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A New Domain Decomposition Parallel Algorithm for Convection–Diffusion Problem*

Jiansong Zhang[a] **and Danping Yang**[b]

[a] *Department of Applied Mathematics, China University of Petroleum*
266580 Qingdao, China
[b] *Department of Mathematics, East China Normal University*
200062 Shanghai, China
E-mail(*corresp.*): `upczjs@gmail.com`
E-mail: `dpyang@math.ecnu.edu.cn`

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Abstract. Basing on overlapping domain decomposition, we construct a new parallel algorithm combined the method of subspace correction with least-squares procedure for solving time-dependent convection–diffusion problem. This algorithm is fully parallel. We analyze the convergence of approximate solution, and study the dependence of the convergent rate on the spacial mesh size, time increment, iteration number and sub-domains overlapping degree. Both theoretical analysis and numerical results suggest that only one or two iterations are needed to reach to given accuracy at each time step.

Keywords: parallel algorithm, finite element method, convection–diffusion problem, convergence analysis.

AMS Subject Classification: 65M55; 65M60.

1 Introduction

With the rapid development of super parallel computers and parallel algorithms, parallel computation based upon domain decomposition has become a powerful tool for solving a large-scale system of partial differential equations. Overlapping domain decomposition methods are one kind of important numerical methods widely used in engineering. For elliptic problems, many numerical methods based on overlapping domain decomposition have been developed. The earliest overlapping domain decomposition method is Schwarz alternating method. The classical Schwarz alternating method is successive. To parallelize the Schwarz alternating method, many new ideas and techniques have been introduced and developed, such as parallel multilevel precondition algorithms,

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additive Schwarz methods, parallel subspace correction (PSC) methods, parallel weighted Schwarz algorithms, substructuring domain decomposition algorithms and etc., (for example, see [2, 3, 4, 9, 10, 11, 13, 19, 20, 21, 22]). Recently, overlapping domain decomposition algorithms are applied to time-dependent parabolic problems (see [5, 6, 12, 14, 15, 16, 17, 25]). Generally speaking, by using difference method in time, time-dependent problems can be discretized as a set of elliptic problems at each time step. One can use any parallel algorithms based on overlapping domain decomposition, which are effective for elliptic problems, to solve these resulted elliptic problems step by step in time. In [5, 6], Cai proposed several additive Schwarz algorithms to solve parabolic problems. In [17], Tai applied parallel weighted Schwarz algorithms to parabolic equations and analyzed iterative number needed to reach given accuracy at each time step. In [15, 16], Rui and Yang proposed and analyzed classical Schwarz algorithm of parabolic problems and proved a convergent rate depended on mesh size. In [14, 25], Sun and Yang gave improved D-D type methods for parabolic problems and proved an almost optimal error estimate, without the factor $H^{-\frac{1}{2}}$ given in Dawson–Dupont's error estimate [8]. But all of these parallel algorithms are iterative algorithms so that many iteration steps are needed to reach given accuracy, which produce much more global amount of computational works. On the basis of the idea of the parallel subspace correction method, the authors established a new family parallel algorithm combined with characteristic finite element scheme and characteristic finite difference scheme for convection–diffusion problem in [26, 27], where the partition functions of unity was used to distribute the corrections in the overlapping domains reasonably. Both theoretical analysis and numerical results suggest that when overlapping degree has a positive lower bound independent of mesh size, only one or two iterations is needed to reach the optimal convergence precision at each time level.

It is well known, parallel subspace correction method for the symmetric positive definite system is similar to the Jacobi method. The idea is to correct the residue equation on each subspace in parallel. The least-squares method for partial differential equations has two typical advantages: it is not subject to LBB condition and it results in symmetric positive definite system. For sufficiently using the advantages of these methods, we combined the least-squares method with parallel subspace correction method and constructed a new parallel algorithm for solving multi-dimensional convection–diffusion problem. We analyze the convergence of approximate solution, and study the dependence of the convergent rate on the spacial mesh size, time increment, iteration number and sub-domains overlapping degree. Both theoretical analysis and numerical experiments indicate the full parallelization and very good approximate property of the algorithm.

2 Formulation of Parallel Algorithm

Let Ω be an open bounded domain $\mathbf{R}^d$ $(1 \leq d \leq 3)$, with a Lipschitz continuous boundary $\Gamma = \Gamma_D \cup \Gamma_N$. To illustrate our method, we consider the following initial–boundary value problem of a first-order time-dependent convection–

diffusion problem: $x \in \Omega$, $0 < t \le T$,

$$\begin{cases} c(x)\dfrac{\partial u(x,t)}{\partial t} + \nabla \cdot \sigma(x,t) + q(x,t)u(x,t) = f(x,t), \\ \sigma(x,t) + A(x)\nabla u(x,t) + \mathbf{b}(x,t)u(x,t) = 0, \\ u(x,t) = 0, \quad x \in \Gamma_D, \quad \sigma(x,t)\cdot\nu(x) = 0, \quad x \in \Gamma_N, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{2.1}$$

where flow field $\mathbf{b} = (b_1(x,t), b_2(x,t), \ldots, b_d(x,t))^T$, source term $q = q(x,t) \ge 0$ and exterior flow function $f = f(x,t)$ in (2.1) are some given functions, the coefficient $c = c(x)$ is positive function and the diffusion coefficient matrix $A = (a(i,j))_{d\times d}$ is a symmetric uniformly positive definite matrix, i.e., there exist some positive constants c_* and a_* such that

$$a_* \sum_{i=1}^{d} \xi_i^2 \le \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j, \quad c_* \le c(x), \ \forall \xi \in \mathbf{R}^d, \, x \in \Omega, \tag{2.2}$$

and ν is the unit vector normal to Γ_N.

Throughout this paper we use usual definitions and notations of Sobolev spaces as in [1]. Let $W^{k,p}(\Omega)$ ($k \ge 0$, $1 \le p \le \infty$) be Sobolev spaces defined on Ω with usual norms $\|\cdot\|_{W^{k,p}(\Omega)}$ and $H^k(\Omega) = W^{k,2}(\Omega)$. Define inner products as follows:

$$(u,v) = \int_\Omega u(x)v(x)dx \quad \forall u,v \in L^2(\Omega),$$
$$(\sigma,\omega) = \sum_{i=1}^{d} (\sigma_i,\omega_i) \quad \forall \sigma,\omega \in \left[L^2(\Omega)\right]^d.$$

Introduce the spaces $V = \{\omega \in [L^2(\Omega)]^d;\ \nabla\cdot\omega \in L^2(\Omega),\ \omega\cdot\nu = 0 \text{ on } \Gamma_N\}$ and $M = \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_D\}$. Make a time partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_J = T$ and set $\tau_n = t_n - t_{n-1}$ and $\tau = \max_{1\le n\le J}\tau_n$. Let $w^n(x) = w(x,t_n)$. By use of the difference technique with first-order accuracy to discretize the first-order system (2.1), we can rewrite the system (2.1) as the following semi-discrete form (see [23])

$$\begin{aligned} & c(x)\bar{\partial}_t u^n(x) + \nabla\cdot\sigma^n(x) + q^n(x)u^n(x) = f^n(x) + R^n(x), \\ & \sigma^n(x) + A(x)\nabla u^n(x) + \mathbf{b}^n(x)u^n(x) = 0, \\ & u^n(x) = 0, \quad x \in \Gamma_D, \qquad \sigma^n(x)\cdot\nu(x) = 0, \quad x \in \Gamma_N, \\ & u^0(x) = u_0(x), \quad x \in \Omega, \end{aligned} \tag{2.3}$$

where

$$R^n(x) = c(x)\big(\bar{\partial}_t u^n(x) - u_t(x)\big) = c(x)\left(\frac{u^n - u^{n-1}}{\tau_n} - u_t(x)\right) = O\left(\tau_n \frac{\partial^2 u}{\partial t^2}\right).$$

