A Maximum Principle for a Fractional Boundary Value Problem with Convection Term and Applications

Mohammed Al-Refai and Kamal Pal

Department of Mathematical Sciences, UAE University
Al Ain Abu Dhabi, UAE, P.O.Box 15551
E-mail(*corresp.*): m_alrefai@uaeu.ac.ae
E-mail: kamal.k@uaeu.ac.ae

Abstract. We consider a fractional boundary value problem with Caputo-Fabrizio fractional derivative of order $1 < \alpha < 2$. We prove a maximum principle for a general linear fractional boundary value problem. The proof is based on an estimate of the fractional derivative at extreme points and under certain assumption on the boundary conditions. A prior norm estimate of solutions of the linear fractional boundary value problem and a uniqueness result of the nonlinear problem have been established. Several comparison principles are derived for the linear and nonlinear fractional problems.

Keywords: fractional differential equations, Caputo-Fabrizio fractional derivative, maximum principle.

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1 Introduction

Recently, Caputo and Fabrizio have introduced a new fractional derivative with nonsingular kernel [12]. They replaced the power-law kernel by a decreasing exponential kernel. The new fractional derivative has been applied to model several science and engineering problems [1,11,15,16,17,18,21]. The novelty of the new derivative is that, there is no singular kernel and it has the ability to describe the material heterogeneities and the fluctuations of different scales [9, 10,14], which cannot be well described by classical local theories or by fractional models with singular kernel [12,13]. Recently, certain classes of Caputo-Fabrizio differential equations were transformed to differential equations with integer

derivatives. And the theory of second order differential equations can be used to study such classes, see [5, 21, 24]. However, it is not possible to transform all types of Caputo-Fabrizio differential equations to ones with integer derivative.

In this paper, we consider the following fractional boundary value problem of order $1 < \alpha < 2$,

$$(^{CFC}D_a^\alpha z)(t) + k_1(t)z'(t) + k_2(t)z(t) = f(t, z), \quad t \in (a, b), \qquad (1.1)$$

$$z(a) - \beta z'(a) = 0 \quad \text{and} \quad z(b) + \gamma z'(b) = 0, \quad \beta, \gamma \geq 0, \qquad (1.2)$$

where $f(t, z)$ is a smooth function, $k_1, k_2 \in C[a, b]$, and $^{CFC}D_a^\alpha$ is the Caputo-Fabrizio fractional derivative of order α. The problem was discussed at first by Al-Refai [3] and it is a particular case of the wide class of boundary value problems considered in Pedas and Tamme [22]. The maximum principle is an important analytical tool to study fractional differential equations. Several existence and uniqueness results have been established using maximum principles for linear and nonlinear fractional diffusion equations, see [3, 6, 7, 8, 19].

The aim of the manuscript is, to analyze the solutions of the above problem analytically by applying certain maximum principles. The paper is organized as follows. In Section 2, we estimate the fractional derivative of a function at its extreme points and obtain a new maximum principle for a linear fractional boundary value problem of order $1 < \alpha < 2$. In Section 3, we apply the obtained maximum principle to study linear and nonlinear fractional boundary value problems. Certain comparison results, norm estimates of solutions and uniqueness results will be discussed. Finally, we close up with some concluding remarks in Section 4.

2 A maximum principle

We start with the definition of the Caputo-Fabrizio fractional derivative.

DEFINITION 1. For $p \in [1, \infty]$ and Ω an open subset of R, the Sobolev space $H^p(\Omega)$ is defined by

$$H^p(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega), \text{ for all } |\alpha| \leq p\}.$$

DEFINITION 2. [12] Let $z \in H^1(a, b)$, $a < b$, $a \in (-\infty, t)$, $0 < \alpha < 1$, the Caputo-Fabrizio fractional derivative of Caputo sense is defined by

$$(^{CFC}D_a^\alpha z)(t) = \frac{N(\alpha)}{1 - \alpha} \int_a^t z'(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}ds,$$

where $N(\alpha) > 0$ is a normalization function satisfying $N(0) = N(1) = 1$.

