Mathematical Modelling and Analysis
Volume 19 Number 3, June 2014, 299–308
http://dx.doi.org/10.3846/13926292.2014.923944

Publisher: Taylor&Francis and VGTU
http://www.tandfonline.com/TMMA
Print ISSN: 1392-6292
Online ISSN: 1648-3510

On the Self-Regularization of Ill-Posed Problems by the Least Error Projection Method*

Alina Ganina, Uno Hämarik and Urve Kangro

University of Tartu
Liivi 2, 50409 Tartu, Estonia
E-mail: alina.ganina@gmail.com
E-mail(*corresp.*): uno.hamarik@ut.ee
E-mail: urve.kangro@ut.ee

Received September 15, 2013; revised April 22, 2014; published online June 1, 2014

Abstract. We consider linear ill-posed problems where both the operator and the right hand side are given approximately. For approximate solution of this equation we use the least error projection method. This method occurs to be a regularization method if the dimension of the projected equation is chosen properly depending on the noise levels of the operator and the right hand side. We formulate the monotone error rule for choice of the dimension of the projected equation and prove the regularization properties.
Keywords: ill-posed problems, self-regularization, projection methods, least error method, monotone error rule.

AMS Subject Classification: 47A52; 65J20.

1 Introduction

In this paper we consider linear ill-posed problems

$$A_0 u = f_0, \tag{1.1}$$

where $A_0 \in \mathcal{L}(H, F)$ is a bounded operator and H, F are infinite dimensional real Hilbert spaces with inner products $(\cdot,\cdot)$ and norms $\|\cdot\|$, respectively. Typical examples of ill-posed problems are equations (1.1) where the operator has a non-closed range $\mathcal{R}(A_0)$. We assume that $f_0 \in \mathcal{R}(A_0)$, $\mathcal{N}(A_0) = \{0\}$ and $\mathcal{N}(A_0^*) = \{0\}$, where A^* is the adjoint operator of A. Denote the solution of problem (1.1) by u_*. It is assumed that instead of exact data f_0 and A_0 there are given noisy data $f \in F$ and $A \in \mathcal{L}(H, F)$ with

$$\|f - f_0\| \le \delta, \qquad \|A - A_0\| \le \eta \tag{1.2}$$

* This work was supported by the Estonian Science Foundation grant 9120 and Estonian Institutional Research Project IUT20-57.

and known noise levels δ, η. The typical feature of ill-posed problems is instability of the solution of the problem $Au = f$ with respect to noise in data. To diminish the influence of noise in data to the approximate solution ill-posed problems are typically solved by iteration methods or by special regularization methods [3,12,24]. For using computers in solution procedures the discretization of the problem is unavoidable. For some discretization methods additional regularization is not needed. Namely, if the solution of the discretized equation converges in the case of exact data to the solution u_*, then in the case of noisy data this method can be viewed as regularization method, when the discretization level as a regularization parameter is properly chosen according to the noise level. This effect is called as self-regularization by discretization [1,5,10,16,17,18,23].

In this paper we consider the projection method for the problem (1.1). For ill-posed problems the convergence conditions of projection methods in case of exact data were stated in [2,4,5,11,13,17,18,21,22,23]. These conditions for traditional projection methods are quite restricting. These restrictions are minimal in the least error method [1,2,4,5,7,13,19,22,23] and therefore we devote this paper namely to this method. For the choice of the dimension of the projected equation the discrepancy principle was used in [5,13,14,15,23], the balancing principle in [1] and the monotone error rule (ME-rule) in [7]. From these rules the discrepancy principle needs serious additional assumptions and the balancing principle needs huge amount of computations. Note that from mentioned papers only in [5,23] the case of noisy operator was considered. The aim of this paper is to extend the results of paper [7] to the case of noisy operator and to prove the regularizing properties of the ME-rule in the least error method.