To construct parallel subspace correction algorithm, we firstly make a domain decomposition. Assume that $\{\Omega_i'\}_{i=1}^N$ is a non-overlapping domain decomposition of Ω. In order to obtain an overlapping domain decomposition, we extend each subregion Ω_i' to a larger region Ω_i such that $\Omega_i' \subset \Omega_i \subset \Omega$ and $dist(\partial\Omega_i'\backslash\partial\Omega, \partial\Omega_i\backslash\partial\Omega) \geq H$ for each $1 \leq i \leq N$, where $H > 0$ is called as overlapping degree. Let $\mathcal{T}_{h_u}$ and $\mathcal{T}_{h_\sigma}$ be two families of quasi-regular finite element partitions of the domain Ω such that the elements in the partitions have the diameters bounded by h_u and h_σ, respectively. Assume that $\mathcal{T}_{h_u,i} = \mathcal{T}_{h_u} \cap \Omega_i$ and $\mathcal{T}_{h_\sigma,i} = \mathcal{T}_{h_\sigma} \cap \Omega_i$ are finite element partitions of Ω_i for $1 \leq i \leq N$. Let $\mathcal{V}_{h_\sigma} \subset V$, and $\mathcal{M}_{h_u} \subset M$ be piecewise r-degree and k-degree polynomial spaces defined on the partitions $\mathcal{T}_{h_\sigma}$ and $\mathcal{T}_{h_u}$, respectively.

Define $\tilde{A} = A^{-1}$, and a bilinear form as follows

$$\begin{aligned} a_n\big((\sigma, w), (\omega, v)\big) = {} & \left(\frac{1}{c}\big(cw + \tau_n\big(\nabla\cdot\sigma + q^n w\big)\big), cv + \tau_n\big(\nabla\cdot\omega + q^n v\big)\right) \\ & + \tau_n\big(\tilde{A}\big(\sigma + A\nabla w + \mathbf{b}^n w\big), \omega + A\nabla v + \mathbf{b}^n v\big). \end{aligned}$$

From (2.3) and [23], we get the standard least-squares finite element procedure:

Least-square scheme. *Given an initial approximation $w_h^0 \in \mathcal{M}_{h_u}$. For $n = 1, 2, \ldots, J$, seek $(\varrho_h^n, w_h^n) \in \mathcal{V}_{h_\sigma} \times \mathcal{M}_{h_u}$ such that*

$$\begin{aligned} a_n\big((\varrho_h^n, w_h^n), (\omega_h, v_h)\big) = {} & \left(\frac{1}{c}\big(cw_h^{n-1} + \tau_n f^n\big), cv_h + \tau_n\big(\nabla\cdot\omega_h + q^n v_h\big)\right), \\ & \forall(\omega_h, v_h) \in \mathcal{V}_{h_\sigma} \times \mathcal{M}_{h_u}. \end{aligned} \tag{2.4}$$

In the following part of this section, we propose a parallel domain decomposition algorithm of the system (2.4). Define finite element sub-spaces:

$$\begin{aligned} \mathcal{M}_{h_u}^i &= \{v_h \in \mathcal{M}_{h_u};\ v_h = 0 \text{ in } \Omega\backslash\Omega_i\}, \quad 1 \leq i \leq N \\ \mathcal{V}_{h_\sigma}^i &= \{\sigma_h \in \mathcal{V}_{h_u};\ \sigma_h = 0 \text{ in } \Omega\backslash\Omega_i\}, \quad 1 \leq i \leq N. \end{aligned}$$

It is clear that

$$\mathcal{M}_{h_u} = \mathcal{M}_{h_u}^1 + \mathcal{M}_{h_u}^2 + \cdots + \mathcal{M}_{h_u}^N, \quad \mathcal{V}_{h_\sigma} = \mathcal{V}_{h_\sigma}^1 + \mathcal{V}_{h_\sigma}^2 + \cdots + \mathcal{V}_{h_\sigma}^N.$$

It is easily seen that there exists a finite open covering family $\{O^i\}_{i=1}^N$ of the domain Ω such that $O^i \cap \Omega \subset \Omega_i$. We know that there exists a partition of unity $\{\varphi_i\}_{i=1}^N$ (see [18, Lemma 3.4]) such that

(a) $\operatorname{supp}(\varphi_i) \subset O^i$, $0 \leq \varphi_i \leq 1$, $\|\varphi_i\|_{W^{r,\infty}} \leq CH^{-r}$, $1 \leq i \leq N$;

(b) $\varphi_1 + \varphi_2 + \cdots + \varphi_N = 1$ in Ω.

Let $\varphi_{h_u}^i$ and $\varphi_{h_\sigma}^i$ be the nodal piecewise linear interpolation of φ_i on the finite element meshes $\mathcal{T}_{h_u}$ and $\mathcal{T}_{h_\sigma}$, and $\mathcal{I}_{h_u}$ and $\mathcal{I}_{h_\sigma}$ be the interpolating operators on $\mathcal{M}_{h_u}$ and $\mathcal{V}_{h_\sigma}$.

Based on (2.4), we can define a parallel subspace correction algorithm.

Parallel Algorithm. *Let m denote the iteration number at each time step. Give an initial approximation $u_h^0 = w_h^0 \in \mathcal{M}_{h_u}$. For $n = 1, 2, \ldots, J$, seek $(\sigma_h^n, u_h^n) \in \mathcal{V}_{h_\sigma} \times \mathcal{M}_{h_u}$ by four steps:*

Step 1. *Set $(\tilde{\sigma}_0^n, \tilde{u}_0^n) = (\sigma_h^{n-1}, u_h^{n-1})$ and $j := 1$.*

Step 2. *For* $i = 1, 2, \dots, N$, *seek* $(\varepsilon_j^i, e_j^i) \in \mathcal{V}_{h_\sigma}^i \times \mathcal{M}_{h_u}^i$, *in parallel, such that*

$$\begin{aligned} a_n\big((\varepsilon_j^i, e_j^i), (\omega_h, v_h)\big) &= \bigg(\frac{1}{c}\big(cu_h^{n-1} + \tau_n f^n\big), c\mathcal{I}_{h_u}\big(\varphi_{h_u}^i v_h\big) \\ &+ \tau_n\big(\nabla\cdot\mathcal{I}_{h_\sigma}\big(\varphi_{h_\sigma}^i\omega_h\big) + q^n\mathcal{I}_{h_u}\big(\varphi_{h_u}^i v_h\big)\big)\bigg) \\ &- a_n\big((\widetilde{\sigma}_{j-1}^n, \widetilde{u}_{j-1}^n), \big(\mathcal{I}_{h_\sigma}\big(\varphi_{h_\sigma}^i\omega_h\big), \mathcal{I}_{h_u}\big(\varphi_{h_u}^i v_h\big)\big)\big), \quad \forall(\omega_h, v_h) \in \mathcal{V}_{h_\sigma} \times \mathcal{M}_{h_u}. \end{aligned} \tag{2.5}$$

Step 3. *Set*

$$\tilde{\sigma}_j^n = \tilde{\sigma}_{j-1}^n + \sum_{i=1}^N \varepsilon_j^i, \quad \tilde{u}_j^n = \tilde{u}_{j-1}^n + \sum_{i=1}^N e_j^i.$$

Step 4. *If* $j < m$, *then set* $j := j+1$ *and return the step 2; or set* $\sigma_h^n = \tilde{\sigma}_m^n$, $u_h^n = \tilde{u}_m^n$ *and then back to the first step to start iteration at the next time step.*

3 Some Lemmas

Let

$$\begin{aligned} \big\|(\omega, v)\big\|_{a_n}^2 = \bigg(\frac{1}{c}\big(cv + \tau_n(\nabla\cdot\omega + q^n v)\big), cv + \tau_n\big(\nabla\cdot\omega + q^n v\big)\bigg) \\ + \tau_n\big(\tilde{A}\big(\omega + A\nabla v + \mathbf{b}^n v\big), \omega + A\nabla v + \mathbf{b}^n v\big). \end{aligned}$$

In order to analyze the convergence of Parallel algorithm, we introduce projection operators $P_{h_\sigma}^i : \mathcal{V}_{h_\sigma} \to \mathcal{V}_{h_\sigma}^i$ and $Q_{h_u}^i : \mathcal{M}_{h_u} \to \mathcal{M}_{h_u}^i$ such that

$$\begin{gathered} a_n\big(\big(P_{h_\sigma}^i\omega, Q_{h_u}^i v\big), (\omega_h, v_h)\big) = a_n\big((\omega, v), (\omega_h, v_h)\big) \\ \forall(\omega_h, v_h) \in \mathcal{V}_{h_\sigma}^i \times \mathcal{M}_{h_u}^i,\ i = 1, 2\dots, N. \end{gathered}$$

Now, we give some important lemmas which are used to analyze the convergence of Parallel algorithm.