For $n \geq 1$, and $0 < \alpha < 1$, the Caputo-Fabrizio fractional derivative $(^{CFC}D_a^{\alpha+n}z)(t)$ of order $(n + \alpha)$ is defined by

$$(^{CFC}D_a^{\alpha+n}z)(t) = (^{CFC}D_a^\alpha z^{(n)})(t).$$

Thus, for $z \in H^2(a,b)$, $a < b$, $1 < \alpha < 2$, the Caputo-Fabrizio fractional derivative of Caputo sense is defined by

$$({}^{CFC}D_a^{\alpha}z)(t) = \frac{N(\alpha-1)}{2-\alpha} \int_a^t z''(s)e^{-\frac{\alpha-1}{2-\alpha}(t-s)}ds.$$

Let $B(\alpha) = \frac{N(\alpha)}{2-\alpha}$, and $\mu_\alpha = \frac{\alpha-1}{2-\alpha}$, then

$$({}^{CFC}D_a^{\alpha}z)(t) = B(\alpha) \int_a^t z''(s)e^{-\mu_\alpha(t-s)}ds, \ 1 < \alpha < 2.$$

We refer the reader to [2, 12, 13], for more details about definition and properties of higher order Caputo-Fabrizio fractional derivatives. In the following result, we estimate the Caputo-Fabrizio fractional derivative of a function at an extreme point. The result is analogous to the ones obtained in [4] for the Caputo and Riemann-Liouville fractional derivatives.

Theorem 1. *Let $z \in H^2(a,b)$ attain its minimum at $t_0 \in [a,b)$, then*

$$({}^{CFC}D_a^{\alpha}z)(t_0) \geq B(\alpha)e^{-\mu_\alpha(t_0-a)}[\mu_\alpha(z(a) - z(t_0)) - z'(a)], \tag{2.1}$$

for all $1 < \alpha < 2$.

Proof. Let $t_0 \in (a,b)$. We define the auxiliary function $h(t) = z(t) - z(t_0)$, $t \in [a,b]$. Then $h(t)$ satisfies the following on $[a,b]$,

$$h(t) \geq 0, \ h(t_0) = h'(t_0) = 0, \text{ and } {}^{CFC}D_a^{\alpha}h(t) = {}^{CFC}D_a^{\alpha}z(t). \tag{2.2}$$

Integration by parts twice of

$$({}^{CFC}D_a^{\alpha}h)(t_0) = B(\alpha) \int_a^{t_0} h''(s)e^{-\mu_\alpha(t_0-s)}ds$$

yields

$$({}^{CFC}D_a^{\alpha}h)(t_0) = B(\alpha)\left(e^{-\mu_\alpha(t_0-s)}h'(s)|_a^{t_0} - \mu_\alpha \int_a^{t_0} h'(s)e^{-\mu_\alpha(t_0-s)}ds\right)$$

$$= B(\alpha)\left(h'(t_0) - e^{-\mu_\alpha(t_0-a)}h'(a) - \mu_\alpha\left[e^{-\mu_\alpha(t_0-s)}h(s)|_a^{t_0}\right.\right.$$

$$\left.\left. - \mu_\alpha \int_a^{t_0} h(s)e^{-\mu_\alpha(t_0-s)}ds\right]\right) = B(\alpha)\left(h'(t_0) - e^{-\mu_\alpha(t_0-a)}h'(a)\right.$$

$$\left. - \mu_\alpha[h(t_0) - e^{-\mu_\alpha(t_0-a)}h(a)] + \mu_\alpha^2 \int_a^{t_0} h(s)e^{-\mu_\alpha(t_0-s)}ds\right)$$

Applying the results in (2.2) we have

$$({}^{CFC}D_a^{\alpha}z)(t_0) = ({}^{CFC}D_a^{\alpha}h)(t_0) \geq B(\alpha)\left(-e^{-\mu_\alpha(t_0-a)}h'(a)\right.$$

$$\left. + \mu_\alpha e^{-\mu_\alpha(t_0-a)}h(a)\right) = B(\alpha)e^{-\mu_\alpha(t_0-a)}\left(\mu_\alpha(z(a) - z(t_0)) - z'(a)\right),$$

which proves the result in (2.1) for $t_0 \in (a,b)$. If $t_0 = a$, then $({}^{CFC}D_a^{\alpha}z)(a) = 0$, and by simple maximum principle we have $z'(a^+) \geq 0$. Thus, the inequality (2.1) holds true for $t_0 = a$, which completes the proof. $\square$

Remark 1. In the above result, if $t_0 = b$, then by simple maximum principle we have $h'(b^-) = z'(b^-) \leq 0$, and thus

$$({}^{CFC}D_a^\alpha z)(b) \geq B(\alpha)z'(b^-) + B(\alpha)e^{-\mu_\alpha(b-a)}[\mu_\alpha(z(a) - z(b)) - z'(a)].$$

Corollary 1. Assume $z \in H^2(a, b)$ attains its minimum at $t_0 \in [a, b]$ and $z'(a) \leq 0$. Then $({}^{CFC}D_a^\alpha z)(t_0) \geq 0$, for all $1 < \alpha < 2$.

