Mathematical Modelling and Analysis
Volume 19 Number 3, June 2014, 417–429
http://dx.doi.org/10.3846/13926292.2014.927014

Publisher: Taylor&Francis and VGTU
http://www.tandfonline.com/TMMA
Print ISSN: 1392-6292
Online ISSN: 1648-3510

On Solvability of the Damped Fučík Type Problem with Integral Condition*

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Received October 30, 2013; revised May 18, 2014; published online June 1, 2014

Abstract. The solvability results are established for the boundary value problem with a damping term $x'' + 2\delta x' = -\mu x^+ + \lambda x^- + h(t, x, x')$, $x(0) = 0$, $\int_0^1 x(s)\,ds = 0$, where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, h is a bounded nonlinearity, μ, λ real parameters. The existence results are based of the knowledge of the Fučík type spectrum for the problem with $h \equiv 0$.

Keywords: Fučík problem, spectrum, regions of solvability, damping term.

AMS Subject Classification: 34B15.

1 Introduction

Consider the problem

$$x'' = -\mu x^+ + \lambda x^-, \quad x(0) = 0, \quad x(1) = 0, \tag{1.1}$$

where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$ and $\mu \in R$, $\lambda \in R$.

The Fučík spectrum of the problem (1.1) is the set of such points (μ, λ) that the problem has a nontrivial solution. The spectrum for the problem (1.1) is well known and consists of infinite set of curves (branches) F_i^+ and F_i^- $(i = 0, 1, 2, \ldots)$.

The lower index shows how many zeroes the respective solution has in the interval $(0, 1)$, but the upper index shows the sign of the derivative of the solution at $t = 0$. All the branches of the spectrum, except F_0^+ and F_0^-, are located in the first quadrant, but F_0^+ and F_0^- can be continued outside the first quadrant in the manner given in the work [2].

Consider the problem

$$x'' = -\mu x^+ + \lambda x^-, \quad x(0) = 0, \quad \int_0^1 x(s)\,ds = 0. \tag{1.2}$$

* This work was supported by Latvian Council of Science through the project 345/2012.

The spectrum of this problem is a union of two symmetric branches stretched along the bisectrix of the first and the third quadrants.

Both analytical and graphical description of the spectrum of the problem (1.2) was given in the work [9]. Some of the branches of the spectrum for the problems (1.1) and (1.2) are shown in Fig. 1, 2.

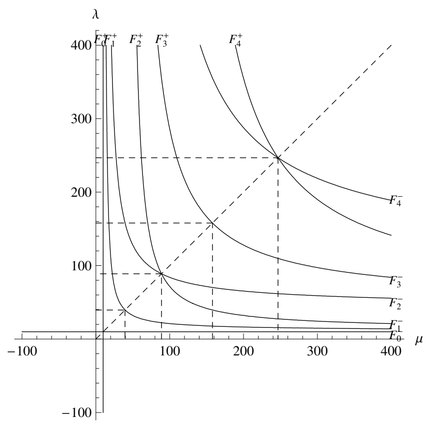


Figure 1. The spectrum of the problem (1.1).

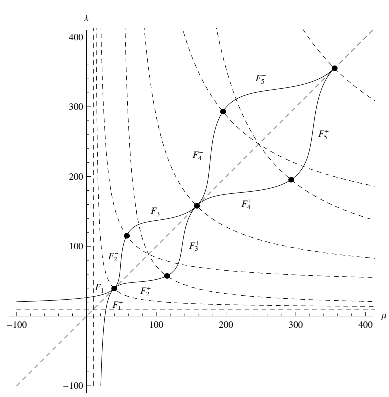


Figure 2. The spectrum of the problem (1.2).

The knowledge of the spectra can be used to define regions of solvability ("good" regions) for the problems

$$x'' = -\mu x^+ + \lambda x^- + h(t, x, x'), \quad x(0) = 0, \quad x(1) = 0, \tag{1.3}$$

$$x'' = -\mu x^+ + \lambda x^- + h(t, x, x'), \quad x(0) = 0, \quad \int_0^1 x(s)\,ds = 0, \tag{1.4}$$

where h is bounded continuous function. It was obtained in the works [2] and [4] respectively. We call the regions "good" regions if the following is true: if (μ, λ) belongs to the "good" region then the problem is solvable, if (μ, λ) does not belong to the "good" region then the problem is solvable or not. The "good" regions of (μ, λ)-plane for the problems (1.3) and (1.4) are shown in Fig. 3 and 4. If (μ, λ) are in the shared region, but not on the Fučík spectrum, then the problem (1.3) (or (1.4)) is certainly solvable for any bounded continuous $h(t, x, x')$.

Fučík spectra of various differential operators and solvability of corresponding nonlinear problems both in resonance and non-resonance cases have been studied by many authors, let us mention [1, 5, 8], and the references therein.