Lemma 1. *For any function* $\varphi \in W^{1,\infty}(\Omega)$ *and* $\omega_h \in \mathcal{V}_{h_\sigma}$, *we have the following estimate*

$$\big\|\varphi\omega_h - \mathcal{I}_{h_\sigma}(\varphi\omega_h)\big\|_{[L^2]^d} \le Kh_\sigma \min\big(\|\varphi\|_{W^{1,\infty}}\|\omega_h\|_{[L^2]^d}, \|\varphi\|_{H^1}\|\omega_h\|_{[L^\infty]^d}\big),$$

where $d = 1, 2, 3$.

Proof. For the case that $\mathcal{V}_{h_\sigma}$ is one of the classical mixed finite element spaces, Yang gave the proof in [24]. On the other hand, for the case that $\mathcal{V}_{h_\sigma}$ is a usual continuous finite element space, using the technique in [24], we can also obtain the above result. □

Lemma 2. *For $1 \le i \le N$, we have*

$$\left\|(\mathcal{I}-\mathcal{I}_{h_\sigma})\left(\varphi^i_{h_\sigma}\omega_h\right)\right\|_{[L^2]^d} \le K\frac{h_\sigma}{H}\|\omega_h\|_{[L^2]^d} \quad \forall \omega_h \in \mathcal{V}_{h_\sigma}, \tag{3.1}$$
$$\left\|(\mathcal{I}-\mathcal{I}_{h_u})\left(\varphi^i_{h_u}v_h\right)\right\|_{L^2} \le K\frac{h_u}{H}\|v_h\|_{L^2} \quad \forall v_h \in \mathcal{M}_{h_u}.$$

Proof. Using Lemma 1, we know that

$$\left\|(\mathcal{I}-\mathcal{I}_{h_\sigma})\left(\varphi^i_{h_\sigma}\omega_h\right)\right\|_{[L^2]^d} \le Kh_\sigma\left\|\varphi^i_{h_\sigma}\right\|_{W^{1,\infty}}\|\omega_h\|_{[L^2]^d} \le K\frac{h_\sigma}{H}\|\omega_h\|_{[L^2]^d}.$$

This the first inequality of (3.1) is proved. In addition, by use of the technique of Theorem 3.1 in [24], we can easily obtain

$$\begin{aligned}\left\|(\mathcal{I}-\mathcal{I}_{h_u}\left(\varphi^i_{h_u}v_h\right)\right\|_{L^2} &\le Kh_u\min\left(\left\|\varphi^i_{h_u}\right\|_{W^{1,\infty}}\|v_h\|_{L^2}, \left\|\varphi^i_{h_u}\right\|_{H^1}\|v_h\|_{L^\infty}\right)\\ &\le Kh_u\left\|\varphi^i_{h_u}\right\|_{W^{1,\infty}}\|v_h\|_{L^2} \le K\frac{h_u}{H}\|v_h\|_{L^2}.\end{aligned}$$

That is the second inequality of (3.1). The proof of Lemma 2 is completed. □

Lemma 3. *Let $h = \max(h_\sigma, h_u)$. The following estimate*

$$\begin{aligned}&\left|a_n\left((\psi,w),(\omega,v)\right) - \sum_{i=1}^N a_n((\psi,w),\left(\mathcal{I}_{h_\sigma}\left(\varphi^h_i P^i_{h_\sigma}\omega\right),\mathcal{I}_{h_u}\left(\varphi^i_{h_u}Q^i_{h_u}v\right)\right)\right|\\ &\qquad \le K\left(\frac{h}{H}+\frac{\sqrt{\tau}}{H}\right)\left\|(\psi,w)\right\|_{a_n}\left\|(\omega_h,v_h)\right\|_{a_n}\end{aligned} \tag{3.2}$$

holds for each (ψ, w) and (ω, v) in $\mathcal{V}_{h_\sigma}\times\mathcal{M}_{h_u}$.

Proof. It is easily seen that

$$\begin{aligned}&a_n\left((\psi,w),\left(\mathcal{I}_{h_\sigma}\left(\varphi^i_{h_\sigma}P^i_{h_\sigma}\omega\right),\mathcal{I}_{h_u}\left(\varphi^i_{h_u}Q^i_{h_u}v\right)\right)\right)\\ &\quad = a_n\left((\psi,w),\left(\varphi^i_{h_\sigma}P^i_{h_\sigma}\omega,\varphi^i_{h_u}Q^i_{h_u}v\right)\right)\\ &\qquad + a_n\left((\psi,w),\left((\mathcal{I}_{h_\sigma}-\mathcal{I})\left(\varphi^i_{h_\sigma}P^i_{h_\sigma}\omega\right),(\mathcal{I}_{h_u}-\mathcal{I})\left(\varphi^i_{h_u}Q^i_{h_u}v\right)\right)\right),\end{aligned}$$

$$\begin{aligned}&a_n\left((\psi,w),\left(\varphi^i_{h_\sigma}P^i_{h_\sigma}\omega,\varphi^i_{h_u}Q^i_{h_u}v\right)\right)\\ &\quad = a_n\left(\left(P^i_{h_\sigma}\left(\varphi^i_{h_\sigma}\psi\right),Q^i_{h_u}\left(\varphi^i_{h_u}w\right)\right),(\omega,v)\right)\\ &\qquad + \tau_n\Bigg[\left(\frac{1}{c}\left(cw+\tau_n\left(\nabla\cdot\psi+q^n w\right)\right),\left(P^i_{h_\sigma}\omega\right)\nabla\varphi^i_{h_\sigma}\right)\\ &\qquad - \left(\frac{1}{c}\psi\nabla\varphi^i_{h_\sigma}, cQ^i_{h_u}v+\tau_n\left(\nabla\cdot\left(P^i_{h_\sigma}\omega\right)+q^nQ^i_{h_u}v\right)\right)\\ &\qquad + \left(\tilde{A}\left(\psi+A\nabla w+\mathbf{b}^n w\right),A\nabla\varphi^h_i Q^i_{h_u}v\right)\\ &\qquad - \left(\nabla\varphi^i_{h_u}w, P^i_{h_\sigma}\omega+A\nabla\left(Q^i_{h_u}v\right)+\mathbf{b}^nQ^i_{h_u}v\right)\Bigg]\end{aligned}$$

$$a_n\left((\psi,w),(\omega,v)\right) = \sum_{i=1}^N a_n\left(\left(\varphi^i_{h_\sigma}\psi,\varphi^i_{h_u}w\right),(\omega,v)\right).$$