The plan of this paper is the following. Section 2 is a short review (based on papers [5,23]) of the results of the least error projection methods for ill-posed problems in the Hilbert spaces, concerning convergence conditions in the case of exact data, but also self-regularization conditions, if the dimension of the discretized equation is chosen a priori or by the discrepancy principle. In Section 3 we formulate the ME-rule for the choice of dimension of the discretized equation in the least error method and prove the regularizing properties. In last Section 4 numerical examples are given.

2 Known Rules for Choice of the Dimension in the Least Error Projection Method

Let H, F be Hilbert spaces and $H_n \subset H$, $F_n \subset F$ ($n \in \mathbb{N}$) be finite-dimensional subspaces with $\dim H_n = \dim F_n$. We denote the corresponding orthoprojectors by P_n and Q_n: $P_nH = H_n$, $Q_nF = F_n$. In the projection method the approximate solution u_n of equation (1.1) is found from the condition

$$u_n \in H_n, \quad (Au_n - f, z_n) = 0 \qquad (\forall z_n \in F_n)\,. \tag{2.1}$$

The last condition is equivalent to the equation

$$Q_nAu_n = Q_nf, \quad u_n \in H_n. \tag{2.2}$$

In case of $H_n = A^* F_n$ the corresponding projection method is called the least error method. This name is justified by the property that in case $f = f_0$, $A = A_0$ we have $u_n = P_n u_*$, while element $u_n \in A^* F_n$, which minimizes $\|u_n - u_*\|$, satisfies condition $(u_n - u_*, A^* z_n) = 0$ $(\forall z_n \in F_n)$. This condition is the same condition as (2.1).

In the following we formulate a theorem [5, 23] about the convergence conditions of least error method in the case of exact data and also in the case of noisy data with a priori choice of n and with the choice of n by the discrepancy principle. We denote the norm of the inverse of A_0^* in the finite dimensional spaces by κ_n:

$$\kappa_n := \sup_{z_n \in F_n} \|z_n\| / \|A_0^* z_n\| \quad (\forall n \in \mathbb{N}).$$

Theorem 1. *The least error method determines in case $\eta \kappa_n < 1$ the unique approximation u_n. Assume $\|z - Q_n z\| \to 0$ as $n \to \infty$ $(\forall z \in F)$. If $\delta = \eta = 0$ then $u_n \to u_*$ as $n \to \infty$. If $\delta > 0$ or $\eta > 0$ and $n = n(\delta, \eta)$ is chosen a priori by conditions*

$$n(\delta, \eta) \to \infty, \quad (\delta + \eta)\kappa_{n(\delta,\eta)} \to 0 \text{ as } \delta, \eta \to 0, \tag{2.3}$$

then $u_{n(\delta,\eta)} \to u_*$ as $\delta, \eta \to 0$. The last convergence holds also in the case when there exists $\alpha \in \mathbb{R}$, $\alpha > 0$ such that

$$(\kappa_{n+1})^\alpha \left\| (I - Q_n) (AA^*)^{\alpha/2} \right\| \le const \quad (\forall n \in \mathbb{N}) \tag{2.4}$$

and $n = n(\delta, \eta)$ is chosen by the discrepancy principle: $n(\delta, \eta)$ is the first index $n \in \mathbb{N}$ satisfying

$$\|Au_n - f\| \le b\big(\delta + \|u_n\|\eta\big), \quad b = const > \left(1 + \gamma^2\right)^{\alpha/2}.$$

Here γ satisfies the condition

$$(\kappa_n)^\alpha \left\| (I - Q_n) (AA^*)^{\alpha/2} \right\| \le \gamma = const \quad (\forall n \in \mathbb{N}).$$

The essential advantage of the least error method in comparison with other projection methods is that in the case of exact data it guarantees the convergence under very mild conditions. This convergence was stated in [2,4,5,22,23]. A discrepancy principle for the choice of n is practical, but for the least error method it is restricted by additional assumptions (2.4). In [1] a balancing principle for the choice of n is proposed but this choice needs more information about the solution and more calculations. In the next section we propose another a posteriori rule for the choice of n for the least error method.

