Mathematical Modelling and Analysis
Volume 19 Number 4, September 2014, 509–523
http://dx.doi.org/10.3846/13926292.2014.956827

Publisher: Taylor&Francis and VGTU
http://www.tandfonline.com/TMMA
Print ISSN: 1392-6292
Online ISSN: 1648-3510

Existence of Traveling Wave Fronts for a Generalized KdV–mKdV Equation*

Ying Xu and Zengji Du

School of Mathematics and Statistics, Jiangsu Normal University
Xuzhou, 221116 Jiangsu, China
E-mail: 4385896@qq.com
E-mail(*corresp.*): duzengji@163.com

Received September 18, 2013; revised July 21, 2014; published online September 1, 2014

Abstract. This paper deals with the existence of traveling wave fronts for a generalized KdV–mKdV equation. We first establish the existence of traveling wave solutions for the equation without delay, and then we prove the existence of traveling wave fronts for the equation with a special local delay convolution kernel and a special nonlocal delay convolution kernel by using geometric singular perturbation theory, Fredholm theory and the linear chain trick.

Keywords: KdV–mKdV equation, traveling wave solution, geometric singular perturbation, linear chain trick.

AMS Subject Classification: 35Q53.

1 Introduction

In the past few decades remarkable progresses have been made in understanding the KdV equation, it can be considered as a paradigm in nonlinear science and has many applications in weakly nonlinear and weakly dispersive physical systems. The Korteweg–de Vries (KdV) equation

$$U_t + \alpha U U_x + U_{xxx} = 0,$$

which was first suggested by Korteweg and de Vries in 1895 [11], they used it as a nonlinear model to study the change of form of long waves advancing in a rectangular channel and given the solitary wave solution.

The standard form of the Burgers–KdV equation was first proposed by Johnson in [10] who derived the equation as the governing equation for waves propagating in a liquid-filled elastic tube

$$U_t + \alpha U U_x + \beta U_{xx} + s U_{xxx} = 0, \tag{1.1}$$

* This work is supported by the Natural Science Foundation of China (Grant No. 11471146 and 11101349), and partially supported by PAPD of Jiangsu Higher Education Institutions, postgraduate training project of Jiangsu Province and Jiangsu Normal University.

where α, β and s are real constants with $\alpha\beta s \neq 0$. Moreover, traveling wave solutions of equation (1.1) was analyzed in [6]. Equation (1.1) is usually considered as a combination of the Burgers equation and KdV equation, if we let $\alpha \neq 0$, $\beta \neq 0$, and $s = 0$ in (1.1) we can get the Burgers equation

$$U_t + \alpha U U_x + \beta U_{xx} = 0,$$

which is named after its use by Burgers for studying turbulence in 1939 [3].

The so-called compound Burgers–Korteweg–de Vries (cBKdV) equation [5] has the following form

$$U_t + \alpha U U_x + \beta U^2 U_x + \mu U_{xx} + s U_{xxx} = 0,$$

where α, β, μ and s are real constants. The first term is the instationary term. The second and third ones are two nonlinear convective terms with different orders. The fourth is the so-called viscous dissipative term, and the coefficient μ refers to a positive quantity and denotes the viscosity. The last one is the dispersive term, presents the dispersion effect.

Let us consider the generalized Korteweg–de Vries–modified Korteweg–de Vries (KdV–mKdV) equation [1] – [13]

$$U_t + (\alpha + \beta U^p + \gamma U^{2p}) U_x + U_{xxx} = 0, \tag{1.2}$$

where $p > 0$, α, β and $\gamma \neq 0$ are real constants. When $p = 1$, $\alpha \neq 0$, $\beta \neq 0$ and $\gamma = 0$, Eq. (1.2) becomes the generalized KdV–mKdV equation

$$U_t + \alpha U_x + \beta U U_x + U_{xxx} = 0.$$

In [13] the G'/G-expansion method is introduced to construct more general exact traveling wave solutions to the generalized KdV–mKdV equation with any-order nonlinear terms. As a result, hyperbolic function solution, trigonometric function solution and rational solution with parameters are obtained. When the parameters take up special values, the solitary waves are also derived from the traveling wave solutions.

Li and Wang used sub-ODE method to obtain the bell type solitary wave solution, the kink type solitary wave solution, the algebraic solitary wave solution and the sinusoidal traveling wave solution of a generalized KdV–mKdV equation (GKdV–mKdV) with high-order nonlinear terms in [12].

The soliton perturbation theory is used in [1] to study the solitons that are governed by the generalized Korteweg–de Vries equation in the presence of perturbation terms. The adiabatic parameter dynamics of the solitons in the presence of the perturbation terms are obtained.

Traveling wave solutions to the different types of KdV equation have been studied extensively and many other powerful methods have been established and developed, for example, Bendixson theorem [7], Amplitude ansatz method [19], Qualitative theory [17], Numerical methods [4] and so on.

Geometric singular perturbation theory [8] has received a great deal of interest and has been used by many researchers to obtain the existence of traveling waves for different equations, such as [2, 14, 16, 18, 20].

Song, Peng and Han [18] used geometric singular perturbation theory, Fredholm theory and the linear chain trick to investigate the existence of traveling wave fronts for the following diffusive single species model

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ru\big[1 - a_1(f * u) - a_2(f * u)^2\big],$$

where the parameter D, r, a_1, a_2 are positive constants, the convolution $f * u$ is taken to be different cases.

Motivated by the above reasons, the purpose of this paper is to establish the existence of traveling wave solutions for the following KdV–mKdV equation by the method in [14, 16, 18]

$$U_t + \alpha U_x + \beta(f * U)U_x + \gamma U_{xx} + U_{xxx} = 0, \tag{1.3}$$

where $\alpha\beta\gamma \neq 0$, $c > 0$ is the wave speed, U_{xx} is the viscous dissipative term.