So,

$$\begin{aligned}
&a_n\big((\psi,w),(\omega,\ v)\big)-\sum_{i=1}^{N}a_n\big((\psi,w),\big(\mathcal{I}_{h_\sigma}\big(\varphi^i_{h_\sigma}P^i_{h_\sigma}\omega\big),\mathcal{I}_{h_u}\big(\varphi^i_{h_u}Q^i_{h_u}v\big)\big)\big)\\
&\quad=\sum_{i=1}^{N}a_n\big(\big(\big(\mathcal{I}-P^i_{h_\sigma}\big)\big(\varphi^i_{h_\sigma}\psi\big),\big(\mathcal{I}-Q^i_{h_u}\big)\big(\varphi^i_{h_u}w\big)\big),(\omega,v)\big)\\
&\qquad-\sum_{i=1}^{N}a_n\big((\psi,w),\big((\mathcal{I}_{h_\sigma}-\mathcal{I})\big(\varphi^i_{h_\sigma}P^i_{h_\sigma}\omega\big),(\mathcal{I}_{h_u}-\mathcal{I})\big(\varphi^i_{h_u}Q^i_{h_u}v\big)\big)\big)\\
&\qquad-\tau_n\sum_{i=1}^{N}\bigg[\bigg(\frac{1}{c}\big(cw+\tau_n\big(\nabla\cdot\psi+q^n w\big)\big),\big(P^i_{h_\sigma}\omega\big)\nabla\varphi^i_{h_\sigma}\bigg)\\
&\qquad-\bigg(\frac{1}{c}\psi\nabla\varphi^i_{h_\sigma},cQ^i_{h_u}v+\tau_n\big(\nabla\cdot\big(P^i_{h_\sigma}\omega\big)+q^nQ^i_{h_u}v\big)\bigg)\\
&\qquad+\big(\tilde{A}\big(\psi+A\nabla w+\mathbf{b}^n w\big),A\nabla\varphi^h_iQ^i_{h_u}v\big)\\
&\qquad-\big(\nabla\varphi^i_{h_u}w,P^i_{h_\sigma}\omega+A\nabla\big(Q^i_{h_u}v\big)+\mathbf{b}^nQ^i_{h_u}v\big)\bigg].
\end{aligned}\tag{3.3}$$

Noting that

$$\begin{aligned}
&\big\|(\mathcal{I}_{h_\sigma}-\mathcal{I})\big(\varphi^i_{h_\sigma}P^i_{h_\sigma}\omega\big),(\mathcal{I}_{h_u}-\mathcal{I})\big(\varphi^i_{h_u}Q^i_{h_u}v\big)\big\|_{a_n}\\
&\quad\le K\bigg\{(1+\tau_n)\frac{h_u}{H}\big\|Q^i_{h_u}v\big\|_{L^2(\Omega_i)}+\tau_n\frac{1}{h_\sigma}\frac{h_\sigma}{H}\big\|P^i_{h_\sigma}\omega\big\|_{[L^2(\Omega_i)]^d}\\
&\qquad+\sqrt{\tau_n}\ \bigg[\frac{h_\sigma}{H}\big\|P^i_{h_\sigma}\omega\big\|_{[L^2(\Omega_i)]^d}+\frac{1}{h_u}\frac{h_u}{H}\big\|Q^i_{h_u}v\big\|_{L^2(\Omega_i)}+\frac{h_u}{H}\big\|Q^i_{h_u}v\big\|_{L^2(\Omega_i)}\bigg]\bigg\}\\
&\quad\le K\bigg(\frac{h}{H}+\frac{\sqrt{\tau}}{H}\bigg)\big\|\big(P^i_{h_\sigma}\omega,Q^i_{h_u}v\big)\big\|_{a_n,\Omega_i},
\end{aligned}$$

we have

$$\begin{aligned}
&\bigg|\sum_{i=1}^{N}a_n\big((\psi,w),\big((\mathcal{I}_{h_\sigma}-\mathcal{I})\big(\varphi^i_{h_\sigma}P^i_{h_\sigma}\omega\big),(\mathcal{I}_{h_u}-\mathcal{I})\big(\varphi^i_{h_u}Q^i_{h_u}v\big)\big)\big)\bigg|\\
&\qquad\le K\bigg(\frac{h}{H}+\frac{\sqrt{\tau}}{H}\bigg)\big\|(\psi,w)\big\|_{a_n}\bigg[\sum_{i=1}^{N}\big\|\big(P^i_{h_\sigma}\omega,Q^i_{h_u}v\big)\big\|^2_{a_n,\Omega_i}\bigg]^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
&\bigg|\sum_{i=1}^{N}a_n\big(\big(\big(\mathcal{I}-P^i_{h_\sigma}\big)\big(\varphi^i_{h_\sigma}\psi\big),\big(\mathcal{I}-Q^i_{h_u}\big)\big(\varphi^i_{h_u}w\big)\big),(\omega,v)\big)\bigg|\\
&\qquad\le K\bigg(\frac{h}{H}+\frac{\sqrt{\tau}}{H}\bigg)\big\|(\psi,w)\big\|_{a_n}\bigg[\sum_{i=1}^{N}\big\|\big(\big(\mathcal{I}-P^i_{h_\sigma}\big)\omega,\big(\mathcal{I}-Q^i_{h_u}\big)v\big)\big\|^2_{a_n,\Omega_i}\bigg]^{1/2}
\end{aligned}$$

and

$$
\begin{aligned}
\tau_n \sum_{i=1}^{N} \Big[& \Big| \Big(\frac{1}{c}\big(cw + \tau_n(\nabla\cdot\psi + q^n w)\big), (P^i_{h_\sigma}\omega)\nabla\varphi^i_{h_\sigma} \Big) \Big| \\
& + \Big| \Big(\frac{1}{c}\psi\nabla\varphi^i_{h_\sigma}, cQ^i_{h_u}v + \tau_n\big(\nabla\cdot(P^i_{h_\sigma}\omega) + q^n Q^i_{h_u}v\big) \Big) \Big| \\
& + \big| \big(\tilde{A}(\psi + A\nabla w + \mathbf{b}^n w), A\nabla\varphi^h_i Q^i_{h_u}v\big) \big| \\
& + \big| \big(\nabla\varphi^i_{h_u}w, P^i_{h_\sigma}\omega + A\nabla(Q^i_{h_u}v) + \mathbf{b}^n Q^i_{h_u}v\big) \big| \Big] \\
\le K\frac{\sqrt{\tau_n}}{H} & \Big[\sum_{i=1}^{N} \big\|(\psi,w)\big\|_{a_n} \big\|(P^i_{h_\sigma}\omega, Q^i_{h_u}v)\big\|^2_{a_n,\Omega_i} \Big]^{1/2}.
\end{aligned}
$$

Substituting these estimates into (3.3), we get the estimate (3.2). This ends the proof of Lemma 3. □

We assume that finite element spaces $\mathcal{V}_{h_\sigma}$ and $\mathcal{M}_{h_u}$ have the inverse property (see [7]) and approximate properties that there exist some positive integers r, r_1, $k > 0$, such that, for $1 \le q \le \infty$ and $\forall\omega \in H(\mathrm{div};\Omega) \cap [W^{r+1,q}(\Omega)]^d$,

$$
\begin{aligned}
&\inf_{\omega_h\in\mathcal{V}_{h_\sigma}} \|\omega - \omega_h\|_{[L^q]^d} \le K h_\sigma^{r+1}\|\omega\|_{[W^{r+1,q}]^d},\\
&\inf_{\omega_h\in\mathcal{V}_{h_\sigma}} \big\|\nabla\cdot(\omega - \omega_h)\big\|_{L^q} \le K h_\sigma^{r_1}\|\nabla\cdot\omega\|_{W^{r_1,q}},\\
&\inf_{v_h\in\mathcal{M}_{h_u}} \|v - v_h\|_{L^q} \le K h_u^{k+1}\|v\|_{W^{k+1,q}}, \quad \forall v \in L^2(\Omega)\cap W^{k+1,q}(\Omega).
\end{aligned}
$$

Lemma 4. *(See Theorem 3.1 in [24].) Let* (σ, u) *and* (ϱ_h^n, w_h^n) *be the solutions of* (2.1) *and least-squares scheme* (2.4)*, respectively. Then there holds the priori error estimate*

$$
\begin{aligned}
&\max_n \big\|u^n - w_h^n\big\|^2_{L^2} + \sum_n \tau_n\big[\big\|\sigma^n - \varrho_h^n\big\|^2_{[L^2]^d} + \big\|\nabla(u^n - w_h^n)\big\|^2_{[L^2]^d}\big]\\
&\quad \le K\big\{h_u^{2k} + h_\sigma^{2(r+1)} + \tau^2\big\}.
\end{aligned}
$$

For Parallel Algorithm, we have the following convergence theorem:

Theorem 1. *Let* (σ, u) *and* (σ_h^n, u_h^n) *are the solutions of the system* (2.1) *and Parallel algorithm, respectively. If* $h^2 = O(\tau)$*, then there holds the following a priori error estimate*

$$
\begin{aligned}
&\max_n \big\|u^n - u_h^n\big\|^2_{L^2} + \sum_n \tau_n\big[\big\|\sigma^n - \sigma_h^n\big\|^2_{[L^2]^d} + \big\|\nabla(u^n - u_h^n)\big\|^2_{[L^2]^d}\big]\\
&\quad \le K\Big\{\Big(\frac{h^2}{H^2} + \frac{\tau}{H^2}\Big)^m + h_u^{2k} + h_\sigma^{2(r+1)} + \tau^2\Big\}, \qquad (3.4)
\end{aligned}
$$

where K *denotes a constant independent of the mesh parameters* H, h_σ, h_u *and* τ.