In this article we consider the equation

$$x'' + 2\delta x' = -\mu x^+ + \lambda x^- \tag{1.5}$$

with conditions

$$x(0) = 0, \quad \int_0^1 x(s)\,ds = 0 \tag{1.6}$$

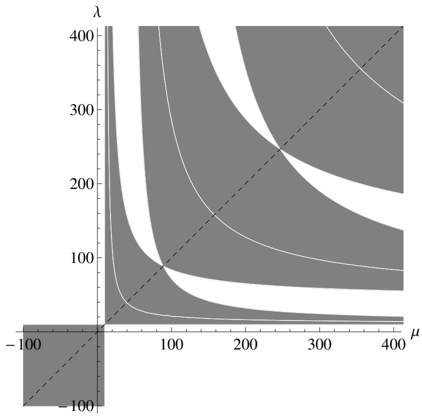


Figure 3. The solvability regions for the problem (1.3).

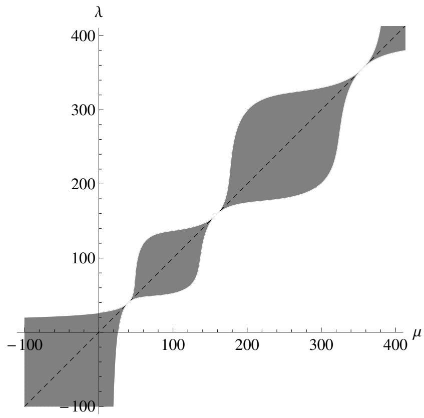


Figure 4. The solvability regions for the problem (1.4).

as well as essentially nonlinear equation

$$x'' + 2\delta x' = -\mu x^+ + \lambda x^- + h(t, x, x') \tag{1.7}$$

with the same conditions (1.6).

The spectrum obtained for the problem (1.5), (1.6) helps to state the existence results for the problem (1.7), (1.6). It is interesting to mention that counting the damping term in equation (1.7) for $\delta > 0$ leads to the enlargement of the region of parameters (μ, λ) for which the problem is solvable.

The equations (1.5) and (1.7) provide a fairly natural generalization of the classical linear oscillator, the restoring force being here piecewise linear. The interest for such equations has been motivated in particular by the models of suspension bridges. The interested reader may consult the works [6, 7] for additional information and references therein.

2 Related Results for the Dirichlet Problem with x'

In order to describe properties of the spectrum for the problem (1.5), (1.6) we provide some information about the problem below

$$x'' + 2\delta x' = -\mu x^+ + \lambda x^-, \quad x(0) = 0, \quad x(1) = 0. \tag{2.1}$$

The spectrum of the problem (2.1) was obtained in the work [3]. The branches of the spectrum for the problem (2.1) can be obtained from the classical Fučík spectrum by translation parallel to the vector (δ^2, δ^2). The spectrum of the problem (2.1) consists of the branches given by the next equations:

$$F_0^+(\delta) = \big\{(\mu, \lambda)\colon \mu = \delta^2 + \pi^2,\ \lambda \in \mathbb{R}\big\},$$
$$F_{2i-1}^+(\delta) = \Big\{(\mu, \lambda)\colon \frac{i\pi}{\sqrt{\mu - \delta^2}} + \frac{i\pi}{\sqrt{\lambda - \delta^2}} = 1\Big\},$$

$$F_{2i}^{+}(\delta) = \left\{(\mu,\lambda)\colon \frac{(i+1)\pi}{\sqrt{\mu-\delta^2}} + \frac{i\pi}{\sqrt{\lambda-\delta^2}} = 1\right\}, \quad F_i^{-}(\delta) = \left\{(\mu,\lambda)\colon (\lambda,\mu) \in F_i^{+}\right\}.$$

The solvability regions for the problem

$$x'' + 2\delta x' = -\mu x^{+} + \lambda x^{-} + h(t,x,x'), \quad x(0)=0, \quad x(1)=0 \tag{2.2}$$

for continuous and bounded $h(t,x,x')$ were considered in the work [3] also.

The spectrum of the problem (2.1) and solvability regions for the problem (2.2) are shown in Fig. 5 and 6. Let us remark that the presence of a damping term in the problem (1.1) does not essentially change the spectrum and solvability regions [3, p. 183].

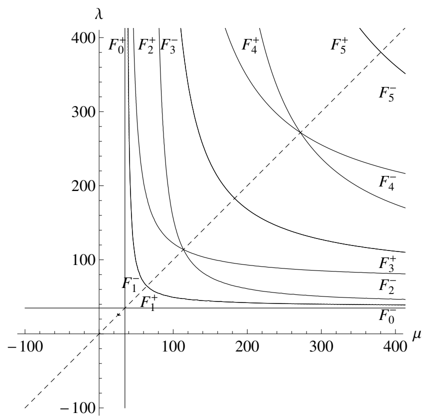


Figure 5. The spectrum of the problem (2.1) for $\delta = 5$.