3 Monotone Error Rule in the Least Error Method

The choice of the regularization parameter is an actual problem in all regularization methods. For the choice of regularization parameter $r = r(\delta, \eta)$ we consider the monotone error rule. For the resulting parameter r_{ME} convergence $u_{r_{ME}} \to u_*$ as $\delta, \eta \to 0$ was shown and some order optimal error estimates were given [6,8,9]. The ME-rule is applicable in algorithms, where in the case of exact data the monotone convergence $u_r \to u_*$ for $r \to \infty$ holds. The idea of the ME-rule is to choose for the regularization parameter $r_{ME} = r(\delta, \eta)$ the largest r-value, for which under information $\|A - A_0\| \le \eta$ and $\|f - f_0\| \le \delta$ we can prove that $\|u_r - u_*\|$ is monotonically decreasing for $r \in (0, r_{ME}]$. For continuous regularization methods as the Tikhonov method $u_r = (A^*A + r^{-1}I)^{-1}A^*f$, where u_r is differentiable with respect to r, this means that $\frac{d}{dr}\|u_r - u_*\|^2 \le 0$ for all $r \in (0, r_{ME})$. For iteration methods where regularization parameter r is the stopping index $n \in \mathbb{N}$, this means that

$$\|u_n - u_*\| < \|u_{n-1} - u_*\| \quad \text{for } n = 1, 2, \ldots, n_{ME}. \tag{3.1}$$

In projection methods the regularization parameter is $n \in \mathbb{N}$ as in iteration methods and the ME-rule should give $n_{ME} = n(\delta, \eta)$ satisfying condition (3.1). The ME-rule for the least error method in case $\eta = 0$ was proposed in [7].

The aim of this section is to develop the ME-rule for the least error method in case of noisy data (1.2). We assume that the subspaces F_n satisfy the condition

$$F_n \subset F_{n+1} \quad (n = 0, 1, \ldots) \tag{3.2}$$

and we show that then ME-rule is applicable in the following form.

ME-rule: choose $n_{ME} = n(\delta, \eta)$ in the least error approximation

$$u_n = A^* v_n \quad (v_n \in F_n)$$

as the first index $n = 1, 2, \ldots$ for which

$$d_{ME}(n) := \frac{(v_{n+1} - v_n, f)}{2\|v_{n+1} - v_n\|} \le \delta + M\eta, \tag{3.3}$$

where $\|u_*\| \le M$.

Note that we get the element $v_n \in F_n$ in a computational procedure automatically without extra work. Note also that the function $d_{ME}(n)$ can be represented in the equivalent form

$$d_{ME}(n) = \frac{\|u_{n+1}\|^2 - \|u_n\|^2}{2\|v_{n+1} - v_n\|}. \tag{3.4}$$

Namely the approximation $u_n = A^* v_n$, $(v_n \in F_n)$ in the least error method satisfies the equality

$$\|u_n\|^2 = \|A^* v_n\|^2 = (AA^* v_n - f + f, v_n) = (Au_n - f, v_n) + (f, v_n),$$

therefore due to equality $(Au_n - f, v_n) = 0$ (see 2.1) also the equality

$$\|u_n\|^2 = (f, v_n). \tag{3.5}$$

From the last equality follows the equality of functionals $d_{ME}(n)$ in (3.3) and (3.4). The following result holds.

Theorem 2. *If $\eta\kappa_n < 1$ then the equation $Q_n A u_n = Q_n f$ in the least error method has a unique solution $u_n = A^* v_n$, $v_n \in F_n$. Assume that the subspaces F_n satisfy condition* (3.2) *and $\|z - Q_n z\| \to 0$ for $n \to \infty$ $(\forall z \in F)$. Then*

1) *u_n is minimum norm solution of the equation $Q_n A u = Q_n f$ $(u \in H)$,*

2) $\|u_n\| \le \|u_{n+1}\|$ $(n \ge 1)$.