The remaining part of this paper is organized as follows. In Section 2, we present geometric singular perturbation theory which is important to obtain our main results. In Section 3, we investigate Eq. (1.3) in three forms: without delay, with a special local delay convolution kernel, with a nonlocal weak generic delay kernel. The existence of traveling wave solutions for it are obtained by using geometric singular perturbation theory, Fredholm theory and the linear chain trick.

2 Preliminaries

In this section, we introduce the following result on invariant manifolds which is due to Fenichel [8].

Lemma 1 [Geometric Singular Perturbation Theorem [8]]. *For the system*

$$\begin{cases} x'(t) = f(x, y, \epsilon), \\ y'(t) = \epsilon g(x, y, \epsilon), \end{cases} \tag{2.1}$$

where $x \in R^n$, $y \in R^l$ and ϵ is a real parameter, f, g are C^∞ on the set $V \times I$ where $V \in R^{n+l}$ and I is an open interval, containing 0. If when $\epsilon = 0$, the system has a compact, normally hyperbolic manifold of critical points M_0, which is contained in the set $\{f(x, y, 0) = 0\}$. Then for any $0 < r < +\infty$, if $\epsilon > 0$, but sufficiently small, there exists a manifold M_ϵ, satisfying

(I) *which is locally invariant under the flow of (2.1);*

(II) *which is C^r in x, y and ϵ;*

(III) *$M_\epsilon = \{(x, y)\colon x = h^\epsilon(y)\}$ for some C^r function $h^\epsilon(y)$ and y in some compact K;*

(IV) *there exist locally invariant stable and unstable manifolds $W^s(M_\epsilon)$ and $W^u(M_\epsilon)$ that lie within $O(\epsilon)$, and are diffeomorphic to, $W^s(M_0)$ and $W^u(M_0)$.*

3 Existence of Traveling Wave Solutions

In this section, we will establish the existence of traveling wave solutions for Eq. (1.3) in three cases: without delay, with a special local delay convolution kernel, with a nonlocal weak generic delay kernel.

3.1 The model without delay

In this section, we will prove the existence of traveling wave solutions for Eq. (1.3) without delay, i.e.,

$$U_t + \alpha U_x + \beta U U_x + \gamma U_{xx} + U_{xxx} = 0. \tag{3.1}$$

Substituting $U(x,t) = \phi(z)$ ($z = x + ct$ and $c > 0$) into the Eq. (3.1), then we obtain the following traveling wave equation

$$(c+\alpha)\phi' + \beta\phi\phi' + \gamma\phi'' + \phi''' = 0.$$

It can be integrated once to yield the equation

$$(c+\alpha)\phi + \frac{\beta}{2}\phi^2 + \gamma\phi' + \phi'' = 0,$$

which is equivalent to the following system of first-order equations

$$\begin{cases} \phi' = \psi, \\ \psi' = -(c+\alpha)\phi - \dfrac{\beta}{2}\phi^2 - \gamma\psi, \end{cases} \tag{3.2}$$

where $' = \frac{d}{dz}$. It has two equilibria $E_1(0,0)$ and $E_2(-\frac{2}{\beta}(c+\alpha), 0)$. The following result shows that there is a traveling wave solution of system (3.2) connecting E_1 and E_2.

Theorem 1. *Assume that* $0 < \alpha + c \le \frac{\gamma^2}{4}$, $c > 0$, $\beta < 0$, $\gamma < 0$, *then, in the* (ϕ, ψ) *phase plane for system* (3.2), *there is a heteroclinic orbit connecting the two equilibria* E_1 *and* E_2. *The heteroclinic connection is confined to* $\psi > 0$ *and the traveling wave* $\phi(z)$ *is strictly monotonically increasing.*

Proof. Linear analysis of system (3.2) shows that E_1 is a node (under the condition on c) and E_2 is a saddle. To establish the existence of a heteroclinic orbit connecting the two equilibria for $\psi > 0$, we shall show that, for a suitable value of $\lambda > 0$, the triangular set

$$\Omega = \left\{ (\phi,\psi) \colon 0 \le \phi \le -\frac{2}{\beta}(c+\alpha),\ 0 \le \psi \le \lambda\phi \right\}$$

is negative invariant. Let f be the vector defined by the right-hand sides of system (3.2) and n be the inward normal vector on the boundary of Ω. We only need to consider the side $\psi = \lambda\phi$, $0 < \phi \le -\frac{2}{\beta}(c+\alpha)$ of the triangle and have

$$f \cdot \overrightarrow{n} = \begin{pmatrix} \psi \\ -(c+\alpha)\phi - \frac{\beta}{2}\phi^2 - \gamma\psi \end{pmatrix} \cdot (\lambda, -1)|_{(\phi,\lambda\phi)}$$
$$= \lambda^2\phi + (c+\alpha)\phi + \frac{\beta}{2}\phi^2 + \gamma\lambda\phi \le \phi\big[\lambda^2 + \gamma\lambda + (c+\alpha)\big].$$

It is obvious that $\lambda^2 + \gamma\lambda + (c+\alpha) = 0$ has two real positive roots with $0 < \lambda_1 \le \lambda_2$ provided $c+\alpha \le \frac{\gamma^2}{4}$. This implies that $f \cdot \overrightarrow{n} \le \phi[\lambda^2+\gamma\lambda+(c+\alpha)] \le 0$ provided $0 < \lambda_1 \le \lambda \le \lambda_2$. Thus, one branch of the unstable manifold of E_2 enters the region Ω and joins E_1 to form a heteroclinic orbit. □

3.2 The model with local delay

In this section, we will prove the existence of traveling wave solutions for Eq. (1.3) with a strong local delay kernel by using geometric singular perturbation theory and the linear chain trick. The convolution $f * U$ is defined by

$$(f * U)(x,t) = \int_{-\infty}^{t} f(t-s)U(x,s)\,ds,$$

where the kernel $f : [0,+\infty) \to [0,+\infty)$ satisfies the following normalization assumption, $f(t) \ge 0$ for all $t \ge 0$ and $\int_0^\infty f(t)\,dt = 1$. Note that the normalization assumption on f ensures that the uniform non-negative steady-state solutions, which are $U_1 = 0$, $U_2 = -\frac{2}{\beta}(c+\alpha)$, are unaffected by the delay.