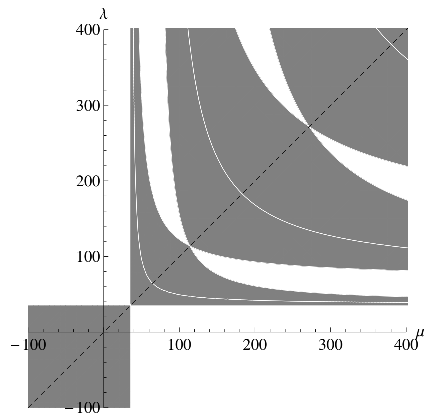


Figure 6. The solvability regions for the problem (2.2) for $\delta = 5$.

3 The Spectrum of the Problem Depending of x' with Integral Condition

Now consider the problem (1.5), (1.6). This problem is a generalization of the problem (1.2). For $\delta = 0$ the problem (1.5), (1.6) reduces to the above mentioned problem (1.2).

The description of the spectrum for the problem (1.5), (1.6) is given in the following results.

Lemma 1. *The branches $F_0^{\pm}$ of the spectrum of the problem* (1.5), (1.6) *do not exist.*

Proof. It is clear that the solution of the problem (1.5), (1.6) must have at least one zero in the interval $(0,1)$ in order to meet the second condition in (1.6). That is why the branches $F_0^{\pm}$ of the spectrum for the problem (1.5), (1.6) do not exist. $\square$

Lemma 2. *The branches $F_1^{\pm}$ of the spectrum of the problem* (1.5), (1.6) *consist of three parts. The branch F_1^{+} is located in the first and fourth quadrants of the (μ,λ)- plane, but the branch F_1^{-} is located in the first and second quadrants of the (μ,λ)- plane.*

Proof. Now suppose that $(\mu,\lambda)\in F_1^{+}$ and let $x(t)$ be a respective nontrivial solution of the problem (1.5), (1.6). The solution has only one zero in the interval $(0,1)$ and $x'(0)=\alpha>0$. Let this zero be denoted by τ. Consider the problem (1.5), (1.6) in the interval $(0,\tau)$. We obtain that it reduces to the problem

$$x''+2\delta x'+\mu x=0,\quad x(0)=0,\ \ x(\tau)=0.$$

The solution

$$x(t;\alpha)=\frac{\alpha}{\sqrt{\mu-\delta^2}}\exp(-\delta t)\sin\bigl(\sqrt{\mu-\delta^2}\,t\bigr) \tag{3.1}$$

of the Cauchy problem

$$x''+2\delta x'+\mu x=0,\quad x(0)=0,\ \ x'(0)=\alpha>0$$

has the first zero at the point $t=\frac{\pi}{\sqrt{\mu-\delta^2}}$. So $\tau=\frac{\pi}{\sqrt{\mu-\delta^2}}$. In view of $\tau<1$ we obtain that $\mu>\pi^2+\delta^2$.

We will use τ in the next formulas instead of $\frac{\pi}{\sqrt{\mu-\delta^2}}$ to simplify them. The calculations show that

$$x'(\tau)=-\alpha\exp(-\delta\tau)=\alpha_-, \tag{3.2}$$

$$\int_0^{\tau}x(s)\,ds=\frac{\alpha}{\mu}\bigl(1+\exp(-\delta\tau)\bigr). \tag{3.3}$$

Now consider the problem (1.5), (1.6) in the interval $(\tau,1)$. In this interval we obtain the problem

$$x''+2\delta x'+\lambda x=0,\quad x(\tau)=0,\quad x'(\tau)=\alpha_-. \tag{3.4}$$

The solution of the problem (3.4) must satisfy the next condition

$$\int_{\tau}^{1}x(s)\,ds=-\frac{\alpha}{\mu}\bigl(1+\exp(-\delta\tau)\bigr)$$

in order to meet the second condition in (1.6). According to $\delta^2-\lambda$ value we obtain three different cases.

Case 1. Consider $\delta^2-\lambda>0$ or $\lambda<\delta^2$. We obtain that the solution which satisfies the last problem is

$$x(t;\alpha_-)=\frac{\alpha_-}{\sqrt{\delta^2-\lambda}}\exp(-\delta(t-\tau))\sinh(\sqrt{\delta^2-\lambda}(t-\tau))$$

or in view of (3.2)

$$x(t;\alpha)=\frac{-\alpha\exp(-\delta\tau)}{\sqrt{\delta^2-\lambda}}\exp(-\delta(t-\tau))\sinh(\sqrt{\delta^2-\lambda}(t-\tau)).$$

It follows that

$$\int_\tau^1 x(s)\,ds = \frac{\alpha}{\lambda}\Big(-\exp(-\delta\tau) + \exp(-\delta)\cosh\big(\sqrt{\delta^2-\lambda}(1-\tau)\big) + \frac{\delta\exp(-\delta)}{\sqrt{\delta^2-\lambda}}\sinh\big(\sqrt{\delta^2-\lambda}(1-\tau)\big)\Big). \tag{3.5}$$

From (3.3) and (3.5) we obtain the following equation

$$\lambda\big(1+\exp(-\delta\tau)\big) + \mu\Big(-\exp(-\delta\tau) + \exp(-\delta)\cosh\big(\sqrt{\delta^2-\lambda}(1-\tau)\big) + \frac{\delta\exp(-\delta)}{\sqrt{\delta^2-\lambda}}\sinh\big(\sqrt{\delta^2-\lambda}(1-\tau)\big)\Big) = 0. \tag{3.6}$$

This equation makes it possible to calculate (μ,λ) values which correspond to the branch $F^+_{1,<}$. Let us remark that we introduce new notation here, the additional "$<$" sign means that $\lambda < \delta^2$.