4 Convergence Analysis

It is easily seen that Parallel algorithm is equivalent to use an iteration with initial values $(\sigma_h^{n-1}, u_h^{n-1})$ to solve the following equation: $(\hat{\sigma}_h^n, \hat{u}_h^n) \in \mathcal{V}_{h_\sigma} \times \mathcal{M}_{h_u}$ such that for any $(\omega_h, v_h) \in \mathcal{V}_{h_\sigma} \times \mathcal{M}_{h_u}$

$$a_n\big((\hat{\sigma}_h^n, \hat{u}_h^n), (\omega_h, v_h)\big) = \bigg(\frac{1}{c}\big(cu_h^{n-1} + \tau_n f^n\big), cv_h + \tau_n\big(\nabla \cdot \omega_h + q^n v_h\big)\bigg). \quad (4.1)$$

From (4.1) we have

$$a_n\big((\sigma_h^n, u_h^n), (\omega_h, v_h)\big) = \bigg(\frac{1}{c}\big(cu_h^{n-1} + \tau_n f^n\big), cv_h + \tau_n\big(\nabla \cdot \omega_h + q^n v_h\big)\bigg) + a_n\big((\sigma_h^n - \hat{\sigma}_h^n, u_h^n - \hat{u}_h^n), (\omega_h, v_h)\big). \quad (4.2)$$

Subtracting (2.4) from (4.2), we can obtain

$$a_n\big((\sigma_h^n - \varrho_h^n, u_h^n - w_h^n), (\omega_h, v_h)\big) = \big(u_h^{n-1} - w_h^{n-1}, cv_h + \tau_n\big(\nabla \cdot \omega_h + q^n v_h\big)\big) + a_n\big((\sigma_h^n - \hat{\sigma}_h^n, u_h^n - \hat{u}_h^n), (\omega_h, v_h)\big). \quad (4.3)$$

We want to obtain the bounds of $\sigma_h^n - \sigma^n$ and $u_h^n - u^n$. It is easily seen that we must estimate the bounds of $\sigma_h^n - \varrho_h^n$ and $u_h^n - w_h^n$. The above equation suggests that the bounds of $\sigma_h^n - \hat{\sigma}_h^n$ and $u_h^n - \hat{u}_h^n$, which reflect the contribution of the iteration error to the global approximation error, are crucial.

Lemma 5. *There holds the estimate*

$$\big\|(\sigma_h^n - \hat{\sigma}_h^n, u_h^n - \hat{u}_h^n)\big\|_{a_n} \le K\bigg(\frac{h^2}{H^2} + \frac{\tau}{H^2}\bigg)^{\frac{m}{2}} \big\|(\sigma_h^{n-1} - \hat{\sigma}_h^n, u_h^{n-1} - \hat{u}_h^n)\big\|_{a_n}. \quad (4.4)$$

Proof. From (2.5), we have

$$\begin{aligned} a_n\big((\varepsilon_j^i, e_j^i), (\omega_h, v_h)\big) &= a_n\big((\varepsilon_j^i, e_j^i), \big(P_{h_\sigma}^i \omega_h, Q_{h_u}^i v_h\big)\big) \\ &= a_n\big((\hat{\sigma}^n - \tilde{\sigma}_{j-1}^n, \hat{u}^n - \tilde{u}_{j-1}^n), \big(\mathcal{I}_{h_\sigma}\big(\varphi_i^h P_{h_\sigma}^i \omega_h\big), \mathcal{I}_{h_u}\big(\varphi_i^h Q_{h_u}^i v_h\big)\big)\big). \quad (4.5) \end{aligned}$$

In addition, from Parallel Algorithm we can obtain the following equation

$$\begin{aligned} &a_n\big((\tilde{\sigma}_j^n - \hat{\sigma}_h^n, \tilde{u}_j^n - \hat{u}_h^n), (\omega_h, v_h)\big) \\ &= a_n\big((\tilde{\sigma}_{j-1}^n - \hat{\sigma}_h^n, \tilde{u}_{j-1}^n - \hat{u}_h^n), (\omega_h, v_h)\big) + a_n\Bigg(\bigg(\sum_{i=1}^N \varepsilon_j^i, \sum_{i=1}^N e_j^i\bigg), (\omega_h, v_h)\Bigg) \\ &= a_n\big((\tilde{\sigma}_{j-1}^n - \hat{\sigma}_h^n, \tilde{u}_{j-1}^n - \hat{u}_h^n), (\omega_h, v_h)\big) \\ &+ a_n\Bigg((\hat{\sigma}_h^n - \tilde{\sigma}_{j-1}^n, \hat{u}_h^n - \tilde{u}_{j-1}^n), \bigg(\sum_{i=1}^N \mathcal{I}_{h_\sigma}\big(\varphi_{h_\sigma}^i P_{h_\sigma}^i \omega_h\big), \sum_{i=1}^N \mathcal{I}_{h_u}\big(\varphi_{h_u}^i Q_{h_u}^i v_h\big)\bigg)\Bigg). \end{aligned} \quad (4.6)$$

Taking $(\omega_h, v_h) = (\tilde{\sigma}_j^n - \hat{\sigma}_h^n, \tilde{u}_j^n - \hat{u}_h^n)$ in (4.6) and using Lemma 3, we have

$$\big\|(\tilde{\sigma}_j^n - \hat{\sigma}_h^n, \tilde{u}_j^n - \hat{u}_h^n)\big\|_{a_n}^2 \le K\bigg(\frac{h^2}{H^2} + \frac{\tau}{H^2}\bigg)\big\|(\tilde{\sigma}_{j-1}^n - \hat{\sigma}_h^n, \tilde{u}_{j-1}^n - \hat{u}_h^n)\big\|_{a_n}^2.$$

Thus, we have

$$\left\|\left(\tilde{\sigma}_m^n - \hat{\sigma}_h^n, \tilde{u}_m^n - \hat{u}_h^n\right)\right\|_{a_n}^2 \le K\left(\frac{h^2}{H^2} + \frac{\tau}{H^2}\right)^m \left\|\left(\tilde{\sigma}_0^n - \hat{\sigma}_h^n, \tilde{u}_0^n - \hat{u}_h^n\right)\right\|_{a_n}^2.$$

That is the inequality (4.4). This ends the proof of Lemma 5. □

Lemma 6. *Let* $\theta^n = u_h^n - w_h^n$, $\rho^n = w_h^n - u^n$, $\pi^n = \sigma_h^n - \varrho_h^n$ *and* $\eta^n = \varrho_h^n - \sigma^n$. *There holds the following estimate*

$$\begin{aligned}
&\left\|\left(\sigma_h^{n-1} - \hat{\sigma}_h^n, u_h^{n-1} - \hat{u}_h^n\right)\right\|_{a_n}^2 \\
&\quad \le K\tau_n\big\{\tau_n\big(\|f^n\|_{L^2}^2 + \|\nabla\cdot\sigma^{n-1}\|_{L^2}^2\big) + \|u^{n-1}\|_{H^1}^2 + \|\sigma^{n-1}\|_{[L^2]^d}^2 \\
&\qquad + \|\theta^{n-1}\|_{L^2}^2 + \|\nabla\theta^{n-1}\|_{[L^2]^d}^2 + \|\xi^{n-1}\|_{[L^2]^d}^2 + \|\rho^{n-1}\|_{L^2}^2 + \|\nabla\rho^{n-1}\|_{[L^2]^d}^2 \\
&\qquad + \|\eta^{n-1}\|_{[L^2]^d}^2 + \tau_n\big[\|\nabla\cdot\pi^{n-1}\|_{L^2}^2 + \|\nabla\cdot\eta^{n-1}\|_{L^2}^2\big]\big\}. \qquad (4.7)
\end{aligned}$$