The kernel

$$f(t) = \frac{1}{\tau}e^{-\frac{t}{\tau}} \quad \text{and} \quad f(t) = \frac{t}{\tau^2}e^{-\frac{t}{\tau}}$$

are frequently seen in the literature on delay differential equations (see [16, 18, 20]). The first of two kernels above is sometimes called the weak generic delay kernel and the second one is the strong general delay kernel.

Theorem 2. *Assume that* $0 < \alpha + c \le \frac{\gamma^2}{4}$, $c > 0$, $\beta < 0$, $\gamma < 0$, *then for any sufficiently small* $\tau > 0$, *the equation* (1.3) *with the strong local delay kernel has a traveling wave solution* $U(x,t) = \phi(x+ct)$ *connecting two equilibria* 0 *and* $-\frac{2}{\beta}(c+\alpha)$.

Proof. We consider the diffusive integro-differential equation (1.3) with the strong local delay kernel

$$f(t) = \frac{t}{\tau^2}e^{-\frac{t}{\tau}}, \quad \tau > 0.$$

The parameter $\tau > 0$ measures the delay, which implies that a particular time in the past, namely, τ time units ago, is more important than any other, since the kernel achieves its unique maximum when $t = \tau$. Equation (1.3) becomes

$$U_t + \alpha U_x + \beta U_x \int_{-\infty}^{t} \frac{t-s}{\tau^2} e^{-\frac{t-s}{\tau}} U(x,s)\,ds + \gamma U_{xx} + U_{xxx} = 0.$$

A traveling wave front of it is a solution of the form $U(x,t) = \phi(z)$, where $z = x + ct$, $c > 0$ is called wave speed, $\phi(z)$ is monotone increasing and it satisfies

$$(c+\alpha)\phi' + \beta(f * \phi)\phi' + \gamma\phi'' + \phi''' = 0.$$
$$\phi(-\infty) = U_1, \qquad \phi(\infty) = U_2.$$

It can be integrated once to yield the equation

$$(c+\alpha)\phi + \frac{\beta}{2}\phi(f * \phi) + \gamma\phi' + \phi'' = 0, \tag{3.3}$$

where

$$f * \phi = \int_0^{\infty} \frac{t}{\tau^2} e^{-\frac{t}{\tau}} \phi(z - ct)\, dt. \tag{3.4}$$

If we define $\eta = (f * \phi)(z)$, then differentiating with respect to z, we can obtain that

$$\frac{d\eta}{dz} = \frac{1}{c\tau}(\xi - \eta), \tag{3.5}$$

where

$$\xi(z) = \int_0^{\infty} \frac{1}{\tau} e^{-\frac{t}{\tau}} \phi(z - ct)\, dt. \tag{3.6}$$

Differentiating both side of it with respect to z, we obtain

$$\frac{d\xi}{dz} = \frac{1}{c\tau}(\phi - \xi). \tag{3.7}$$

Define $\phi' = \psi$. Then from (3.5) and (3.7), the traveling wave equation (3.3) can be replaced by the following system

$$\begin{cases} \phi_z = \psi, \\ \psi_z = -(c+\alpha)\phi - \dfrac{\beta}{2}\phi^2 - \gamma\psi, \\ c\tau\eta_z = \xi - \eta, \\ c\tau\xi_z = \phi - \xi. \end{cases} \tag{3.8}$$

Note that when $\tau \to 0$, then it can be seen that $\eta \to \phi$ (this can be most easily seen by examining (3.4) and (3.6). Thus, in this limit, we arrive at the non-delay version of the model

$$\begin{cases} \phi' = \psi, \\ \psi' = -(c+\alpha)\phi - \dfrac{\beta}{2}\phi^2 - \gamma\psi. \end{cases} \tag{3.9}$$

It has been proved that the system (3.9) has a traveling wave solution connecting E_1 and E_2 in Theorem 1. When the τ is non-zero, we will show that (3.8) has traveling wave fronts for sufficiently small $\tau > 0$ by the geometric singular perturbation theory. For $\tau > 0$, (3.8) has two equilibria in the (ϕ, ψ, η, ξ) phase space, i.e.

$$(\phi, \psi, \eta, \xi) = (0, 0, 0, 0), \quad (\phi, \psi, \eta, \xi) = (U_2, 0, U_2, U_2).$$

Note that when $\tau = 0$, (3.8) does not define a dynamical system in R^4. This problem may be overcome by the transformation $z = \tau s$, under which the system becomes

$$\begin{cases} \phi_s = \tau\psi, \\ \psi_s = \tau\Big[-(c+\alpha)\phi - \dfrac{\beta}{2}\phi\eta - \gamma\psi\Big], \\ c\eta_s = \xi - \eta, \\ c\xi_s = \phi - \xi. \end{cases} \tag{3.10}$$

We refer to (3.8) as the slow system and (3.10) as the fast system. The two system are equivalent when $\tau > 0$. If τ is set to zero in (3.8), then the flow of that system is confined to the set

$$M_0 = \big\{(\phi,\psi,\eta,\xi) \in R^4,\ \eta = \xi,\ \xi = \phi\big\},$$

which is a two-dimensional invariant manifold for system (3.8) with $\tau = 0$. In order to obtain a two-dimensional invariant manifold for sufficiently small $\tau > 0$ by geometric singular theory, it suffices to verify the normal hyperbolicity of M_0. The linearized matrix of (3.10) restricted to M_0 is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{c} & \frac{1}{c} \\ \frac{1}{c} & 0 & 0 & -\frac{1}{c} \end{pmatrix}.$$