Case 2. Consider $\delta^2 - \lambda = 0$ or $\lambda = \delta^2$. We obtain such solution of the problem (3.4) $x(t;\alpha_-) = \alpha_-(t-\tau)\exp(-\delta(t-\tau))$ or in view of (3.2) $x(t;\alpha) = -\alpha\exp(-\delta\tau)(t-\tau)\exp(-\delta(t-\tau))$. It follows that

$$\int_\tau^1 x(s)\,ds = \frac{\alpha}{\delta^2}\big(-\exp(-\delta\tau) + \exp(-\delta)(1+\delta-\delta\tau)\big). \tag{3.7}$$

From (3.3) and (3.7) we obtain the equation

$$\delta^2\big(1+\exp(-\delta\tau)\big) + \mu\big(-\exp(-\delta\tau) + \exp(-\delta)(1+\delta-\delta\tau)\big) = 0. \tag{3.8}$$

The last equation makes it possible to calculate μ value which corresponds to the point (μ,δ^2) in the (μ,λ) plane. Let us introduce the notation $F^+_{1,=}$ for this point where additional "$=$" sign means that $\lambda = \delta^2$.

Case 3. Consider $\delta^2 - \lambda < 0$ or $\lambda > \delta^2$. We obtain that the solution of the problem (3.4) is

$$x(t;\alpha) = \frac{-\alpha\exp\big(-\delta\tau\big)}{\sqrt{\lambda-\delta^2}}\exp\big(-\delta(t-\tau)\big)\sin(\sqrt{\lambda-\delta^2}(t-\tau)).$$

It follows that

$$\int_\tau^1 x(s)\,ds = \frac{\alpha}{\lambda}\Big(-\exp(-\delta\tau) + \exp(-\delta)\cos\big(\sqrt{\lambda-\delta^2}(1-\tau)\big) + \frac{\delta\exp(-\delta)}{\sqrt{\lambda-\delta^2}}\sin\big(\sqrt{\lambda-\delta^2}(1-\tau)\big)\Big). \tag{3.9}$$

From (3.3) and (3.9) we obtain the equation

$$\lambda\big(1+\exp(-\delta\tau)\big) + \mu\Big(-\exp(-\delta\tau) + \exp(-\delta)\cos\big(\sqrt{\lambda-\delta^2}(1-\tau)\big) + \frac{\delta\exp(-\delta)}{\sqrt{\lambda-\delta^2}}\sin\big(\sqrt{\lambda-\delta^2}(1-\tau)\big)\Big) = 0. \tag{3.10}$$

The equation (3.10) makes it possible to calculate (μ, λ) values which correspond to the branch $F^+_{1,>}$ where the additional “$>$” sign means that $\lambda > \delta^2$.

In view of all the above mentioned we obtain that $F_1^+ = F^+_{1,<} \cup F^+_{1,=} \cup F^+_{1,>}$ and F_1^+ is located in the first and fourth quadrants of (μ, λ) plane.

The proof for F_1^- is similar. $\square$

Lemma 3. *Each branch of the spectrum for the problem* (1.5), (1.6) *is bounded with the branches of the spectrum for the problem* (2.1).

Proof. Consider the solution of the problem (1.5), (1.6) with two zeroes in the interval $(0,1)$. Let us denote them τ_1 and τ_2. We must consider the corresponding problems in the intervals $(0,\tau_1)$, (τ_1,τ_2) and $(\tau_2, 1)$. Similarly as in Lemma 2 we obtain that $\tau_1 = \frac{\pi}{\sqrt{\mu-\delta^2}}$, $\tau_2 = \frac{\pi}{\sqrt{\mu-\delta^2}} + \frac{\pi}{\sqrt{\lambda-\delta^2}}$.

The solution of the corresponding Cauchy problem in the interval $(\tau_2, 1)$ has the first zero $\frac{2\pi}{\sqrt{\mu-\delta^2}} + \frac{\pi}{\sqrt{\lambda-\delta^2}}$ outside the interval $(0,1)$. It follows that

$$\frac{\pi}{\sqrt{\mu-\delta^2}} + \frac{\pi}{\sqrt{\lambda-\delta^2}} < 1 \le \frac{2\pi}{\sqrt{\mu-\delta^2}} + \frac{\pi}{\sqrt{\lambda-\delta^2}}$$

for solutions of the problem (1.5), (1.6) with two zeroes in the interval $(0,1)$. In view of it the branch F_2^+ of the spectrum for the problem (1.5), (1.6) is bounded with the branches F_1^+ and F_2^+ of the spectrum for the problem (2.1). $\square$

The explicit formulas of the branches of the spectrum for the problem (1.5), (1.6) are given in the next theorem.