Proof. By Theorem 1 we have

$$({}^{CFC}D_a^\alpha z)(t_0) \geq B(\alpha)e^{-\mu_\alpha(t_0-a)}\big(\mu_\alpha(z(a) - z(t_0)) - z'(a)\big).$$

Since $z(t_0) \leq z(a)$ and $z'(a) \leq 0$, we have $({}^{CFC}D_a^\alpha z)(t_0) \geq 0$. $\square$

Consider the linear fractional operator

$$(P_\alpha z)(t) = ({}^{CFC}D_a^\alpha z)(t) + k_1(t)z'(t) + k_2(t)z(t) = f(t), t \in (a, b), 1 < \alpha < 2 \tag{2.3}$$

with boundary conditions

$$B_1(z) \quad = \quad z(a) - \beta z'(a), \quad B_2(z) = z(b) + \gamma z'(b), \quad \beta, \gamma \geq 0. \tag{2.4}$$

We have the following maximum principle for the linear fractional boundary value problem (2.3)–(2.4).

Lemma 1. *Let $z(t) \in H^2(a, b) \cap C[a, b]$, satisfy the inequalities*

$$(P_\alpha z)(t) = {}^{CFC}D_a^\alpha z(t) + k_1(t)z'(t) + k_2(t)z(t) \leq 0, \quad t \in (a, b), \tag{2.5}$$
$$B_1(z) \geq 0, \quad B_2(z) \geq 0, \quad \beta, \gamma \geq 0, \tag{2.6}$$

where $k_1(t)$, $k_2(t) \in C[a, b]$ and $k_2(t) \leq 0$, $t \in [a, b]$. If $\beta \geq \frac{1}{\alpha-1}$, then it holds that $z(t) \geq 0$, $t \in [a, b]$.

Proof. Assume that the statement is not true. Since $z(t)$ is continuous, then $z(t)$ has absolute minimum at some point t_0 with $z(t_0) < 0$. Let $t_0 \in (a, b)$, then $z'(t_0) = 0$. In the following, we prove that $({}^{CFC}D_a^\alpha z)(t_0) > 0$. By Corollary 1, the result is true if $z'(a) \leq 0$. If $z'(a) > 0$, by Theorem 1 there holds

$$({}^{CFC}D_a^\alpha z)(t_0) \geq B(\alpha)e^{-\mu_\alpha(t_0-a)}[\mu_\alpha(z(a) - z(t_0)) - z'(a)].$$

Since $\beta(\alpha - 1) \geq 1$, and $z(a) \geq \beta z'(a)$, we have

$$(\alpha - 1)[z(a) - z(t_0)] \geq (\alpha - 1)[\beta z'(a) - z(t_0)]$$
$$= \beta(\alpha - 1)z'(a) - (\alpha - 1)z(t_0) \geq z'(a) - (\alpha - 1)z(t_0).$$

Thus,

$$(\alpha - 1)(z(a) - z(t_0)) - (2 - \alpha)z'(a) \geq z'(a) - (\alpha - 1)z(t_0) - (2 - \alpha)z'(a)$$
$$= (\alpha - 1)z'(a) - (\alpha - 1)z(t_0) = (\alpha - 1)(z'(a) - z(t_0)).$$

Because $z'(a) > 0$, and $z(t_0) < 0$, the term $(\alpha - 1)(z'(a) - z(t_0)) > 0$, and thus $(^{CFC}D_a^\alpha z)(t_0) > 0$. The above results together with $k_2(t) \leq 0$, imply

$$(P_\alpha z)(t_0) = (^{CFC}D_a^\alpha z)(t_0) + k(t_0)z'(t_0) + k_2(t)z(t_0)$$
$$\geq (^{CFC}D_a^\alpha z)(t_0) + k_2(t_0)z(t_0) > 0,$$

which contradicts inequality (2.5). If $t_0 = a$, by simple maximum principle $z'(a^+) \geq 0$. Applying the boundary conditions $z(a) - \beta z'(a) \geq 0$ we have $z(a) \geq 0$ and a contradiction is reached. Similarly, if $t_0 = b$ then simple maximum principle implies $z'(b^-) \leq 0$. The boundary condition $z(b) + \gamma z'(b^-) \geq 0$ yields $z(b) \geq 0$ and a contradiction is reached. □