An easy calculation shows that the matrix has four eigenvalues: $0, 0, -\frac{1}{c}, -\frac{1}{c}$, the number of the eigenvalues with zero real part is equal to $\dim M_0$ and the other eigenvalues are hyperbolic. So it is easy to check that the manifold M_0 is normally hyperbolic by the method of [9]. From geometric singular perturbation theory, it is obvious that there exists a sub-manifold M_τ of the perturbed system (3.8) of R^4 for sufficiently small $\tau > 0$, which can be written as

$$M_\tau = \big\{(\phi,\psi,\eta,\xi) \in R^4,\ \eta = \xi + g(\phi,\psi,\tau),\ \xi = \phi + h(\phi,\psi,\tau)\big\},$$

where the functions g, h satisfy

$$g(u,\widetilde{u},0) = h(u,\widetilde{u},0) = 0.$$

Thus the functions g and h can be expanded into the form of a Taylor series about τ

$$\begin{aligned} g(\phi,\psi,\tau) &= \tau g_1(\phi,\psi) + \tau^2 g_2(\phi,\psi) + \cdots, \\ h(\phi,\psi,\tau) &= \tau h_1(\phi,\psi) + \tau^2 h_2(\phi,\psi) + \cdots. \end{aligned} \tag{3.11}$$

Substituting $\eta = \xi + g(\phi,\psi,\tau)$, $\xi = \phi + h(\phi,\psi,\tau)$ into the slow system (3.8), we have

$$\begin{aligned} c\tau\Big\{\Big(1 + \frac{\partial h}{\partial\phi} + \frac{\partial g}{\partial\phi}\Big)\psi + \Big(\frac{\partial h}{\partial\psi} + \frac{\partial g}{\partial\psi}\Big)\Big[-(c+\alpha)\phi - \frac{\beta}{2}\phi(\phi+h+g) - \gamma\psi\Big]\Big\} &= -g, \\ c\tau\Big\{\Big(1 + \frac{\partial h}{\partial\phi}\Big)\psi + \frac{\partial h}{\partial\psi}\Big[-(c+\alpha)\phi - \frac{\beta}{2}\phi(\phi+h+g) - \gamma\psi\Big]\Big\} &= -h. \end{aligned} \tag{3.12}$$

Combining (3.11) with (3.12) and comparing coefficients of τ and τ^2, we have

$$g_1 = -c\psi, \quad g_2 = 2c^2\Big[-(c+\alpha)\phi - \frac{\beta}{2}\phi^2 - \gamma\psi\Big],$$
$$h_1 = -c\psi, \quad h_2 = c^2\Big[-(c+\alpha)\phi - \frac{\beta}{2}\phi^2 - \gamma\psi\Big]. \tag{3.13}$$

The slow system (3.8) restricted to M_τ is given by

$$\begin{cases} \phi' = \psi, \\ \psi' = -(c+\alpha)\phi - \dfrac{\beta}{2}\phi(\phi + h + g) - \gamma\psi, \end{cases} \tag{3.14}$$

where g and h is given by (3.11) and (3.13). Obviously, the system (3.14) is simplified to the corresponding non-local system (3.9) when $\tau = 0$. It is easy to see system (3.14) has two equilibria points $(\phi, \psi) = (0, 0)$ and $(-\frac{2}{\beta}(c+\alpha), 0)$ for any sufficiently small $\tau > 0$.

In the following, we will prove that there exists a heteroclinic orbit connects the two equilibria, then the equation (1.3) has a traveling wave solution connect 0 and $-\frac{2}{\beta}(c+\alpha)$.

The system (3.14) can be written as

$$\begin{cases} \phi' = \psi, \\ \psi' = \Phi(\phi, \psi, c, \tau), \end{cases} \tag{3.15}$$

where $\tau = \varepsilon^2$. Notice that $\Phi(\phi, \psi, c, \tau) = -(c+\alpha)\phi - \frac{\beta}{2}\phi^2 - \gamma\psi$. From Theorem 1, we know that traveling wavefronts of (3.15) exist when $\tau = 0$. Therefore, in the (ϕ, ψ) phase plane, it can be characterized as the graph of some function w, which means when $\tau = 0$, $\psi = w(\phi, c)$. By the stable manifold theorem, for sufficiently small $\tau > 0$, we can still characterize the stable manifold at $(-\frac{2}{\beta}(c+\alpha), 0)$ as the graph of some function $\psi = w_1(\phi, c, \tau)$, where $w_1(-\frac{2}{\beta}(c+\alpha), c, \tau) = 0$. Furthermore, based on continuous dependence of solution trajectories on parameters, the manifold must still cross the line $\phi = -\frac{1}{\beta}(c+\alpha)$ somewhere provided τ is sufficiently small.

Similarly, let $\psi = w_2(\phi, c, \tau)$ be the equation for the unstable manifold at the origin. It satisfies $w_2(0, c, \tau) = 0$ and it must also cross the line $\phi = -\frac{1}{\beta}(c+\alpha)$ somewhere for suitable sufficiently small τ. Hence

$$w_1(\phi, c, 0) = w_2(\phi, c, 0) = w(\phi, c). \tag{3.16}$$

For the unperturbed problem, fix a value of $c = c^* \leq \frac{\gamma^2}{4} - \alpha$, so that the equation of corresponding front in the phase plane is $\psi = w(\phi, c^*)$. In order to show that a heteroclinic connection exists in the perturbed problem $(\tau > 0)$, we only need to prove that there exists a value of $c = c(\tau)$, near to c^*, so that the manifold w_1 and w_2 cross the line $\phi = -\frac{1}{\beta}(c+\alpha)$ at a point. We will use the implicit function theorem to prove there exists a unique wave speed $c = c(\tau)$. For this purpose, we construct auxiliary function

$$F(c, \tau) = w_1\Big(-\frac{1}{\beta}(c+\alpha), c, \tau\Big) - w_2\Big(-\frac{1}{\beta}(c+\alpha), c, \tau\Big).$$