Theorem 1. *The spectrum of the problem for different* δ *values consists of the branches given by*

$$\begin{aligned}
F_1^+ &= F^+_{1,>} \cup F^+_{1,=} \cup F^+_{1,<};\\
F^+_{1,>} &= \Big\{(\mu,\lambda)\colon \lambda\Big(1+\exp\Big(\frac{-\delta\pi}{\sqrt{\mu-\delta^2}}\Big)\Big) + \mu\Big(-\exp\Big(\frac{-\delta\pi}{\sqrt{\mu-\delta^2}}\Big)\\
&\quad + \exp(-\delta)\cos\Big(\sqrt{\lambda-\delta^2}\Big(1-\frac{\pi}{\sqrt{\mu-\delta^2}}\Big)\Big) + \frac{\delta\exp(-\delta)}{\sqrt{\lambda-\delta^2}}\\
&\quad \times \sin\Big(\sqrt{\lambda-\delta^2}\Big(1-\frac{\pi}{\sqrt{\mu-\delta^2}}\Big)\Big)\Big) = 0,\ \mu > \pi^2+\delta^2,\\
&\quad \frac{\pi}{\sqrt{\mu-\delta^2}} + \frac{\pi}{\sqrt{\lambda-\delta^2}} \ge 1\Big\};\\
F^+_{1,=} &= \Big\{(\mu,\lambda)\colon \lambda\Big(1+\exp\Big(\frac{-\delta\pi}{\sqrt{\mu-\delta^2}}\Big)\Big) + \mu\Big(-\exp\Big(\frac{-\delta\pi}{\sqrt{\mu-\delta^2}}\Big)\\
&\quad + \exp(-\delta)\Big(1+\delta-\frac{\delta\pi}{\sqrt{\mu-\delta^2}}\Big)\Big) = 0,\ \mu > \pi^2+\delta^2,\ \lambda=\delta^2\Big\};
\end{aligned}$$

$$\begin{aligned}
F_{1,<}^{+} = \Big\{(\mu,\lambda)\colon\ &\lambda\Big(1+\exp\Big(\frac{-\delta\pi}{\sqrt{\mu-\delta^2}}\Big)\Big)+\mu\Big(-\exp\Big(\frac{-\delta\pi}{\sqrt{\mu-\delta^2}}\Big)\\
&+\exp(-\delta)\cosh\Big(\sqrt{\delta^2-\lambda}\Big(1-\frac{\pi}{\sqrt{\mu-\delta^2}}\Big)\Big)+\frac{\delta\exp(-\delta)}{\sqrt{\delta^2-\lambda}}\\
&\times\sinh\Big(\sqrt{\delta^2-\lambda}\Big(1-\frac{\pi}{\sqrt{\mu-\delta^2}}\Big)\Big)\Big)=0,\ \mu>\pi^2+\delta^2,\ \ \lambda<\delta^2\Big\};
\end{aligned}$$

$$\begin{aligned}
F_{2i}^{+} = \Big\{(\mu,\lambda)\colon\ &\lambda\Big(\Big(1+\exp\Big(\frac{\pi\delta}{\sqrt{\mu-\delta^2}}\Big)\Big)\sum_{j=1}^{i}\exp\Big(\frac{-\delta j\pi}{\sqrt{\mu-\delta^2}}+\frac{-\delta(j-1)\pi}{\sqrt{\lambda-\delta^2}}\Big)\\
&+\exp\Big(\frac{-\delta i\pi}{\sqrt{\mu-\delta^2}}+\frac{-\delta i\pi}{\sqrt{\lambda-\delta^2}}\Big)-\exp(-\delta)\cos\Big(\sqrt{\mu-\delta^2}\Big(1-\frac{i\pi}{\sqrt{\lambda-\delta^2}}\Big)\\
&+i\pi\Big)-\frac{\delta e^{-\delta}}{\sqrt{\mu-\delta^2}}\sin\Big(\sqrt{\mu-\delta^2}\Big(1-\frac{i\pi}{\sqrt{\lambda-\delta^2}}\Big)+i\pi\Big)\Big)\\
&-\mu\Big(1+\exp\Big(\frac{\pi\delta}{\sqrt{\lambda-\delta^2}}\Big)\Big)\sum_{j=1}^{i}\exp\Big(\frac{-\delta j\pi}{\sqrt{\mu-\delta^2}}+\frac{-\delta j\pi}{\sqrt{\lambda-\delta^2}}\Big)=0,\\
&\frac{i\pi}{\sqrt{\mu-\delta^2}}+\frac{i\pi}{\sqrt{\lambda-\delta^2}}<1,\ \frac{(i+1)\pi}{\sqrt{\mu-\delta^2}}+\frac{i\pi}{\sqrt{\lambda-\delta^2}}\ge 1\Big\};
\end{aligned}$$