If $n = n_{ME}$ is chosen as first $n \in \mathbb{N}$ for which $d_{ME}(n) \le \delta + M\eta$, where $\|u_*\| \le M$, then

3) $\|u_n - u_*\| \le \|u_{n-1} - u_*\|$ $(n = 1, 2, \ldots, n_{ME})$;

4) *if $n_{ME} \to \infty$ as $\delta \to 0$, $\eta \to 0$, then $\|u_{n_{ME}} - u_*\| \to 0$ as $\delta \to 0$, $\eta \to 0$,*

5) $0 \le d_{ME}(n) \le \|Au_n - f\|/2$ $(\forall n \in \mathbb{N})$.

Proof. Unique solvability of the equation under condition $\eta\kappa_n < 1$ is proved in [5]. From the equality

$$\mathcal{N}(Q_n A) = \big(\mathcal{R}\big((Q_n A)^*\big)\big)^\perp = \big(\mathcal{R}(A^* Q_n)\big)^\perp = (A^* F_n)^\perp \tag{3.6}$$

follows that approximation $u_n \in A^* F_n$ in the least error method is the minimum-norm solution of the equation $Q_n A u = Q_n f$ $\quad (u \in H)$. Indeed, due to (3.6) all solutions of equation $Q_n A u = Q_n f$ have form $u = u_n + u_n'$ with $u_n \in A^* F_n$ and $u_n' \in (A^* F_n)^\perp$, but $\|u_n + u_n'\|^2 = \|u_n\|^2 + \|u_n'\|^2 \ge \|u_n\|^2$. From (3.2) follows that $u_{n+1} \in A^* F_{n+1}$ solves both equations $Q_{n+1}(Au - f) = 0$ and $Q_n(Au - f) = 0$. This fact with property 1) $u_n = \arg\min\{\|u\| \colon u \in H,$ $Q_n(Au - f) = 0\}$ gives $\|u_n\| \le \|u_{n+1}\|$, hence the assertion 2) is proved.

To prove assertion 3) we use equality (3.5), the inequality

$$\|f - Au_*\| = \big\|f - f_0 + (A_0 - A)u_*\big\| \le \delta + M\eta$$

(see (1.2)) and get

$$\begin{aligned}
\|u_n - u_*\|^2 - \|u_{n-1} - u_*\|^2 &= \|u_n\|^2 - \|u_{n-1}\|^2 - 2(u_n - u_{n-1}, u_*) \\
&= (v_n - v_{n-1}, f) - 2\big(A^*(v_n - v_{n-1}), u_*\big) \\
&= (v_n - v_{n-1}, f - 2Au_*) \\
&= (v_n - v_{n-1}, 2(f - Au_*) - f) \\
&\le 2\|v_n - v_{n-1}\|(\delta + M\eta) - (v_n - v_{n-1}, f) \\
&= 2\|v_n - v_{n-1}\|\big((\delta + M\eta) - d_{ME}(n-1)\big).
\end{aligned} \tag{3.7}$$

As $d_{ME}(n-1) > \delta + M\eta$, inequality 3) is proved.

To prove assertion 4) we start with error estimate

$$\|u_n - u_*\| \le \|u_* - P_n u_*\| + (\delta + M\eta)\kappa_n/(1 - \eta\kappa_n) \tag{3.8}$$

from [5]. This estimate guarantees convergence $\|u_{n_0} - u_*\| \to 0$ as $\delta \to 0$, $\eta \to 0$ for every a priori parameter choice $n_0 = n_0(\delta, \eta)$ under conditions (2.3).