Next, we only need to verify $\frac{\partial F}{\partial c}|_{(c^*,0)} \neq 0$. Use the system (3.14) and (3.16), it is easy to know

$$\frac{d\psi}{d\phi} = \frac{\psi'}{\phi'} = \frac{\Phi(\phi,\psi,c,\tau)}{\psi},$$

$$i\frac{d}{d\phi}\left(\frac{\partial w_1}{\partial c}(\phi,c^*,0)\right)=\frac{\partial}{\partial c}\left(\frac{\partial w_1}{\partial \phi}(\phi,c,0)\right)\Big|_{c=c^*}=\frac{\partial}{\partial c}\left(\frac{\Phi(\phi,w_1(\phi,c,0),c,\tau)}{w_1(\phi,c,0)}\right)\Big|_{c=c^*}$$
$$=\frac{\partial}{\partial c}\left(-\frac{c+\alpha}{w_1(\phi,c,0)}\phi-\frac{\beta\phi^2}{2w_1(\phi,c,0)}-\gamma\right)\Big|_{c=c^*}$$
$$=\left(\frac{\phi(c+\alpha)}{w_1^2}+\frac{\beta\phi^2}{2w_1^2}\right)\left(\frac{\partial w_1}{\partial c}(\phi,c^*,0)\right)-\frac{\phi}{w_1(\phi,c^*)}.$$

Integrating it from $\phi = -\frac{1}{\beta}(c+\alpha)$ to $\phi = -\frac{2}{\beta}(c+\alpha)$, we have

$$\frac{\partial w_1}{\partial c}\left(-\frac{c+\alpha}{\beta},c^*,0\right)$$
$$=\frac{s}{w_1(s,c^*)}\int_{-\frac{c+\alpha}{\beta}}^{-\frac{2(c+\alpha)}{\beta}}\exp\left(-\int_{-\frac{c+\alpha}{\beta}}^{z}\frac{\beta s^2+2s(c+\alpha)}{2w^2(s,c^*)}\,ds\right)dz. \tag{3.17}$$

Similarly, we have

$$\frac{\partial w_2}{\partial c}\left(-\frac{c+\alpha}{\beta},c^*,0\right)$$
$$=-\frac{s}{w_1(s,c^*)}\int_{0}^{-\frac{c+\alpha}{\beta}}\exp\left(-\int_{-\frac{c+\alpha}{\beta}}^{z}\frac{\beta s^2+2s(c+\alpha)}{2w^2(s,c^*)}\,ds\right)dz. \tag{3.18}$$

From (3.17) and (3.18), one has

$$\frac{\partial F}{\partial c}(c^*,0)=\frac{\partial w_1}{\partial c}\left(-\frac{c+\alpha}{\beta},c^*,0\right)-\frac{\partial w_2}{\partial c}\left(-\frac{c+\alpha}{\beta},c^*,0\right)$$
$$=\frac{s}{w_1(s,c^*)}\int_{0}^{-\frac{2(c+\alpha)}{\beta}}\exp\left(-\int_{-\frac{c+\alpha}{\beta}}^{z}\frac{\beta s^2+2s(c+\alpha)}{2w^2(s,c^*)}\,ds\right)dz$$
$$>0.$$

Thus the equation (1.3) with the strong general delay kernel has a traveling wave solution $U(x,t) = \phi(x+ct)$ connecting two equilibria 0 and $-\frac{2}{\beta}(c+\alpha)$ for any sufficiently small $\tau > 0$. □

3.3 The model with nonlocal delay

In this section, we will prove the existence of traveling wave solutions for Eq. (1.3) with a nonlocal weak generic kernel by using geometric singular perturbation theory, the Fredholm theory and the linear chain trick. The convolution $f * U$ is denoted by

$$(f*U)(x,t)=\int_{-\infty}^{t}\int_{-\infty}^{\infty}f(x-y,t-s)U(y,s)dyds,$$

where the kernel $f : [0, +\infty) \to [0, +\infty)$ satisfies the following normalization assumption, $f(t) \geq 0$ for all $t \geq 0$ and

$$\int_{-\infty}^{t} \int_{-\infty}^{\infty} f(x,t)\, dxdt = 1.$$

Note that equations of various types can be derived from Eq. (1.3) by taking different delay kernels. For example, when taking the kernel to be $f(x,t) = \delta(x)\delta(t)$, where δ denotes Dirac's delta function, then Eq. (1.3) becomes the corresponding undelayed perturbed KdV–mKdV equation

$$U_t + \alpha U_x + \beta U U_x + \gamma U_{xx} + U_{xxx} = 0.$$

While taking $f(x,t) = \delta(x)\delta(t-\tau)$, Eq. (1.3) becomes the following equation with discrete delay

$$U_t + \alpha U_x + \beta U(x, t-\tau) U_x + \gamma U_{xx} + U_{xxx} = 0.$$

We consider the two special kernels

$$f_1(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \frac{1}{\tau} e^{-\frac{t}{\tau}}, \quad f_2(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \frac{t}{\tau^2} e^{-\frac{t}{\tau}}, \tag{3.19}$$

where $\tau > 0$ in each case. The first of two kernels above is sometimes called the weak generic delay kernel and the second one is the strong general delay kernel. The two kernels have been frequently used in the literature (see [2, 9, 15, 18]).

In this section, we will consider the condition that Eq. (1.3) with the following weak generic delay kernel $f_1(x.t)$.

Theorem 3. *For any sufficiently small $\tau > 0$, there exists the speed $c \leq \frac{\gamma^2}{4} - \alpha$ such that equation* (1.3) *with the weak generic kernel has a traveling wave solution $U(x,t) = \phi(x+ct)$ connecting its two equilibria* 0 *and* $-\frac{2}{\beta}(c+\alpha)$.