$$\begin{aligned}
F_{2i+1}^{+} = \Big\{(\mu,\lambda)\colon\ &\lambda\Big(1+\exp\Big(\frac{\pi\delta}{\sqrt{\mu-\delta^2}}\Big)\Big)\sum_{j=1}^{i+1}\exp\Big(\frac{-\delta j\pi}{\sqrt{\mu-\delta^2}}+\frac{-\delta(j-1)\pi}{\sqrt{\lambda-\delta^2}}\Big)\\
&-\mu\Big(\Big(1+\exp\Big(\frac{\pi\delta}{\sqrt{\lambda-\delta^2}}\Big)\Big)\sum_{j=1}^{i}\exp\Big(\frac{-\delta j\pi}{\sqrt{\mu-\delta^2}}+\frac{-\delta j\pi}{\sqrt{\lambda-\delta^2}}\Big)\\
&+\exp\Big(\frac{-\delta(i+1)\pi}{\sqrt{\mu-\delta^2}}+\frac{-\delta i\pi}{\sqrt{\lambda-\delta^2}}\Big)\\
&-\exp(-\delta)\cos\Big(\sqrt{\lambda-\delta^2}\Big(1-\frac{(i+1)\pi}{\sqrt{\mu-\delta^2}}\Big)+i\pi\Big)\\
&-\frac{\delta\exp(-\delta)}{\sqrt{\lambda-\delta^2}}\sin\Big(\sqrt{\lambda-\delta^2}\Big(1-\frac{(i+1)\pi}{\sqrt{\mu-\delta^2}}\Big)+i\pi\Big)\Big)=0,\\
&\frac{(i+1)\pi}{\sqrt{\mu-\delta^2}}+\frac{i\pi}{\sqrt{\lambda-\delta^2}}<1,\ \frac{(i+1)\pi}{\sqrt{\mu-\delta^2}}+\frac{(i+1)\pi}{\sqrt{\lambda-\delta^2}}\ge 1\Big\};
\end{aligned}$$

$$F_i^{-} = \big\{(\mu,\lambda)\colon (\lambda,\mu)\in F_i^{+}\big\},\quad i=1,2,\ldots$$

Proof. The proof of this theorem for branches $F_1^{\pm}$ follows from Lemma 2. Now we will prove the theorem for the case of F_{2i}^{+}. Let $x(t)$ be a respective nonlinear solution of the problem (1.5), (1.6). It has $2i$ zeroes in the interval $(0,1)$ and $x'(0)>0$. Let these zeroes be denoted by τ_1, τ_2 and so on.

Consider the problem (1.5), (1.6) in the intervals $(0,\tau_1)$, (τ_1,τ_2), $\ldots$, $(\tau_{2i},1)$. We obtain the equations

$$x''+2\delta x'+\mu x=0 \quad\text{and}\quad x''+2\delta x'+\lambda x=0 \tag{3.11}$$

in the the odd and even intervals respectively.

The direct calculations show that $\tau_1 = \frac{\pi}{\sqrt{\mu-\delta^2}}$, $\tau_2 = \frac{\pi}{\sqrt{\mu-\delta^2}} + \frac{\pi}{\sqrt{\lambda-\delta^2}}$, $\ldots$, $\tau_{2i-1} = \frac{i\pi}{\sqrt{\mu-\delta^2}} + \frac{(i-1)\pi}{\sqrt{\lambda-\delta^2}}$ and $\tau_{2i} = \frac{i\pi}{\sqrt{\mu-\delta^2}} + \frac{i\pi}{\sqrt{\lambda-\delta^2}}$. It is easy to show that the derivatives $x'(\tau_j)$ $(j = 0, \ldots, 2i)$ of the nontrivial solution $x(t)$ are:

$$
\begin{aligned}
&x'(0) = \alpha > 0, \quad x'(\tau_1) = -\alpha \exp\Big(\frac{-\delta\pi}{\sqrt{\mu-\delta^2}}\Big),\\
&x'(\tau_2) = \alpha \exp\Big(\frac{-\delta\pi}{\sqrt{\mu-\delta^2}} + \frac{-\delta\pi}{\sqrt{\lambda-\delta^2}}\Big),\\
&x'(\tau_3) = -\alpha \exp\Big(\frac{-2\delta\pi}{\sqrt{\mu-\delta^2}} + \frac{-\delta\pi}{\sqrt{\lambda-\delta^2}}\Big), \quad \ldots\\
&x'(\tau_{2i}) = \alpha \exp\Big(\frac{-i\delta\pi}{\sqrt{\mu-\delta^2}} + \frac{-i\delta\pi}{\sqrt{\lambda-\delta^2}}\Big).
\end{aligned}
$$