Proof. It is easily seen that the equation (2.5) has the following equivalent form:

$$\begin{aligned}
&a_n\big(\big(\hat{\sigma}_h^n - \sigma_h^{n-1}, \hat{u}_h^n - u_h^{n-1}\big), (\omega_h, v_h)\big) \\
&\quad = \tau_n\left(\frac{1}{c}\big(f^n - \nabla\cdot\sigma_h^{n-1} - q^n u_h^{n-1}\big), cv_h + \tau_n\big(\nabla\cdot\omega_h + q^n v_h\big)\right) \\
&\qquad - \tau_n\big(\tilde{A}\big(\sigma_h^{n-1} + A\nabla u_h^{n-1} + \mathbf{b}^n u_h^{n-1}\big), \omega_h + A\nabla v_h + \mathbf{b}^n v_h\big). \qquad (4.8)
\end{aligned}$$

We know that

$$\begin{aligned}
&\tau_n\left(\frac{1}{c}\big(\nabla\cdot\sigma_h^{n-1} + q^n u_h^{n-1}\big), cv_h + \tau_n\big(\nabla\cdot\omega_h + q^n v_h\big)\right) \\
&\quad = \tau_n\left(\frac{1}{c}\big(\nabla\cdot\pi^{n-1} + q^n\theta^{n-1}\big), cv_h + \tau_n\big(\nabla\cdot\omega_h + q^n v_h\big)\right) \\
&\qquad + \tau_n\left(\frac{1}{c}\big(\nabla\cdot\eta^{n-1} + q^n\rho^{n-1}\big), cv_h + \tau_n\big(\nabla\cdot\omega_h + q^n v_h\big)\right) \\
&\qquad + \tau_n\left(\frac{1}{c}\big(\nabla\cdot\sigma^{n-1} + q^n u^{n-1}\big), cv_h + \tau_n\big(\nabla\cdot\omega_h + q^n v_h\big)\right) \qquad (4.9)
\end{aligned}$$

and

$$\begin{aligned}
&\tau_n\big(\tilde{A}\big(\sigma_h^{n-1} + A\nabla u_h^{n-1} + b^n u_h^{n-1}\big), \omega_h + A\nabla v_h + b^n v_h\big) \\
&\quad = \tau_n\big(\tilde{A}\big(\pi^{n-1} + A\nabla\theta^{n-1} + b^n\theta^{n-1}\big), \omega_h + A\nabla v_h + b^n v_h\big) \\
&\qquad + \tau_n\big(\tilde{A}\big(\eta^{n-1} + A\nabla\rho^{n-1} + b^n\rho^{n-1}\big), \omega_h + A\nabla v_h + b^n v_h\big) \\
&\qquad + \tau_n\big(\tilde{A}\big(\sigma^{n-1} + A\nabla u^{n-1} + b^n u^{n-1}\big), \omega_h + A\nabla v_h + b^n v_h\big). \qquad (4.10)
\end{aligned}$$

Substituting (4.9) and (4.10) into (4.8) and using the inequality $ab \le \frac{1}{\delta}a^2 + \delta b^2$, we can easily obtain

$$\begin{aligned}
&a_n\big(\big(\hat{\sigma}_h^n - \sigma_h^{n-1}, \hat{u}_h^n - u_h^{n-1}\big), (\omega_h, v_h)\big) \\
&\quad \le K\tau_n\big\{\tau_n\big(\|f^n\|_{L^2}^2 + \|\nabla\cdot\sigma^{n-1}\|_{L^2}\big) + \|\sigma^{n-1}\|_{[L^2]^d}^2 + \|\nabla u^{n-1}\|_{[L^2]^d}^2
\end{aligned}$$

$$+\|u^{n-1}\|^2_{L^2}+\|\theta^{n-1}\|^2_{L^2}+\|\nabla\theta^{n-1}\|^2_{[L^2]^d}+\|\pi^{n-1}\|^2_{[L^2]^d}+\|\rho^{n-1}\|^2_{L^2}$$
$$+\|\pi^{n-1}\|^2_{[L^2]^d}+\|\rho^{n-1}\|^2_{L^2}+\|\nabla\rho^{n-1}\|^2_{[L^2]^d}+\|\eta^{n-1}\|^2_{[L^2]^d}$$
$$+\tau_n\big[\|\nabla\cdot\pi^{n-1}\|^2_{L^2}+\|\nabla\eta^{n-1}\|^2_{[L^2]^d}\big]\big\}+\delta\|(\omega_h,v_h)\|^2_{a_n}. \tag{4.11}$$

Choosing $(\omega_h,v_h)=(\hat\sigma_h^n-\sigma_h^{n-1},\hat u_h^n-u_h^{n-1})$ in (4.11), and choosing sufficiently small δ, we can easily get the inequality (4.7). □

Now, we prove Theorem 1.

Proof. From (4.3), we see that (π^n,θ^n) satisfies the following error equation:

$$\begin{aligned}&\Big(\frac{1}{c}\big(c\theta^n+\tau_n\big(\nabla\cdot\pi^n+q^n\theta^n\big)\big),cv_h+\tau_n\big(\nabla\cdot\omega_h+q^nv_h\big)\Big)\\&\quad+\tau_n\big(\tilde A\big(\pi^n+A\nabla\theta^n+\mathbf b^n\theta^n\big),\omega_h+A\nabla v_h+\mathbf b^nv_h\big)\\&=\big(\theta^{n-1},cv_h+\tau_n\big(\nabla\cdot\omega_h+q^nv_h\big)\big)\\&\quad+a_n\big(\big(\sigma_h^n-\hat\sigma_h^n,u_h^n-\hat u_h^n\big),(\omega_h,v_h)\big).\end{aligned}\tag{4.12}$$

Taking $(\omega_h,v_h)=(\pi^n,\theta^n)$ in (4.12), we have

$$\begin{aligned}&\big(c\theta^n,\theta^n\big)+\tau_n\big[\big(A\nabla\theta^n,\nabla\theta^n\big)+\big(\tilde A\mathbf b^n\theta^n,\mathbf b^n\theta^n\big)+\big(\tilde A\pi^n,\pi^n\big)\big]\\&\quad+\tau_n^2\Big[\Big(\frac1c\nabla\cdot\pi^n,\nabla\cdot\pi^n\Big)+\Big(\frac1c q^n\theta^n,q^n\theta^n\Big)\Big]\\&=\big(c\theta^{n-1},\theta^n\big)+\tau_n\big[\big(\theta^{n-1},\nabla\cdot\pi^n\big)+\big(\theta^{n-1},q^n\theta^n\big)\big]\\&\quad-2\tau_n\Big[\Big(\frac1c\big(c\theta^n+\tau_n\nabla\cdot\pi^n\big),q^n\theta^n\Big)+\big(\tilde A\pi^n+\nabla\theta^n,\mathbf b^n\theta^n\big)\Big]\\&\quad+a_n\big(\big(\sigma_h^n-\hat\sigma_h^n,u_h^n-\hat u_h^n\big),\big(\pi^n,\theta^n\big)\big).\end{aligned}$$