Proof. We can define that

$$V(x,t) = (f * U)(x,t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} \frac{1}{\tau} e^{-\frac{t-s}{\tau}} U(y,s)\, dy\, ds.$$

Through the direct calculation (see [18]) we have

$$V_t = V_{xx} + \frac{1}{\tau}(U - V).$$

So Eq. (1.3) can be written as

$$\begin{cases} U_t + \alpha U_x + \beta V U_x + \gamma U_{xx} + U_{xxx} = 0, \\ V_t - V_{xx} - \dfrac{1}{\tau}(U - V) = 0. \end{cases}$$

In order to find the traveling wave solution, taking $U(x,t) = \phi(z), V(x,t) = \psi(z)$, $z = x + ct$ and $c > 0$, then we obtain the flowing system

$$\begin{cases} (c+\alpha)\phi' + \beta\psi\phi' + \gamma\phi'' + \phi''' = 0, \\ c\psi' - \psi'' - \dfrac{1}{\tau}(\phi - \psi) = 0, \end{cases} \tag{3.20}$$

where $' = \frac{d}{dz}$, under the boundary value conditions

$$\begin{cases} \lim\limits_{z\to-\infty} \big(\phi(z), \psi(z)\big) = (0,0), \\ \lim\limits_{z\to+\infty} \big(\phi(z), \psi(z)\big) = \Big(-\dfrac{2}{\beta}(c+\alpha), -\dfrac{2}{\beta}(c+\alpha)\Big). \end{cases}$$

The first equation of the system (3.20) can be integrated once to yield the equation

$$(c+\alpha)\phi + \frac{\beta}{2}\phi\psi + \gamma\phi' + \phi'' = 0,$$

then we get the following two second order ordinary differential equations

$$\begin{cases} (c+\alpha)\phi + \dfrac{\beta}{2}\phi\psi + \gamma\phi' + \phi'' = 0, \\ \psi'' - c\psi' + \dfrac{1}{\tau}(\phi - \psi) = 0. \end{cases}$$

By defining new variables $\phi' = \phi_1$, $\psi' = \psi_1$ the system can be formulated as

$$\begin{cases} \phi' = \phi_1, \\ \phi_1' = -(c+\alpha)\phi - \dfrac{\beta}{2}\phi\psi - \gamma\phi_1, \\ \psi' = \psi_1, \\ \psi_1' = c\psi_1 - \dfrac{1}{\tau}(\phi - \psi). \end{cases} \tag{3.21}$$

Let $\varepsilon = \sqrt{\tau}$ and define new variables

$$u_1 = \phi, \quad u_2 = \phi_1, \quad v_1 = \psi, \quad v_2 = \varepsilon\psi_1,$$

thus the system (3.21) can be cast into standard form for a singular perturbation problem

$$\begin{cases} u_1' = u_2, \\ u_2' = -(c+\alpha)u_1 - \dfrac{\beta}{2}u_1v_1 - \gamma u_2, \\ \varepsilon v_1' = v_2, \\ \varepsilon v_2' = c\varepsilon v_2 - (u_1 - v_1), \end{cases} \tag{3.22}$$

which is called a slow system. When $\varepsilon = 0$, the system becomes

$$\begin{cases} u_1' = u_2, \\ u_2' = -(c+\alpha)u_1 - \dfrac{\beta}{2}u_1v_1 - \gamma u_2, \end{cases} \tag{3.23}$$

which has a heteroclitic orbit connecting its two equilibria $(0,0)$ and $(-\frac{2}{\beta}(c+\alpha), 0)$ from Theorem 1. What we will prove is that the system (3.22) has a traveling solution connecting $(0,0,0,0)$ and $(-\frac{2}{\beta}(c+\alpha), 0, -\frac{2}{\beta}(c+\alpha), 0)$ for sufficiently small $\tau > 0$. Note that when $\varepsilon \neq 0$, it do not define a dynamic system in R^4. This problem can be overcome through the transformation $\xi = \varepsilon\eta$, under which the system (3.22) becomes

$$\begin{cases} \dot{u_1} = \varepsilon u_2, \\ \dot{u_2} = -(c+\alpha)\varepsilon u_1 - \dfrac{\beta}{2}\varepsilon u_1 v_1 - \gamma\varepsilon u_2, \\ \dot{v_1} = v_2, \\ \dot{v_2} = c\varepsilon v_2 - (u_1 - v_1), \end{cases} \tag{3.24}$$

where $\cdot$ denotes the derivative with respect to η. The system (3.24) is called a fast system. The slow system (3.22) and fast system (3.24) are equivalent when $\varepsilon > 0$. Let $\varepsilon \to 0$ in system (3.22), then the flow of system (3.22) is confined to the set

$$M_0 = \{(u_1, u_2, v_1, v_2) \in R^4,\ v_2 = 0,\ v_1 = u_1\},$$

which is a two-dimensional invariant manifold for system (3.22) with $\varepsilon = 0$. In order to obtain a two-dimensional invariant manifold for efficiently small $\varepsilon > 0$ by geometric singular theory, it suffices to verify the normal hyperbolicity of M_0. The linearized matrix of (3.24) restricted to M_0 is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix},$$

which has four eigenvalues: $0, 0, 1, -1$, the number of the eigenvalues with zero real part is equal to $\dim M_0$ and the other eigenvalues are hyperbolic. So it is easy to check that the manifold M_0 is normally hyperbolic by the method of [9]. From geometric singular perturbation theory, it is obvious that there exists a sub-manifold M_ε of the perturbed system (3.22) of R^4 for sufficiently small $\varepsilon > 0$, which can be written as

$$M_\varepsilon = \{(u_1, u_2, v_1, v_2) \in R^4,\ v_1 = u_1 + f(u_1, u_2, \varepsilon),\ v_2 = g(u_1, u_2, \varepsilon)\},$$

where f, g depend smoothly on ε and satisfy

$$f(u_1, u_2, 0) = g(u_1, u_2, 0) = 0.$$

Thus the functions f and g can be expanded into the form of a Taylor series about ε

$$\begin{aligned} f(u_1, u_2, \varepsilon) &= \varepsilon f_1 + \varepsilon^2 f_2 + \cdots, \\ g(u_1, u_2, \varepsilon) &= \varepsilon g_1 + \varepsilon^2 g_2 + \cdots. \end{aligned} \tag{3.25}$$