Consider the solutions of the respective Cauchy problems with respect to (3.11). In view of the last derivatives we obtain:

$$
\begin{aligned}
&\int_0^{\tau_1} x(s)\,ds = \frac{\alpha}{\mu}\Big(1+\exp\Big(\frac{-\delta\pi}{\sqrt{\mu-\delta^2}}\Big)\Big); \qquad (3.12)\\
&\int_{\tau_1}^{\tau_2} x(s)\,ds = -\frac{\alpha}{\lambda}\Big(1+\exp\Big(\frac{\delta\pi}{\sqrt{\lambda-\delta^2}}\Big)\Big)\exp\Big(\frac{-\delta\pi}{\sqrt{\mu-\delta^2}} + \frac{-\delta\pi}{\sqrt{\lambda-\delta^2}}\Big);\\
&\int_{\tau_2}^{\tau_3} x(s)\,ds = \frac{\alpha}{\mu}\Big(1+\exp\Big(\frac{\delta\pi}{\sqrt{\mu-\delta^2}}\Big)\Big)\exp\Big(\frac{-2\delta\pi}{\sqrt{\mu-\delta^2}} + \frac{-\delta\pi}{\sqrt{\lambda-\delta^2}}\Big);\\
&\cdots\cdots\cdots\\
&\int_{\tau_{2i}}^{1} x(s)\,ds = \frac{\alpha}{\mu}\Big(\exp\Big(\frac{-\delta i\pi}{\sqrt{\mu-\delta^2}} + \frac{-\delta i\pi}{\sqrt{\lambda-\delta^2}}\Big)\\
&\qquad - \exp(-\delta)\cos\Big(\sqrt{\mu-\delta^2} - \frac{i\pi\sqrt{\mu-\delta^2}}{\sqrt{\lambda-\delta^2}} + i\pi\Big)\\
&\qquad - \frac{\delta e^{-\delta}}{\sqrt{\mu-\delta^2}}\sin\Big(\sqrt{\mu-\delta^2} - \frac{i\pi\sqrt{\mu-\delta^2}}{\sqrt{\lambda-\delta^2}} + i\pi\Big)\Big).
\end{aligned}
$$

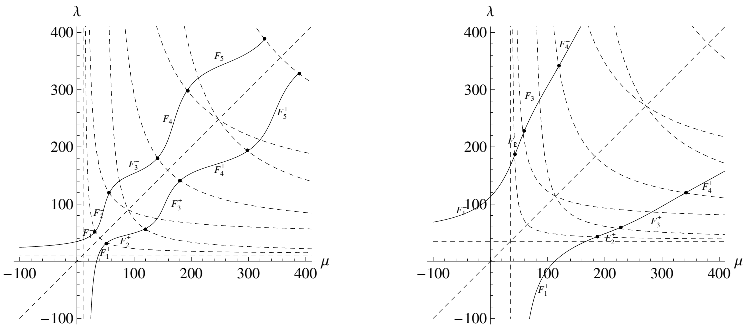
In view of the equations (3.12) we obtain the expression for F^+_{2i}.

The ideas of the proof for all other branches are similar. We consider the problem (1.5), (1.6) in the intervals $(0, \tau_1)$, (τ_1, τ_2), $\ldots$, $(\tau_{2i+1}, 1)$ and in the same manner as before we obtain the expression for F^+_{2i+1}. $\square$

The graphical description of the spectrum for some values of δ is given in Fig. 7 and 8.

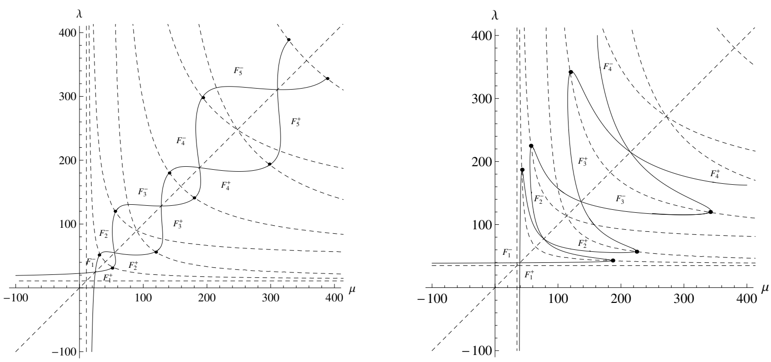
4 The Solvability Regions

The knowledge of the spectrum of the problem (1.5), (1.6) can be used to define regions of solvability for the problem (1.7), (1.6).



$\delta = 1$ $\delta = 5$

Figure 7. The spectrum of the problem (1.5), (1.6) for some positive δ values.



$\delta = -1$ $\delta = -5$

Figure 8. The spectrum of the problem (1.5), (1.6) for some negative δ values.

So, consider the problem (1.7), (1.6).

We suppose that $h : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded Lipschitz (with respect to x and x') function.

Consider the Cauchy problem

$$x'' + 2\delta x' = -\mu x^+ + \lambda x^- + h(t, x, x'), \quad x(0) = 0, \quad x'(0) = \alpha. \tag{4.1}$$

Let $x(t;\alpha)$ be a solution of (4.1).

