{\pm 1}_{d,+}(\xi)| \leq c_1(1+|\hat{\zeta}^2|)^{\pm\frac{æ}{2}}, \quad |A^{\pm 1}_{d,-}(\xi)| \leq c_2(1+|\hat{\zeta}^2|)^{\pm\frac{\alpha - æ}{2}}$$

with constants c_1, c_2 non-depending on h,

$$\hat{\zeta}^2 \equiv \hbar^2 \left(\sum_{k=1}^{m-1} (e^{-ih\xi_k} - 1)^2 + (e^{-ih(\xi_m + i\tau)} - 1)^2 \right), \quad \xi_m + i\tau \in \hbar\Pi_\pm.$$

The number $æ \in \mathbb{R}$ is called an index of periodic factorization.

Remark 3. For an elliptic symbol $A_d(\xi)$, such periodic factorization always exists(see [3, 20]).

For some simple cases one can use the topological formula [3, 20]

$$æ = \frac{1}{2\pi} \int_{-\hbar\pi}^{\hbar\pi} d \arg A_d(\cdot, \xi_m),$$

where $A_d(\cdot, \xi_m)$ means that $\xi' \in \hbar\mathbb{T}^{m-1}$ is fixed, and the integral is the integral in Stieltjes sense. It means that we need to calculate divided by 2π variation of the argument of the symbol $A_d(\xi)$ when ξ_m varies from $-\hbar\pi$ to $\hbar\pi$ under fixed ξ'.

Example 2. Let $A_d(\xi) = k^2 + \hat{\xi}^2, k \in \mathbb{R}$, such that the condition (3.1) is satisfied, in other words A_d is the discrete Laplacian plus $k^2 I$. The variation of an argument mentioned above can be calculated immediately, and it equals to 1.

As we will see the index of factorization very influences on the solvability picture of the equation (3.1). For special case we have the following result.

Theorem 1. *If the elliptic symbol $\tilde{A}_d(\xi) \in E_\alpha$ admits periodic factorization with index æ so that $|æ - s| < 1/2$, then the the equation* (3.2) *has unique solution in the space $H^s(D_d)$ for arbitrary right-hand side $v_d \in H^{s-\alpha}(D_d)$,*

$$\tilde{u}_d(\xi) = \tilde{A}^{-1}_{d,+}(\xi) P^{per}_{\xi'}(\tilde{A}^{-1}_{d,-}(\xi) \widetilde{\ell v}_d(\xi)). \tag{4.1}$$

Remark 4. It is easy to see that the solution does not depend on choice of continuation ℓv_d.

Here we consider more complicated case when the condition $|æ - s| < 1/2$ does not hold. There are two possibilities in this situation, and we consider one case which leads to typical boundary value problems.

Theorem 2. *Let* $æ - s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$. *Then a general solution of the equation* (3.2) *in Fourier images has the following form*

$$\tilde{u}_d(\xi) = \tilde{A}^{-1}_{d,+}(\xi) X_n(\xi) P^{per}_{\xi'}(X_n^{-1}(\xi)\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi)) + \tilde{A}^{-1}_{d,+}(\xi)\sum_{k=0}^{n-1} c_k(\xi')\hat{\zeta}^k_m,$$

where $X_n(\xi)$ *is an arbitrary polynomial of order* n *of variables* $\hat{\zeta}_k = \hbar(e^{-ih\xi_k} - 1), k = 1, \dots, m$, *satisfying the condition* (3.1), $c_k(\xi'), j = 0, 1, \dots, n-1$, *are arbitrary functions from* $H^{s_k}(h\mathbb{T}^{m-1}), s_k = s - æ + k - 1/2$.

The a priori estimate

$$||u_d||_s \le a\big(||f||^+_{s-\alpha} + \sum_{k=0}^{n-1}[c_k]_{s_k}\big)$$

holds, where $[\cdot]_{s_k}$ *denotes a norm in the space* $H^{s_k}(h\mathbb{T}^{m-1})$, *and the constant* a *does not depend on* h.

Proof. We will use factorization method proving the theorem according to [3], although the same statement can be obtained by the method of periodic Riemann boundary value problem [19, 20, 22]. Since $v_d \in H^{s-\alpha}_0(Q_d)$, we can continue it to $lv_d \in H^{s-\alpha}(h\mathbb{Z}^m)$. Let us introduce

$$w_d(\tilde{x}) = lv_d(\tilde{x}) - (A_d u_d)(\tilde{x}),$$

so that $w_d(\tilde{x}) \equiv 0, \ \forall \tilde{x} \in D_d$. Further we write

$$(A_d u_d)(\tilde{x}) + w_d(\tilde{x}) = lv_d(\tilde{x})$$

and apply the discrete Fourier transform

$$A_d(\xi)\tilde{u}_d(\xi) + \tilde{w}_d(\xi) = \widetilde{lv_d}(\xi).$$

After factorization of our symbol $A_d(\xi)$, we have

$$A_{d,+}(\xi)\tilde{u}_d(\xi) + A^{-1}_{d,-}(\xi)\tilde{w}_d(\xi) = A^{-1}_{d,-}(\xi)\widetilde{lv_d}(\xi).$$

Now we need to study functional spaces in the last equality. Since $\widetilde{lv_d}(\xi) \in \widetilde{H}^{s-\alpha}(h\mathbb{Z}^m)$, then according to properties of $A^{-1}_{d,-}(\xi)$ we obtain $A^{-1}_{d,-}(\xi)\widetilde{lv_d}(\xi) \in \widetilde{H}^{s-æ}(h\mathbb{Z}^m)$. Let $X_n(\xi)$ be an arbitrary polynomial of order n of variables $\hat{\zeta}_k = \hbar(e^{-ih\xi_k} - 1), k = 1, \dots, m$, satisfying the condition (3.1).