Substituting $v_1 = u_1 + f$ and $v_2 = g$ into the slow system (3.22), we have

$$\begin{aligned} \varepsilon\left[\frac{\partial f}{\partial u_1}u_2 + \frac{\partial f}{\partial u_2}\left(-(c+\alpha)u_1 - \frac{\beta}{2}u_1(u_1+f) - \gamma u_2\right) + u_2\right] &= g, \\ \varepsilon\left[\frac{\partial g}{\partial u_1}u_2 + \frac{\partial g}{\partial u_2}\left(-(c+\alpha)u_1 - \frac{\beta}{2}u_1(u_1+f) - \gamma u_2\right)\right] &= c\varepsilon g + f. \end{aligned} \tag{3.26}$$

Combining (3.25) with (3.26) and comparing coefficients of ε and ε^2, we have

$$f_1 = 0, \qquad f_2 = -cu_1 - \alpha u_1 - \frac{\beta}{2}u_1^2 - cu_2 - \gamma u_2, \qquad g_1 = u_2, \qquad g_2 = 0.$$

Hence we have

$$f = \varepsilon^2\Big(-cu_1 - \alpha u_1 - \frac{\beta}{2}u_1^2 - cu_2 - \gamma u_2\Big) + o\big(\varepsilon^2\big), \qquad g = \varepsilon u_2 + o\big(\varepsilon^2\big). \quad (3.27)$$

The slow system (3.23) restricted to M_ε is given by

$$\begin{cases} u_1' = u_2, \\ u_2' = -(c+\alpha)u_1 - \frac{\beta}{2}u_1(u_1 + f) - \gamma u_2, \end{cases} \quad (3.28)$$

where f is given by (3.27). Obviously, the system (3.28) is simplified to (3.23) when $\varepsilon = 0$. It is easy to see system (3.28) has two equilibria $(u_1, u_2) = (0, 0)$ and $(-\frac{2}{\beta}(c+\alpha), 0)$ for any sufficiently small $\varepsilon > 0$. In the following, we will prove that there exists a heteroclinic orbit connects the two equilibria, then the equation (1.3) has a traveling wave solution connect 0 and $-\frac{2}{\beta}(c+\alpha)$. We set

$$u_1 = u_0 + \epsilon^2\phi + \cdots, \qquad u_2 = \widetilde{u}_0 + \epsilon^2\psi + \cdots$$

Substituting into (3.28), we find that, comparing the coefficients of ϵ^2, the differential equation system determining ϕ and ψ is

$$\frac{d}{dz}\begin{pmatrix}\phi\\ \psi\end{pmatrix} + \begin{pmatrix} 0 & -1 \\ (c+\alpha)+\beta u_0 & \gamma \end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\beta}{2}u_0 F(u_0, \widetilde{u}_0) \end{pmatrix}, \quad (3.29)$$

where $F(u_0, \widetilde{u}_0) = -cu_0 - \alpha u_0 - \frac{\beta}{2}u_0^2 - c\widetilde{u}_0 - \gamma\widetilde{u}_0$ and we shall show that this system has a solution satisfying $\phi(\pm\infty) = 0$ and $\psi(\pm\infty) = 0$.

Let L^2 denote the space of square integral functions, with inner production

$$\int_{-\infty}^{\infty}\big(x(z), y(z)\big)\,dz,$$

$(\cdot,\cdot)$ being the Euclidean inner product on R^2. From the Fredholm theory, we know that (3.29) will have a solution if and only if

$$\int_{-\infty}^{\infty}\left(x(z), \begin{pmatrix} 0 \\ -\frac{\beta}{2}u_0 F(u_0, \widetilde{u}_0) \end{pmatrix}\right)dz = 0,$$

for all functions $x(z) \in R^2$ in the kernel of the adjoint of operator l defined by the left-hand side of (3.29). It is easy to verify that the adjoint operator l^* is given by

$$l^* = -\frac{d}{dz} + \begin{pmatrix} 0 & (c+\alpha)+\beta u_0 \\ -1 & \gamma \end{pmatrix}, \quad (3.30)$$

and thus to compute $Kerl^*$ we have to find all $x(z)$ satisfying

$$\frac{dx(z)}{dz} = \begin{pmatrix} 0 & (c+\alpha)+\beta u_0 \\ -1 & \gamma \end{pmatrix} x(z), \quad (3.31)$$

the general solution of which will be difficult to find because the matrix is nonconstant. However, we are only looking for solutions satisfying $x(\pm\infty) = 0$ and, in fact, the only such solution is the zero solution. Recall that $u_0(z)$ is the solution of the unperturbed problem and although we have no explicit express for it, we do know that it tends to zero as $z \to -\infty$. Letting $z \to -\infty$ in (3.31), the matrix becomes a constant matrix, with eigenvalues λ satisfying

$$\lambda^2 - \gamma\lambda + c + \alpha = 0,$$

and, since $c + \alpha \leq \frac{\gamma^2}{4}$, the eigenvalues are therefore both real and negative. Hence, as $z \to -\infty$, the solution of (3.31) other than the zero solution must decreasing exponentially for small z. So the only solution satisfying $x(\pm\infty) = 0$ is the zero solution. This means, of course, that the Fredholm orthogonality condition trivially holds and so solutions of (3.30) exist, which satisfy $\phi(\pm\infty) = 0$ and $\psi(\pm\infty) = 0$. Therefore a heteroclinic connection exists between the two non-negative equilibrium points $(0, 0)$ and $(-\frac{2}{\beta}(c+\alpha), 0)$ of (3.29) for sufficiently small $\epsilon > 0$. Furthermore, when $\tau > 0$ is sufficiently small, a heteroclinic connection exists between the two non-negative equilibrium points $(0, 0, 0, 0)$ and $(-\frac{2}{\beta}(c + \alpha), 0, -\frac{2}{\beta}(c + \alpha), 0)$ of (3.21). Therefore, Eq. (1.3) with (3.19) exists traveling wavefronts, which connect the two trivial equilibria u_1 and u_2. From the above discussion, we can obtain Theorem 3. □