Consider the Cauchy problems

$$z'' + 2\delta z' = -\mu z^+ + \lambda z^-, \quad z(0) = 0, \quad z'(0) = 1 \tag{4.2}$$

and

$$z'' + 2\delta z' = -\mu z^+ + \lambda z^-, \quad z(0) = 0, \quad z'(0) = -1. \tag{4.3}$$

Let the functions $z_+(t)$ and $z_-(t)$ be solutions of the Cauchy problems (4.2) and (4.3) respectively.

Lemma 4. *The functions $u(t;\alpha) = \frac{1}{|\alpha|} x(t;\alpha)$ tend uniformly in $t \in [0,1]$ to the functions $z_+(t)$ as $\alpha \to +\infty$ and to $z_-(t)$ as $\alpha \to -\infty$.*

Proof. The functions $u(t;\alpha)$ satisfy the following problem

$$u'' + 2\delta u' = -\mu u^+ + \lambda u^- + \frac{1}{|\alpha|} h(t, x(t), x'(t)), \quad u(0) = 0, \quad u'(0) = 1.$$

The last term $(\frac{1}{|\alpha|}h)$ in the above equation tends to zero as $|\alpha| \to +\infty$ since h is bounded.

The equation in the beginning of the proof of Lemma 4 is such that all solutions extend on $[0,1]$. Besides, due to properties of $h(t,x,x')$, the Cauchy problems are uniquely solvable and solutions $u(t;\alpha)$ tend uniformly in $t \in [0,1]$ to $z_+(t)$ as $\alpha \to +\infty$ and to $z_-(t)$ as $\alpha \to -\infty$. □

The following result is valid.

Theorem 2. *If*

$$\int_0^1 z_-(s)\,ds \int_0^1 z_+(s)\,ds < 0, \tag{4.4}$$

where meaning of $z_+(t)$ and $z_-(t)$ and restrictions on $h(t,x,x')$ are as above, then there exists $\alpha_0 \in \mathbb{R}$ such that $x(t;\alpha_0)$ solves the problem (1.7), (1.6).

Proof. We consider the Cauchy problem (4.1). Since the functions $u(t;\alpha) = \frac{1}{|\alpha|}x(t;\alpha)$ tend to the functions $z_\pm$ as $\alpha \to \pm\infty$ and the condition (4.4) holds, one has that

$$\int_0^1 x(s;\alpha)\,ds \int_0^1 x(s;-\alpha)\,ds < 0.$$

By continuous dependence of $x(t;\alpha)$ on α one concludes that there exists α_0 such that the condition (1.6) holds. □

We obtain that the problem (1.7), (1.6) is solvable if (μ,λ) does not belong to the spectrum of the problem (1.5), (1.6) but it is such that (4.4) holds ("good" regions for solvability).

The regions of (μ,λ)-plane where the problem (1.7), (1.6) is certainly solvable have been depicted in Fig. 9 for some values of δ.

Remark 1. The regions of (μ,λ)-plane where the problem (1.7), (1.6) is certainly solvable are located between $\sum_- = \bigcup_{i=1}^{+\infty} F_i^-$ and $\sum_+ = \bigcup_{i=1}^{+\infty} F_i^+$.

5 Conclusions

1. The analytical description of the spectrum for the problem (1.5), (1.6) was obtained. The presence of a damping term in the problem (1.2) changes the spectrum essentially (Figs. 7, 8).

2. The visualization of the spectrum for the problem (1.5), (1.6) was obtained for some values of δ.

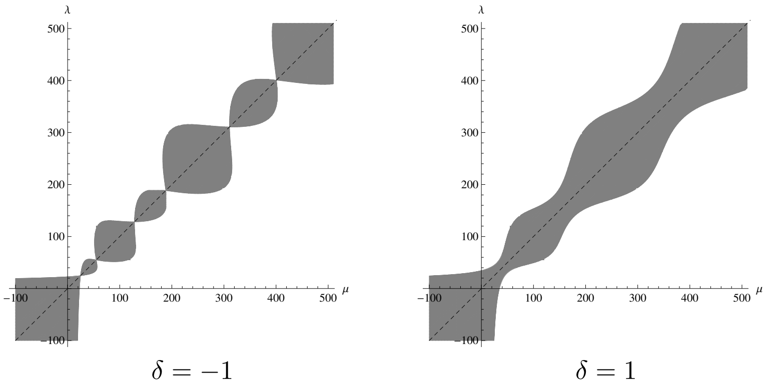


Figure 9. Solvability regions for the the problem (1.7), (1.6).

3. The problem (1.7), (1.6) was considered and the existence of solutions was established by making use of previously studied spectra for the Fučík equation.

4. The presence of a damping term in the problem (1.4) changes the solvability regions essentially. These regions for the problem (1.7), (1.6) enlarge together with $\delta > 0$ (Fig. 9).

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