Then $X_n^{-1}(\xi)A^{-1}_{d,-}(\xi)\widetilde{lv_d}(\xi) \in \widetilde{H}^{-\delta}(h\mathbb{Z}^m)$, so that we can write the following decomposition

$$X_n^{-1}(\xi)A^{-1}_{d,-}(\xi)\widetilde{lv_d}(\xi) = f_+(\xi) + f_-(\xi),$$

where

$$f_+(\xi) = (P^{per}_{\xi'}(X_n^{-1}A^{-1}_{d,-}\widetilde{lv_d}))(\xi), \quad f_-(\xi) = (Q^{per}_{\xi'}(X_n^{-1}A^{-1}_{d,-}\widetilde{lv_d}))(\xi),$$

according to the jump problem and Theorem 1. Moreover, $f_+ \in \widetilde{H}^{-\delta}(Q_d), f_- \in \widetilde{H}^{-\delta}(h\mathbb{Z}^m \setminus Q_d)$. Therefore,

$$A_{d,+}(\xi)\tilde{u}_d(\xi) + A^{-1}_{d,-}(\xi)\tilde{w}_d(\xi) = X_n(\xi)f_+(\xi) + X_n(\xi)f_-(\xi),$$

or in other words,

$$A_{d,+}(\xi)\tilde{u}_d(\xi) - X_n(\xi)f_+(\xi) = X_n(\xi)f_-(\xi) - A^{-1}_{d,-}(\xi)\tilde{w}_d(\xi).$$

Thus, we have that the left-hand side of the last equality belongs to $\widetilde{H}^{s-æ}(Q_d)$, and the right-hand side belongs to $\widetilde{H}^{s-æ}(h\mathbb{Z}^m \setminus Q_d)$. Now, if we take inverse discrete Fourier transform for both left-hand side and right-hand one we obtain that these are discrete distribution supported on the discrete hyper-plane $h\mathbb{Z}^{m-1}$. Therefore, according to Lemma 1, we obtain

$$A_{d,+}(\xi)\tilde{u}_d(\xi) - X_n(\xi)f_+(\xi) = \sum_{k=0}^{n} c_k(\xi')\hat{\zeta}^k_m,$$

or after re-writing

$$\tilde{u}_d(\xi) = \tilde{A}^{-1}_{d,+}(\xi)X_n(\xi)P^{per}_{\xi'}(X^{-1}_n(\xi)\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi)) + \tilde{A}^{-1}_{d,+}(\xi)\sum_{k=0}^{n} c_k(\xi')\hat{\zeta}^k_m.$$

The left question is how much summands we need in the right-hand side. Counting principle is a very simple because every summand should belong to the space $\widetilde{H}^s(\hbar\mathbb{T}^m)$.

Let us consider the summand $c_k(\xi')\hat{\zeta}^k_m$. Taking into account that order of $A^{-1}_{d,+}(\xi)$ is $-æ$, we need to verify the finiteness of the $H^{s-æ}$-norm for $c_k(\xi')\hat{\zeta}^k_m$. We have

$$\begin{aligned}
||c_k(\Delta^{(k)}_m\delta)||^2_{s-æ} &= \int_{\hbar\mathbb{T}^m}(1+|\zeta^2_h|)^{s-æ}||c_k(\xi')\hat{\zeta}^k_m|^2 d\xi \\
&= \int_{\hbar\mathbb{T}^m}(1+|\zeta^2_h|)^{s-æ}||c_k(\xi')|^2|\hat{\zeta}^k_m|^2 d\xi \leq a_1\hbar^{2(s-æ+k+1/2)} \\
&\times \int_{\hbar\mathbb{T}^{m-1}}|c_k(\xi')|^2 d\xi' \leq a_2\int_{\hbar\mathbb{T}^{m-1}}(1+|{\zeta'}^2_h|)^{s-æ+k+1/2}|c_k(\xi')|^2 d\xi',
\end{aligned}$$

where $\xi' = (\xi_1,\dots,\xi_{m-1}), {\zeta'}^2_h = \hbar^2\sum\limits_{k=1}^{m-1}(e^{-ih\xi_k}-1)^2$, and the constants a_1, a_2 do not depend on h. The last summand should be $(n-1)$-th because for n-th summand we obtain a positive growth: for $k=n$ we have $s_n = s-æ-n+1/2 = -n-\delta+n+1/2 = -\delta+1/2 > 0$. $\square$

Corollary 2. Let $æ - s = n+\delta, \in \mathbb{N}, |\delta| < 1/2,\ v_d \equiv 0$. A general solution of the equation (3.2) has the following form

$$\tilde{u}_d(\tilde{x}', \tilde{x}_m) = \tilde{A}^{-1}_{d,+}(\xi)\sum_{k=0}^{n-1} c_k(\xi')\hat{\zeta}^k_m.$$

The Theorem 2 implies that if we want to have a unique solution in the case $æ - s = n+\delta, n \in \mathbb{N}, |\delta| < 1/2$, we need some additional conditions to determine uniquely unknown functions $c_k(\xi'), k = 0, 1, \dots, n-1$.

5 Conclusions

The proof of the Theorem 1 and a solvability theorems for simple boundary value problems for the equation (3.2) will appear in Springer Proc. Math. & Stat. Consideration and constructions for the left case $æ - s = -n + \delta, n \in \mathbb{N}, |\delta| < 1/2$, will appear in Tatra Mt. Math. Publ.

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