Since $n_{ME} \to \infty$ as $\delta \to 0,\ \eta \to 0$, convergence $\|u_* - P_n u_*\| \to 0$ as $n \to \infty$ follows from assumption $\|z - Q_n z\| \to 0$ $(\forall z \in F)$. In case $n_{ME} > n_0$ we have $\|u_{n_{ME}} - u_*\| \le \|u_{n_0} - u_*\| \to 0$ as $\delta \to 0,\ \eta \to 0$ due to the monotonicity property 3) of the error. In case $n_{ME} < n_0$ we have $\kappa_{n_{ME}} < \kappa_{n_0}$ due to (3.2) hence convergence $\|u_{n_{ME}} - u_*\| \to 0$ as $\delta \to 0,\ \eta \to 0$ follows from (3.8), (2.3).

In assertion 5) the left inequality holds due to assertion 2) and equality (3.4). Solutions of projected equations $u_n = A^* v_n$ and $u_{n+1} = A^* v_{n+1}$ satisfy due to (2.1) equalities $(Au_n - f, v_n) = 0$, $(Au_{n+1} - f, v_n) = 0$. It yields that $(A(u_{n+1} - u_n), v_n) = 0$ and via relations

$$0 = \big(A(u_{n+1} - u_n), v_n\big) = \big(AA^*(v_{n+1} - v_n), v_n\big) = \big((v_{n+1} - v_n), Au_n\big)$$

we get equality $(Au_n, v_{n+1} - v_n) = 0$. Hence the numerator of the function (3.3) can be written also in the form $(v_{n+1} - v_n, f - Au_n)$ and d_{ME} can be estimated by $\|Au_n - f\|/2$. □

Remark 1. In paper [8] the information about the noise level was given in a more general form

$$\big\|D(f - f_0)\big\| \le \delta, \quad \big\|D(A - A_0)\big\| \le \eta \tag{3.9}$$

instead of (1.2), where D is a linear injective, possibly unbounded operator in F with domain $\mathcal{D}(D)$. We assume that $f, f_0 \in \mathcal{D}(D)$. In this case we can estimate in analogy to (3.7)

$$\begin{aligned}\|u_n - u_*\|^2 - \|u_{n-1} - u_*\|^2 &= (v_n - v_{n-1}, 2(f - Au_*) - f)\\ &= \big(\big(D^{-1}\big)^*(v_n - v_{n-1}), 2D(f - Au_*)\big) - (v_n - v_{n-1}, f)\\ &\le 2\big\|(D^{-1})^*(v_n - v_{n-1})\big\|(\delta + M\eta) - (v_n - v_{n-1}, f).\end{aligned}$$

Therefore under (3.9) the monotonicity of the error for $n{=}1, \dots, n_{ME}$ is guaranteed where n_{ME} is the first index for which

$$d_{ME}(n) := \frac{(v_{n+1} - v_n, f)}{2\|(D^{-1})^*(v_{n+1} - v_n)\|} \le \delta + M\eta. \tag{3.10}$$

In this case Theorem 2 holds with 5) substituted by

5′) $0 \le d_{ME}(n) \le \|D(Au_n - f)\|/2$ $(\forall n \in \mathbb{N})$.

4 Numerical Examples

4.1 Numerical differentiation

We consider Volterra integral equation

$$A_0 u(t) \equiv \int_0^t \frac{(t-s)^{l-1}}{(l-1)!} u(s)\, ds = f_0(t), \quad t \in [0, 1]$$

with $H = F = L_2(0,1)$ and with f_0 such that the solution $u_* = f_0^{(l)} \in L_2(0,1)$. Subspaces F_n consist of piecewise constant functions on $k(n) = 2^n$ subintervals generated by $m = k+1$ mesh points $t_i = (i-1)/(m-1)$, $i = 1,\dots,m$:

$$F_n = span\{\Psi_i(t),\ i = 1,\dots,k(n)\}, \quad \Psi_i(t) = \begin{cases} 1, & t \in [t_i, t_{i+1}], \\ 0, & \text{otherwise.} \end{cases}$$

The approximation u_n is in the case of exact data the best approximation of u_* in subspace $H_n = span\{A^*\Psi_i(t),\ i = 1,\dots,k(n)\}$, where

$$A^*\Psi_i(t) = \begin{cases} \frac{(t_{i+1}-t)^l}{l!} - \frac{(t_i-t)^l}{l!}, & t \in [0, t_i), \\ \frac{(t_{i+1}-t)^l}{l!}, & t \in [t_i, t_{i+1}), \\ 0, & t \in [t_{i+1}, 1]. \end{cases}$$