Acknowledgements

The authors express their sincere thanks to the anonymous reviewers for their valuable comments and suggestions for improving the quality of the paper.

References

[1] M. Antonova and A. Biswas. Adiabatic parameter dynamics of perturbed solitary waves. *Commun. Nonlinear Sci. Numer. Simul.*, **14**:734–748, 2009. http://dx.doi.org/10.1016/j.cnsns.2007.12.004.

[2] P. Ashwin, M.V. Bartuccelli and S.A. Gourley. Traveling fronts for the KPP equation with spatio-temporal delay. *Z. Angew. Math. Phys.*, **53**:103–122, 2002. http://dx.doi.org/10.1007/s00033-002-8145-8.

[3] J.M. Burgers. Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion. *Trans. Roy. Neth. Acad. Sci. Amsterdam*, **17**:1–53, 1939.

[4] J.C. Ceballos, M. Sepulveda and O.P.V. Villagra'n. The Korteweg–de Vries–Kawahara equation in a bounded domain and some numerical results. *Appl. Math. Comput.*, **190**:912–936, 2007. http://dx.doi.org/10.1016/j.amc.2007.01.107.

[5] C. David, R. Fernando and Z.S. Feng. On solitary wave solutions of the compound Burgers–Korteweg–de Vries equation. *Phys. Lett. A*, **375**:44–50, 2007.

[6] Z.S. Feng. On traveling wave solutions of the Burgers–Korteweg–de Vries equation. *Nonlinearity*, **20**:343–356, 2007. http://dx.doi.org/10.1088/0951-7715/20/2/006.

[7] Z.S. Feng and R. Knobel. Traveling waves to a Burgers–Korteweg–de Vries-type equation with higher-order nonlinearities. *J. Math. Anal. Appl.*, **328**:1435–1450, 2007. http://dx.doi.org/10.1016/j.jmaa.2006.05.085.

[8] N. Fenichel. Geometric singular perturbation theory for ordinary differential equations. *J. Differential Equations*, **31**:53–98, 1979. http://dx.doi.org/10.1016/0022-0396(79)90152-9.

[9] S.A. Gourley and S.G. Ruan. Convergence and traveling wave fronts in functional differential equations with nonlocal terms: A competition model. *SIAM. J. Math. Anal.*, **35**:806–822, 2003. http://dx.doi.org/10.1137/S003614100139991.

[10] R.S. Johnson. A nonlinear equation incorporating damping and dispersion. *J. Fluid Mech.*, **42**:49–60, 1970. http://dx.doi.org/10.1017/S0022112070001064.

[11] D.J. Korteweg and G. de Vries. On the change of form of long waves advancing in a rectangular channel and on a new type of long stationary waves. *Philos. Mag. R Soc. London*, **39**:422–413, 1895.

[12] X.Z. Li and M.L. Wang. A sub-ODE method for finding exact solutions of a generalized KdV–mKdV equation with high-order nonlinear terms. *Phys. Lett. A*, **361**:115–118, 2007. http://dx.doi.org/10.1016/j.physleta.2006.09.022.

[13] Z.L. Li. Constructing of new exact solutions to the GKdV–mKdV equation with any-order nonlinear terms by (G'/G)-expansion method. *Appl. Math. Comput.*, **217**:1398–1403, 2010. http://dx.doi.org/10.1016/j.amc.2009.05.034.

[14] M.B.A. Mansour. Traveling wave solutions for a singularly perturbed Burgers–KdV equation. *Ind. Acad. Sci.*, **73**:799–806, 2009.

[15] C.H. Ou and J.H. Wu. Persistence of wavefronts in delayed nonlocal reaction–diffusion equations. *J. Differential Equations*, **238**:219–261, 2007. http://dx.doi.org/10.1016/j.jde.2006.12.010.

[16] S.G. Ruan and D.M. Xiao. Stability of steady states and existence of traveling wave in a vector disease model. *Proc. Roy. Soc. Edinburgh*, **134A**:991–1011, 2004. http://dx.doi.org/10.1017/S0308210500003590.

[17] J.W. Shen. Shock wave solutions of the compound Burgers–Korteweg–de Vries equation. *Appl. Math. Comput.*, **196**:842–849, 2008. http://dx.doi.org/10.1016/j.amc.2007.07.029.

[18] Y.L. Song, Y.H. Peng and M.A. Han. Traveling wavefronts in the diffusive single species model with Allee effect and distributed delay. *Appl. Math. Comput.*, **152**:483–497, 2004. http://dx.doi.org/10.1016/S0096-3003(03)00571-X.

[19] H. Triki, T.R. Taha and A.M. Wazwaz. Solitary wave solutions for a generalized KdV–mKdV equation with variable coefficients. *Math. Comput. Simul.*, **80**:1867–1873, 2010. http://dx.doi.org/10.1016/j.matcom.2010.02.001.

[20] Z.H. Zhao. Solitary waves of the generalized KdV equation with distributed delays. *J. Math. Anal. Appl.*, **344**:32–41, 2008. http://dx.doi.org/10.1016/j.jmaa.2008.02.036.