The condition $\beta \geq \frac{1}{\alpha-1}$ has been introduced at the first time in [3]. Then, later on it has been used by many authors to establish certain comparison principles in the regular and discrete forms, see for instance [20, 23]. In the following example, we show that one cannot simply discard the condition $\beta \geq \frac{1}{\alpha-1}$, to obtain the maximum principle in Lemma 1.

Example 1. Consider $z(t) = at + bt^2 + ct^3 + d$, $0 \leq t \leq 1$. Choose $\mu_\alpha = \frac{1}{9}$, $c = 0.08$, and $d = 0.02$. Let a and b be such that

$$b = 3c\left(\frac{1}{\mu_\alpha} - \frac{1}{1 - e^{-\mu_\alpha}}\right) = -0.122222 \text{ and } a = -(b + c + d) = 0.0222218.$$

For $1 < \alpha < 2$, we have

$$^{CFC}D_0^\alpha t = {}^{CFC}D_0^\alpha 1 = 0, \quad ^{CFC}D_0^\alpha t^2 = \frac{2B(\alpha)}{\mu_\alpha}\left(1 - e^{-\mu_\alpha t}\right),$$

$$^{CFC}D_0^\alpha t^3 = \frac{6B(\alpha)}{\mu_\alpha}\left(t - \frac{1}{\mu_\alpha}(1 - e^{-\mu_\alpha t})\right).$$

Thus,

$$(^{CFC}D_0^\alpha z)(t) = \frac{B(\alpha)}{\mu_\alpha}\left((2b - \frac{6c}{\mu_\alpha})(1 - e^{-\mu_\alpha t}) + 6ct\right)$$
$$= -\frac{6cB(\alpha)}{\mu_\alpha}\left(\frac{1}{1 - e^{-\mu_\alpha}}(1 - e^{-\mu_\alpha t}) - t\right) = -4.32B(\alpha)\left(\frac{1}{1 - e^{-\frac{1}{9}}}(1 - e^{-\frac{1}{9}t}) - t\right).$$

One can easily show that $(^{CFC}D_0^\alpha z)(t) \leq 0$, $0 \leq t \leq 1$. If we choose $\beta = 0.5$ and $\gamma = 1$, then we have

$$B_1(z) = z(0) - \beta z'(0) = d - \beta a = 0.02 - 0.5(0.0222218) \geq 0,$$
$$B_2(z) = z(1) + \gamma z'(1) = a + b + c + d + \gamma(a + 2b + 3c)$$
$$= a + 2b + 3c = 0.0222218 - 0.244444 + 0.24 = 0.0177783 \geq 0.$$

Let $k_1(t) = k_2(t) = 0$, we have

$$(P_\alpha z)(t) = (^{CFC}D_0^\alpha z)(t) \leq 0, \ 0 \leq t \leq 1, B_1(z), B_2(z) \geq 0,$$

while $z(0.9) = -0.00068001 < 0$.

3 Applications

In this section, we establish certain analytical results on the system (1.1–1.2). We have

3.1 The linear problem

Lemma 2. *Let $z \in H^2(a,b) \cap C[a,b]$ be the solution of*

$$
\begin{aligned}
(P_\alpha z)(t) &= f(t), \quad t \in (a,b), \quad 1 < \alpha < 2,\\
B_1(z) &= 0, \qquad B_2(z) = 0,
\end{aligned}
$$

where $k_1(t), k_2(t) \in C[a,b]$, $k_2(t) \leq 0$. If $\beta \geq \frac{1}{\alpha-1}$, then it holds that

$$
||z(t)||_{[a,b]} = \max_{t \in [a,b]} |z(t)| \leq M = \max_{t \in [a,b]} \left\{ \left| \frac{f(t)}{k_2(t)} \right| \right\},
$$

provided M exists.