For computations we used noisy data f with $\|f - f_0\| \le \delta$, where $\delta = 10^{-1},\dots,10^{-5}$. In experiments we computed two parameters: n_{ME} chosen by the monotone error rule and n_{opt} found as the last number for which the inequality $\|u_n - u_*\| \le \|u_{n-1} - u_*\|$ was true.

We give the results for three examples with exact operator A_0 and different right-hand sides f_0, solutions u_* and differentiation order l. We used perturbed data $f(t) = f_0(t) + \delta\cos(10t)/0.723$. The problems are normalized so that $\|f\| \approx 1$. Then in examples 1, 2 and 3 norms of the solutions are approximately 1,1 and 5,5 and 14. In the tables the numbers of subintervals $k_{ME} = 2^{n_{ME}}$, $k_{opt} = 2^{n_{opt}}$ corresponding to the parameters n_{ME}, n_{opt} are given. The relative errors $e_{k(n)} = \|u_n - u_*\|/\|u_*\|$ corresponding to k_{ME}, k_{opt} are also presented. Note that due to (3.1) the inequality $n_{ME} \le n_{opt}$ always holds.

Example 1. $u_*(t) = \frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)$, $f(t) = \sin\left(\frac{\pi t}{2}\right)$, $l = 1$

δ	k_{opt}	k_{ME}	$e_{n_{opt}}$	$e_{n_{ME}}$
10^{-1}	2	1	0.106	0.121
10^{-2}	2	2	0.027	0.027
10^{-3}	4	2	0.010	0.025
10^{-4}	16	8	0.0013	0.0018
10^{-5}	32	16	0.0002	0.0004

Example 2. $u_*(t) = 25(t^4 - 2t^3 + t)$, $f(t) = \frac{5}{6}t^6 - \frac{5}{2}t^5 + \frac{25}{6}t^3$, $l = 2$

δ	k_{opt}	k_{ME}	$e_{n_{opt}}$	$e_{n_{ME}}$
10^{-1}	2	1	0.444	0.800
10^{-2}	4	2	0.235	0.423
10^{-3}	8	8	0.071	0.071
10^{-4}	16	16	0.022	0.022
10^{-5}	32	16	0.007	0.017

Example 3. $u_*(t) = 10(t^2 - 1 - \frac{\pi^3}{32}\cos\left(\frac{\pi t}{2}\right))$, $f(t) = \frac{1}{6}t^5 - \frac{5}{3}t^3 + \frac{5}{2}\sin\left(\frac{\pi t}{2}\right) - \frac{5\pi}{4}t$, $l = 3$

δ	k_{opt}	k_{ME}	$e_{n_{opt}}$	$e_{n_{ME}}$
10^{-1}	1	1	0.521	0.521
10^{-2}	2	1	0.348	0.521
10^{-3}	4	4	0.188	0.188
10^{-4}	8	4	0.104	0.162
10^{-5}	16	8	0.055	0.060

4.2 Phillips problem

We consider the normalized Phillips problem (see [20])

$$\int_{-6}^{6} \phi(t-s)u(s)\,ds = \frac{6-|t|}{15}\left(1+\frac{1}{2}\cos\left(\frac{\pi t}{3}\right)\right) + \frac{3}{10\pi}\sin\left(\frac{\pi |t|}{3}\right),$$

where

$$\phi(x) = \begin{cases} 1+\cos\left(\frac{\pi x}{3}\right), & |x|<3, \\ 0, & |x|\geq 3. \end{cases}$$

We use $H = F = L_2(-6,6)$. The solution of the problem is $\phi(x)/15$ with $\|\phi\| = 0.2$. The subspaces F_n consist again of piecewise constant functions on $k(n) = 2^n$ subintervals of $[-6,6]$. We added the perturbations $\delta\cos 10t/\sqrt{6}$, with $\delta = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$ to the right hand side. The computations were performed with perturbed operator, where $\frac{\eta}{6}\cos 10t\cos s$ was added to the kernel, with three noise levels, $\eta = 1$, $\eta = 0.1$ and $\eta = 0.01$.