Substituting the equality

$$\big(c\big(\theta^n-\theta^{n-1}\big),\theta^n\big)=\frac12\big[\big(c\theta^n,\theta^n\big)-\big(c\theta^{n-1},\theta^{n-1}\big)+\big(c\big(\theta^n-\theta^{n-1}\big),\theta^n-\theta^{n-1}\big)\big]$$

and

$$\begin{aligned}\tau_n\big(\theta^{n-1},\nabla\cdot\pi^n\big)&=\tau_n\big[\big(\theta^{n-1}-\theta^n,\nabla\cdot\pi^n\big)-\big(\nabla\theta^n,\pi^n\big)\big]\\&\le\frac12\Big[\big(c\big(\theta^n-\theta^{n-1}\big),\theta^n-\theta^{n-1}\big)+\tau_n^2\Big(\frac1c\nabla\cdot\pi^n,\nabla\cdot\pi^n\Big)\\&\qquad+\tau_n\big(A\nabla\theta^n,\nabla\theta^n\big)+\tau_n\big(\tilde A\pi^n,\pi^n\big)\Big]\end{aligned}$$

into (4.12), utilizing Lemma 5, Lemma 6 and the assumption $h^2=O(\tau)$, we have

$$\begin{aligned}&\big(c\theta^n,\theta^n\big)+\tau_n\big[\big(A\nabla\theta^n,\nabla\theta^n\big)+\big(\tilde A\mathbf b^n\theta^n,\mathbf b^n\theta^n\big)+\big(\tilde A\pi^n,\pi^n\big)\big]\\&\quad+\tau_n^2\Big[\Big(\frac1c\nabla\cdot\pi^n,\nabla\cdot\pi^n\Big)+\Big(\frac1c q^n\theta^n,q^n\theta^n\Big)\Big]\\&\le\big(c\theta^{n-1},\theta^{n-1}\big)+K\tau_n\Big\{\|\eta^{n-1}\|_{[L^2]^d}+\|\rho^{n-1}\|_{L^2}\end{aligned}$$

$$+ \|\nabla \rho^{n-1}\|_{[L^2]^d} + \|\theta^{n-1}\|_{L^2}^2 + \|\theta^n\|_{L^2}^2 + \left(\frac{h^2}{H^2} + \frac{\tau}{H^2}\right)^m \Big\}$$
$$+ \delta\tau_n \big[\|\nabla\theta^n\|_{(L^2)^d}^2 + \|\pi^n\|_{[L^2]^d} + \tau_n \|\nabla\cdot\pi^n\|_{L^2}\big]. \quad (4.13)$$

Summing (4.13) up from 1 to n, for sufficiently small δ, using Lemma 4, we obtain the estimate

$$\|\theta^n\|_{L^2}^2 + \sum_{j=1}^{n} \tau_j \big[\|\nabla\theta^j\|_{[L^2]^d}^2 + \|\xi^j\|_{(L^2)^d}^2\big]$$
$$\le K\left\{\sum_{j=1}^{n} \tau_j \|\theta^j\|_{L^2}^2 + h_u^{2k} + h_\sigma^{2(r+1)} + \tau^2 + \left(\frac{h^2}{H^2} + \frac{\tau}{H^2}\right)^m\right\}. \quad (4.14)$$

An application of discrete Gronwall's lemma to (4.14) leads to

$$\max_n \|\theta^n\|_{L^2}^2 + \sum_n \tau_n \big[\|\nabla\theta^n\|_{[L^2]^d}^2 + \|\pi^n\|_{[L^2]^d}^2\big]$$
$$\le K\left\{\left(\frac{h^2}{H^2} + \frac{\tau}{H^2}\right)^m + h_u^{2k} + h_\sigma^{2(r+1)} + \tau^2\right\}.$$

Using Lemma 4, we have the error estimate (3.4). The proof of Theorem 1 is complete. □

5 Numerical Examples

Consider the following first-order time-dependent two-dimensional convection–diffusion system:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + \nabla\cdot\sigma(x,t) + u(x,t) = f(x,t), \\ \sigma(x,t) + \mathbf{A}(x)\nabla u(x,t) + \mathbf{b}(x,t)u(x,t) = 0, \end{cases}$$

where $x \in \Omega = [0,1]\times[0,1]$, $t \in (0,T]$, $\mathbf{A} = a\mathbf{E}$, $\mathbf{E}$ is the unit matrix, and $\mathbf{b} = (1,1)^T$. We divide Ω into four sub-domains: $\Omega_1 = [0,0.6]\times[0,0.6]$, $\Omega_2 = [0.4,1]\times[0,0.6]$, $\Omega_3 = [0,0.6]\times[0.4,1]$, $\Omega_4 = [0.4,1]\times[0.4,1]$. See Figure 1.

Throughout all experiments in this section, we use piecewise linear polynomial spaces, and we take the linear unit decomposition functions $\{\varphi_i\}_{i=1}^4$ as follows:

$$\varphi_1(x,y) = \begin{cases} 1, & (x,y)\in[0,0.4]\times[0,0.4], \\ 3-5y, & (x,y)\in[0,0.4]\times[0.4,0.6], \\ 3-5x, & (x,y)\in[0.4,0.6]\times[0,0.4], \\ \dfrac{3}{2} - \dfrac{5}{4}(x+y), & \text{otherwise}, \end{cases}$$

$$\varphi_2(x,y) = \begin{cases} 1, & (x,y)\in[0.6,1]\times[0,0.4], \\ 3-5y, & (x,y)\in[0.6,1]\times[0.4,0.6], \\ 5x-2, & (x,y)\in[0.4,0.6]\times[0,0.4], \\ \dfrac{1}{4} - \dfrac{5}{4}(x-y), & \text{otherwise}, \end{cases}$$

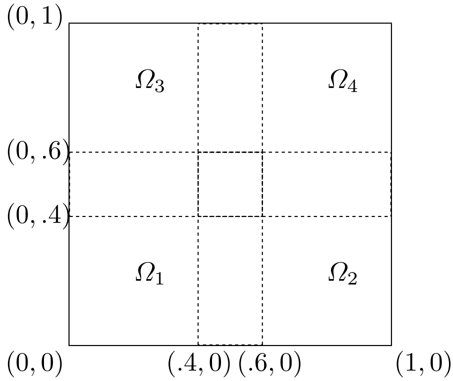

Figure 1. The sub-domains of Ω.

$$\varphi_3(x,y)=\begin{cases}1, & (x,y)\in[0,0.4]\times[0,0.4],\\ 5y-2, & (x,y)\in[0,0.4]\times[0.4,0.6],\\ 3-5x, & (x,y)\in[0.4,0.6]\times[0.6,1],\\ \frac{1}{4}+\frac{5}{4}(y-x), & \text{otherwise},\end{cases}$$

$$\varphi_4(x,y)=\begin{cases}1, & (x,y)\in[0.6,1]\times[0.6,1],\\ 5y-2, & (x,y)\in[0.6,1]\times[0.4,0.6],\\ 5x-2, & (x,y)\in[0.4,0.6]\times[0.6,1],\\ \frac{5}{4}(x+y)-1, & \text{otherwise}.\end{cases}$$

Based on this domain decomposition, we give the triangulation. We first divide Ω into uniform squares with mesh-size h, then we obtain the triangulation by dividing each square into two triangles. The iterative number in each time step is m.

Define the A-norm error:

$$\|(e,E)\|_A^2=\max_{1\le n\le 1/\tau}\|e^n\|_{L^2}^2+\tau\sum_{k=1}^{1/\tau}\{\|E^k\|_{[L^2]^2}^2+\|\nabla e^k\|_{[L^2]^2}^2\},$$

and the L^∞-norm error

$$\|(e,E)\|_\infty=\max_n\big(|e^n|,|E^n|\big),$$

where $e^n=u_{ij}^n-u_{h;ij}^n$, $E^n=\sigma_{ij}^n-\sigma_{h;ij}^n$; u_{ij}^n, $u_{h;ij}^n$, σ_{ij}^n and $\sigma_{h;ij}^n$ denote the exact solution and approximate solution at point $(ih,jh,n\tau)$, respectively.