Proof. We have $M = \max_{[a,b]}\{|\frac{f(t)}{k_2(t)}|\}$, and thus $M \geq |\frac{f(t)}{k_2(t)}|$, $t \in [a,b]$, or

$$
|f(t)| - |k_2(t)|M \leq 0, \quad t \in [a,b].
$$

Let $u_1(t) = M - z(t)$, then it holds that

$$
\begin{aligned}
(P_\alpha u_1)(t) =& ^{CFC} D_a^\alpha u_1(t) + k_1(t)u_1'(t) + k_2(t)u_1(t)\\
=& -^{CFC} D_a^\alpha z(t) - k_1(t)z'(t) + k_2(t)M - k_2(t)z(t) = k_2(t)M - f(t)\\
=& -|k_2(t)|M - f(t), \text{ as } k_2(t) = -|k_2(t)| \leq -|k_2(t)|M + |f(t)| \leq 0,\\
B_1(u_1) =& B_2(u_1) = M \geq 0.
\end{aligned}
$$

Thus by virtue of the Lemma 1 we have $u_1(t) = M - z(t) \geq 0$ or

$$
z(t) \leq M. \tag{3.1}
$$

Analogously, Let $u_2(t) = M + z(t)$, then it holds that

$$
\begin{aligned}
(P_\alpha u_2)(t) &= {}^{CFC} D_a^\alpha u_1(t) + k_1(t)u_1'(t) + k_2(t)u_1(t)\\
&= {}^{CFC} D_a^\alpha z(t) + k_1(t)z'(t) + k_2(t)M + k_2(t)z(t)\\
&= k_2(t)M + f(t) \leq -|k_2(t)|M + |f(t)| \leq 0,\\
B_1(u_2) &= B_2(u_2) = M \geq 0.
\end{aligned}
$$

Thus by virtue of the Lemma 1 we have $u_2(t) = M + z(t) \geq 0$, or

$$
z(t) \geq -M. \tag{3.2}
$$

By combining the two equations (3.1) and (3.2) we have

$$
|z(t)| \leq M, \quad t \in [a,b],
$$

and hence the result.   □

Remark 2. If $k_2(t) \leq -\epsilon$, $t \in [a, b]$ with $\epsilon > 0$, then we guarantee the existence of the maximum M in the above result.

Lemma 3. *Let $z_1, z_2 \in H^2(a, b) \cap C[a, b]$ be the solution of*

$$(P_\alpha z_1)(t) = f_1(t), \quad t \in (a, b), \quad 1 < \alpha < 2, \tag{3.3}$$
$$B_1(z_1) = B_2(z_1) = 0,$$

and

$$(P_\alpha z_2)(t) = f_2(t), \quad t \in [a, b], \quad 1 < \alpha < 2, \tag{3.4}$$
$$B_1(z_2) = B_2(z_2) = 0,$$

where, $k_1(t), k_2(t), f_1(t), f_2(t) \in C[a, b]$, $k_2(t) \leq 0$, and $\beta \geq \frac{1}{\alpha-1}$. If $f_1(t) \leq f_2(t)$, $t \in [a, b]$ then it holds that

$$z_1(t) \geq z_2(t), \ t \in [a, b].$$

Proof. Let $w(t) = z_1(t) - z_2(t)$, $t \in [a, b]$, then by subtracting Equation (3.4) from Equation (3.3) it holds that

$$(P_\alpha w)(t) = ^{CFC} D_a^\alpha w(t) + k_1(t) w'(t) + k_2(t) w(t) = (f_1(t) - f_2(t)) \leq 0$$
$$B_1(w) = w(a) - \beta w'(a) = 0, \quad B_2(w) = w(b) + \gamma w'(b) = 0, \quad \beta, \gamma \geq 0.$$

By virtue of the Lemma 1 we have $w(t) \geq 0$, or

$$z_1(t) \geq z_2(t), \qquad t \in [a, b],$$

and hence the result. □

3.2   The nonlinear problem

We consider the nonlinear fractional boundary value problem

$$(P_\alpha z)(t) = (^{CFC} D_a^\alpha z)(t) + k_1(t) z'(t) + k_2(t) z(t)$$
$$= f(t, z), \quad t \in (a, b), \ 1 < \alpha < 2, \tag{3.5}$$
$$B_1(z) = z(a) - \beta z'(a) = 0, \quad B_2(z) = z(b) + \gamma z'(b) = 0, \beta, \gamma \geq 0, \tag{3.6}$$

where $f(t, z)$ is a smooth function, $k_1(t), k_2(t) \in C[a, b]$, and $k_2(t) \leq 0$.