η	δ	k_{opt}	k_{ME}	$e_{n_{opt}}$	$e_{n_{ME}}$
1	10^{-1}	8	2	0.18	0.78
1	10^{-2}	8	2	0.10	0.78
1	10^{-3}	8	2	0.09	0.78
1	10^{-4}	8	2	0.09	0.78
0.1	10^{-1}	8	4	0.112	0.430
0.1	10^{-2}	16	8	0.035	0.431
0.1	10^{-3}	16	8	0.028	0.035
0.1	10^{-4}	16	8	0.014	0.035
0.01	10^{-1}	8	4	0.105	0.431
0.01	10^{-2}	16	8	0.016	0.035
0.01	10^{-3}	16	8	0.007	0.033
0.01	10^{-4}	16	8	0.005	0.033

For small perturbations of the operator the results obtained were rather good: the discretization levels proposed by the ME-rule were typically k^n with n_{ME} as the optimal n minus 1.

Note that typically the error of the regularized solution decreases monotonically also somewhat further in all regularization methods, up to some $n_{\mathrm{opt}} \geq n_{ME}$ in iteration methods and up to some $\alpha_{\mathrm{opt}} \leq \alpha_{ME}$ in Tikhonov method. Our numerical experiments suggest to use instead of regularization parameters from the ME-rule the post-estimated parameters, using in Tikhonov method the parameter $\alpha_{ME}/2.3$ and in the Landweber iteration method the integer part of $2.3n_{ME}$. In the least error projection method the situation is the same as the tables here show: instead of the discretization level n_{ME} usually the level $n_{ME}+1$ is better to use, but in case of operator perturbations even larger n may be better. Unfortunately the amount of our numerical experiments is not sufficient for more precise recommendation for choice of $n \geq n_{ME}$.

Note also that the derivation of the ME-rule in (3.7) uses only one inequality, which turns to the equality in certain case, in this case the ME-rule gives the optimal parameter. We do not see possibility to get a better estimate.

References

[1] G. Bruckner and S.V. Pereverzev. Self-regularization of projection methods with a posteriori discretization level choice for severely ill-posed problems. *Inverse Problems*, **19**(1):147–156, 2003. http://dx.doi.org/10.1088/0266-5611/19/1/308.

[2] H.W. Engl. On the convergence of regularization methods for ill-posed linear operator equations. In G. Hämmerlin and K.-H. Hoffmann(Eds.), *Improperly Posed Problems and Their Numerical Treatment*, pp. 81–95. Birkhäuser, 1983.

[3] H.W. Engl, M. Hanke and A. Neubauer. *Regularization of Inverse Problems.* Kluwer, Dordrecht, 1996.

[4] V.M. Fridman. New methods of solving a linear operator equation. *Dokl. Akad. Nauk SSSR*, **128**:482–484, 1959.

[5] U. Hämarik. Projection methods for the regularization of linear ill-posed problems. *Proc. Comput. Center Tartu Univ.*, **50**:69–90, 1983.

[6] U. Hämarik. Monotonicity of error and choice of the stopping index in iterative regularization methods. In A. Pedas(Ed.), *Differential and Integral Equations: Theory and Numerical Analysis*, pp. 15–30. Estonian Math. Society, Tartu, 1999.

[7] U. Hämarik, E. Avi and A. Ganina. On the solution of ill-posed problems by projection methods with a posteriori choice of the discretization level. *Math. Model. Anal.*, **7**(2):241–252, 2002. http://dx.doi.org/10.1080/13926292.2002.9637196.