Experiment 1. In this experiment, the right-hand side term, boundary condition and initial condition are selected in such a way that the exact solutions are $u=10tx^2y^2(x-1)^2(y-1)^2$. By using $H=0.2$, $T=1.0$ and different values of parameters $\tau=h$ and m, the errors $\|(e,E)\|_A$ and $\|(e,E)\|_\infty$, with least-square scheme and Parallel algorithm, respectively, are depicted in Tables 1–3, where "$\star$" denote the numerical results by using least-square scheme and "$*$"

Table 1. Numerical results in the case $a = 1e - 1$.

h	m	$\star$	1	2	3	4
$\frac{1}{10}$	$\|\cdot\|_\infty$	3.5666e−3	4.3172e−3	3.3687e−3	2.7670e−3	2.4270e−3
	$\|\cdot\|_A$	4.4078e−3	6.1450e−3	4.2909e−3	3.5379e−3	3.2032e−3
$\frac{1}{20}$	$\|\cdot\|_\infty$	2.3522e−3	1.4946e−3	8.1086e−4	5.1845e−4	4.2320e−4
	$\|\cdot\|_A$	2.2419e−3	2.0368e−3	9.9448e−4	7.1156e−4	6.3565e−4
$\frac{1}{40}$	$\|\cdot\|_\infty$	1.3076e−3	6.5352e−4	2.8280e−4	1.4287e−4	9.5200e−5
	$\|\cdot\|_A$	1.1210e−3	8.7358e−4	2.9010e−4	1.6584e−4	1.4170e−4
$*$	$\|\cdot\|_\infty$	0.7238	1.3619	1.7872	2.1378	2.3360
$*$	$\|\cdot\|_A$	0.9876	1.4072	1.9433	2.2075	2.2493

Table 2. Numerical results in the case $a = 1e - 3$.

h	m	$\star$	1	2	3	4
$\frac{1}{10}$	$\|\cdot\|_\infty$	1.2407e−3	3.7626e−3	2.8825e−3	2.3932e−3	2.1578e−3
	$\|\cdot\|_A$	2.6426e−3	6.6209e−3	4.8104e−3	4.1672e−3	3.9311e−3
$\frac{1}{20}$	$\|\cdot\|_\infty$	2.2918e−4	8.6680e−4	4.3673e−4	3.3434e−4	3.3408e−4
	$\|\cdot\|_A$	5.1842e−4	1.9838e−3	9.2671e−4	7.2032e−4	6.8827e−4
$\frac{1}{40}$	$\|\cdot\|_\infty$	5.2593e−5	3.6747e−4	1.4969e−4	5.8624e−5	5.0109e−5
	$\|\cdot\|_A$	1.3150e−4	1.0209e−3	3.2065e−4	1.7158e−4	1.5216e−4
$*$	$\|\cdot\|_\infty$	2.2801	1.6780	2.1336	2.6757	2.7142
$*$	$\|\cdot\|_A$	2.1644	1.3486	1.9535	2.3011	2.3456

Table 3. Numerical results in the case $a = 1e - 5$.

h	m	$\star$	1	2	3	4
$\frac{1}{10}$	$\|\cdot\|_\infty$	1.2663e−3	3.7591e−3	2.8810e−3	2.3929e−3	2.1646e−3
	$\|\cdot\|_A$	2.6843e−3	6.6409e−3	4.8237e−3	4.1806e−3	3.9459e−3
$\frac{1}{20}$	$\|\cdot\|_\infty$	2.4071e−4	8.6335e−4	4.3367e−4	3.3501e−4	3.3476e−4
	$\|\cdot\|_A$	5.2962e−4	2.0001e−3	9.2990e−4	7.2185e−4	6.9006e−4
$\frac{1}{40}$	$\|\cdot\|_\infty$	4.7969e−5	3.6970e−4	1.5119e−4	5.9602e−5	4.9544e−5
	$\|\cdot\|_A$	1.2798e−4	1.0438e−3	3.2882e−4	1.7363e−4	1.5328e−4
$*$	$\|\cdot\|_\infty$	2.3612	1.6730	2.1261	2.6636	2.7246
$*$	$\|\cdot\|_A$	2.1953	1.3348	1.9374	2.2948	2.3431

denote the convergence rates in the sense $\|\cdot\|_\infty$ and $\|\cdot\|_A$. From the numerical results we can see that using the parallel subspace correction method we can get the similar convergence rate to least-squares method for time-dependent convection–diffusion problem, even iterating only two cycles at each time level.

Experiment 2. In this experiment, we select the right-hand side, the initial and boundary functions with complex structure:

$$\begin{cases} f(t,x,y) = 15\sin(3\pi xt)\sin(1.5\pi y) + 9\cos(\pi xt)\cos(5\pi y), \\ u_0(x,y) = 0. \end{cases} \tag{5.1}$$

Set $a = 1.0$, $T = 1.0$, $H = 0.2$ and $h = \tau = 1/40$. Using Least-squares Scheme and Parallel Algorithm with $m = 1, 2$, we get Figures 2–4. These figures clearly

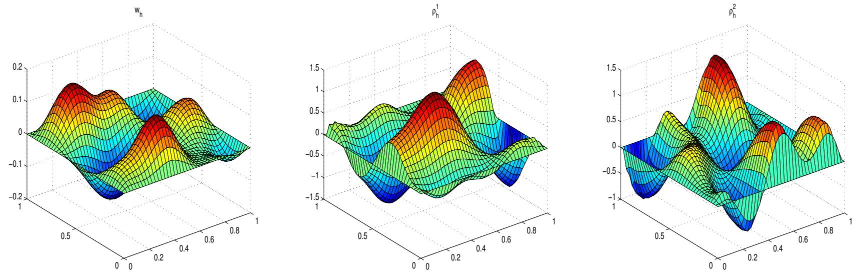

Figure 2. Numerical results by Least-squares algorithm.

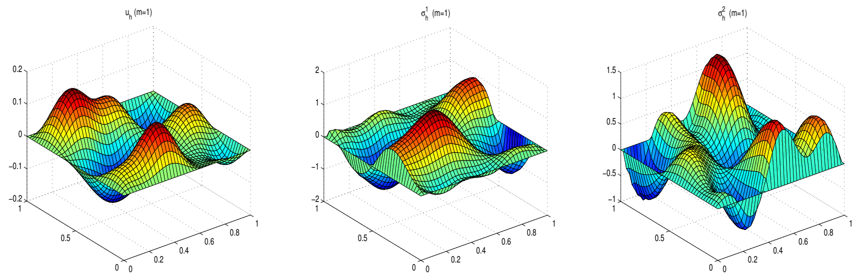

Figure 3. Numerical results by Parallel algorithm, $m = 1$.

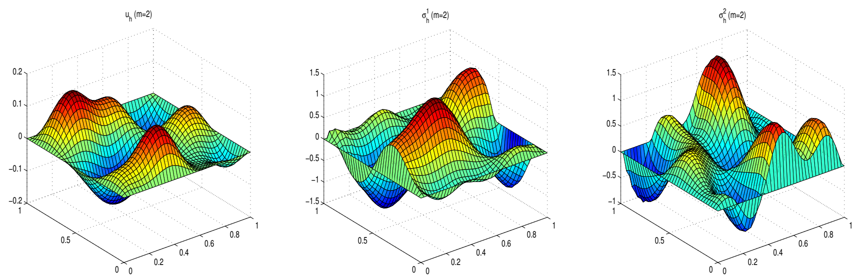

Figure 4. Numerical results by Parallel algorithm, $m = 2$.

show that u_h, σ_h approximate to w_h and ϱ_h at different time, respectively.

6 Conclusion

In this paper, sufficiently utilizing the advantages of parallel subspace correction method and the least-squares method, we have constructed a new parallel algorithm for convection–diffusion equation. We use the partition functions of unity to distribute the corrections in the overlapping domains reasonably. We analyze the convergence of this algorithm, and study the dependence of the convergent rate on the spacial mesh size, time increment, iteration times and sub-domains overlapping degree. Both theoretical analysis and numerical experiments suggest that only one or two iterations are needed to reach to optimal accuracy at each time level, while for a general iterative-type parallel algorithms, many iteration steps are needed to reach given accuracy, which produce much more global amount of computational works.

Moreover, our method is also applied for more general problems and complicated problems, e.g., the Navier-Stokes equations, miscible displacement problems in porous media etc.

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