Lemma 4. *If $f(t, z)$ is non-decreasing with respect to z, then the nonlinear fractional boundary value problem (3.5)–(3.6) has at most one solution $z \in H^2(a, b) \cap C[a, b]$.*

Proof. Assume that $z_1, z_2 \in H^2(a, b) \cap C[a, b]$ be two solutions of the above equation, we shall show that $z_1 = z_2$. Let $w = z_1 - z_2$. Then it holds that

$$(P_\alpha w)(t) = f(t, z_1) - f(t, z_2), \quad B_1(w) = B_2(w) = 0.$$

Applying the mean value theorem we have

$$f(t, z_1) - f(t, z_2) = \frac{\partial f}{\partial z}(z^*)(z_1 - z_2),$$

for some z^* between z_1 and z_2. Thus,

$$(P_\alpha w)(t) - \frac{\partial f}{\partial z}(z^*)w = ({}^{CFC}D_a^\alpha w)(t) + k_1(t)w'(t) + (k_2(t) - \frac{\partial f}{\partial z}(z^*))w(t) = 0. \tag{3.7}$$

Since $-\frac{\partial f}{\partial z}(z^*) \leq 0$, and $k_2(t) \leq 0$, then $w(t) \geq 0$, by virtue of Lemma 1. Also, Equation (3.7) holds true for $-w$ and thus $-w \leq 0$, by virtue of Lemma 1. Thus, $w = 0$ and the result is proved. $\square$

Lemma 5. *Let $z(t) \in H^2(a, b) \cap C[a, b]$ be the solution of (3.5–3.6), and assume that there exists $g_1(t), g_2(t) \in C[a, b]$ such that*

$$\mu_2 z(t) + g_2(t) \leq f(t, z) \leq \mu_1 z(t) + g_1(t), \quad for \quad all \quad t \in (a, b),$$

where $\mu_1, \mu_2 \geq k_2(t)$. Let $u_1, u_2 \in H^2(a, b)$ be the solutions of

$$\begin{aligned}
(P_\alpha u_1)(t) &= \mu_1 u_1(t) + g_1(t), \quad t \in [a, b], \quad 1 < \alpha < 2, \\
B_1(u_1) &= 0, \ B_2(u_1) = 0, \quad \beta, \gamma \geq 0, \\
(P_\alpha u_2)(t) &= \mu_1 u_2(t) + g_2(t), \quad t \in [a, b], \quad 1 < \alpha < 2, \\
B_1(u_2) &= 0, \ B_2(u_2) = 0, \quad \beta, \gamma \geq 0.
\end{aligned} \tag{3.8}$$

Then it holds that $u_1(t) \leq z(t) \leq u_2(t)$, $t \in [a, b]$.

Proof. We shall first prove that $u_1(t) \leq z(t)$. By subtracting Equation (3.8) from Equation (3.5) we have

$$\begin{aligned}
(P_\alpha z)(t) - (P_\alpha u_1)(t) &= f(t, z) - \mu_1 u_1(t) - g_1(t) \\
&\leq \mu_1 z(t) + g_1(t) - \mu_1 u_1(t) - g_1(t) = \mu_1(z(t) - u_1(t)).
\end{aligned}$$

Let $v(t) = z(t) - u_1(t)$, then it holds that

$${}^{CFC}D_a^\alpha v(t) + k_1(t)v'(t) + (k_2(t) - \mu_1)v(t) \leq 0.$$

Since $\mu_1 \geq k_2(t)$ we have $(k_2(t) - \mu_1) \leq 0$. By virtue of the Lemma 1, we have $v(t) \geq 0$, or $u_1(t) \leq z(t)$, $t \in [a, b]$. By applying analogous steps we can show that $z(t) \leq u_2(t)$, $t \in [a, b]$. Combining both estimates we have

$$u_1(t) \leq z(t) \leq u_2(t), \quad t \in [a, b].$$

$\square$

4 Conclusions

We estimated the Caputo-Fabrizio fractional derivative of order $1 < \alpha < 2$, for a function at its extreme points. We then used the result to construct a

maximum principle for a linear fractional boundary value problem under the condition $\beta \geq \frac{1}{\alpha-1}$, where β is defined in (1.2). We proved that this condition is a sufficient and essential condition to obtain such maximum principle. We then implemented the constructed maximum principle to obtain a norm estimate of solutions to the linear problem and to prove a uniqueness result for the nonlinear problem. Several comparison principles are obtained as well.

Acknowledgements

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