[8] U. Hämarik, U. Kangro, R. Palm, T. Raus and U. Tautenhahn. Monotonicity of error of regularized solution and its use for parameter choice. *Inverse Probl. Sci. Eng.*, **22**(1):10–30, 2014. http://dx.doi.org/10.1080/17415977.2013.827185.

[9] U. Hämarik and U. Tautenhahn. On the monotone error rule for parameter choice in iterative and continuous regularization methods. *BIT*, **41**(5):1029–1038, 2001. http://dx.doi.org/10.1023/A:1021945429767.

[10] H. Harbrecht, S. Pereverzev and R. Schneider. Self–regularization by projection for noisy pseudodifferential equations of negative order. *Numer. Math.*, **95**(1):123–143, 2003. http://dx.doi.org/10.1007/s00211-002-0417-x.

[11] B. Hofmann, P. Mathé and S.V. Pereverzev. Regularization by projection: approximation theoretic aspects and distance functions. *J. Inverse Ill-Posed Probl.*, **15**(5):527–545, 2007. http://dx.doi.org/10.1515/jiip.2007.029.

[12] V.K. Ivanov, V.V. Vasin and V.P. Tanana. *The Theory of Linear Ill-Posed Problems and its Applications.* Nauka, Moscow, 1978. (in Russian)

[13] B. Kaltenbacher. Regularization by projection with a posteriori discretization level choice for linear and nonlinear ill-posed problems. *Inverse Problems*, **16**(5):1523–1539, 2000. http://dx.doi.org/10.1088/0266-5611/16/5/322.

[14] B. Kaltenbacher. On the regularizing properties of a full multigrid method for ill-posed problems. *Inverse Problems*, **17**(4):767–788, 2001. http://dx.doi.org/10.1088/0266-5611/17/4/313.

[15] B. Kaltenbacher. V-cycle convergence of some multigrid methods for ill-posed problems. *Math. Comp.*, **72**(244):1711–1730, 2003. http://dx.doi.org/10.1090/S0025-5718-03-01533-3.

[16] B. Kaltenbacher and J. Offtermatt. A convergence analysis of regularization by discretization in preimage space. *Math. Comp.*, **81**(280):2049–2069, 2012. http://dx.doi.org/10.1090/S0025-5718-2012-02596-8.

[17] P. Mathé and S.V. Pereverzev. Optimal discretization of inverse problems in Hilbert scales. regularization and self-regularization of projection methods. *SIAM J. Numer. Anal.*, **38**(6):1999–2021, 2001. http://dx.doi.org/10.1137/S003614299936175X.

[18] P. Mathé and N. Schöne. Regularization by projection in variable Hilbert scales. *Appl. Anal.*, **87**(2):201–219, 2008. http://dx.doi.org/10.1080/00036810701858185.

[19] W.V. Petryshyn. Direct and iterative methods for the solution of linear operator equations in Hilbert space. *Trans. Amer. Math. Soc.*, **105**(1):136–175, 1962. http://dx.doi.org/10.1090/S0002-9947-1962-0145651-8.

[20] D.L. Phillips. A technique for the numerical solution of certain integral equations of the first kind. *J. Assoc. Comput. Mach.*, **9**:84–97, 1962. http://dx.doi.org/10.1145/321105.321114.

[21] G.R. Richter. Numerical solution of integral equations of the first kind with nonsmooth kernels. *SIAM J. Numer. Anal.*, **15**(3):511–522, 1978. http://dx.doi.org/10.1137/0715033.

[22] T.I. Seidman. Convergent approximation methods for ill-posed problems, Part I: general theory. *Control Cybernet.*, **10**:31–49, 1981.

[23] G.M. Vainikko and U. Hämarik. Projection methods and self-regularization in ill-posed problems. *Soviet Math.*, **29**(10):1–20, 1985.

[24] G.M. Vainikko and A.Yu. Veretennikov. *Iteration Procedures in Ill-Posed Problems.* Nauka, Moscow, 1986. (in Russian)