ISSN: 1392-6292/ eISSN: 1648-3510

VILNIUS TECH Vilnius Gediminas Technical University

MATHEMATICAL MODELLING and ANALYSIS

2026
Volume 31
Issue 3
Pages 476–498
https://doi.org/10.3846/mma.2026.25296

New existence and exponential stability results for periodic solutions in recurrent neural networks with generalized piecewise constant delay via coincidence degree theory

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Article History:

- received November 12, 2025
- revised February 12, 2026
- accepted February 20, 2026

Abstract. The present work investigates recurrent neural systems incorporating generalized piecewise constant delay, with particular emphasis on establishing periodic behaviors and verifying their exponential convergence on a global scale. The existence of periodic solutions is established via Mawhin's coincidence degree in combination with sharp *a priori* estimates, while uniqueness and exponential attractivity are derived through a Lyapunov functional approach supported by differential inequalities adapted to the delay structure. The obtained criteria are concise, verifiable, and applicable in practice. Representative computational experiments are provided to substantiate the analytical findings.

Keywords: recurrent neural networks; generalized piecewise constant delay; periodic solutions; coincidence degree theory; global exponential stability; method of Lyapunov functions.

AMS Subject Classification: 92B20; 34K13; 34A36; 34K20.

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1 Introduction

Recurrent neural networks (RNNs), especially cellular architectures, form a fundamental class of nonlinear dynamical systems in which recurrent couplings encode memory and spatial interaction. Their structural regularity makes them well suited for analog and sampled-data implementations [10, 14]. A central problem is to identify conditions ensuring the existence of periodic solutions and their uniform exponential convergence, which together guarantee predictable cyclic behavior and quantitative robustness.

Various analytical techniques have been developed to address periodicity and stability. Lyapunov-based methods, including Krasovskii and Razumikhin constructions, provide explicit decay estimates and have been applied to systems with discontinuous activations [5], time-varying or leakage delays [13], impulsive or memristive effects [17], and neutral or inertial dynamics [11,21]. From the existence perspective, coincidence degree theory, particularly Mawhin's continuation theorem, reformulates periodic boundary value problems as Fredholm operator equations and yields periodic solutions under suitable a priori bounds [3]. Fixed-point approaches [2], semigroup and energy methods [15,20], Gronwall-type inequalities [8,24], matrix inequalities [16,23], and stochastic or almost-periodic formulations [13,22] further expand the class of treatable neural models.

Within this framework, differential equations with generalized piecewise constant arguments (DEGPCD) provide a flexible description of hybrid delay mechanisms. In such systems, trajectories remain continuous while the deviating argument evolves through a sample-and-hold process that may alternate between advanced and retarded phases. Foundational studies appear in [1,19]. Subsequent developments established periodic solvability, operator-theoretic formulations using Green's functions, stability for alternately advanced and retarded neural models with impulses [6], and extensions to nonautonomous or almost-periodic settings [7]. For neural systems, globally exponentially stable periodic solutions under generalized piecewise constant delays have been obtained [9], and related techniques have been applied to hybrid epidemiological models.

Such generalized piecewise constant delays naturally arise in sampled-data and networked implementations, where state information is updated at irregular instants and held constant between updates. This mechanism reflects hybrid analog and digital computations and cannot be captured by smooth time-varying delays.

Despite these advances, several limitations persist. Many criteria rely on global Lipschitz conditions, diagonal dominance, or uniformly bounded delays [16,23], assumptions that may conflict with cellular templates or saturation-type nonlinearities. Inertial, neutral, impulsive, and alternating regimes are often studied separately, and Lyapunov constructions that neglect hold effects may yield conservative decay estimates. Although progress has been made in neutral and inertial periodicity [11,21], impulsive neutral delays [4], fixed-time stabilization [15], and leakage-type contraction for difference equations [2], a unified approach combining cellular structures, generalized sample-and-hold delays, and explicit global exponential rates remains limited.

Motivated by this gap, the present paper studies recurrent neural systems with generalized piecewise constant delays. First, we establish new sufficient conditions for periodic solutions via a Mawhin-type coincidence degree framework adapted to cellular coupling. Second, we construct Lyapunov functionals tailored to the retarded segments to derive explicit global exponential convergence rates, consistent with analyses of discontinuous activations, impulsive effects, and spatial interactions [5,8,17,20,24]. The resulting criteria are concise and verifiable, making them suitable for sampled-data neural implementations.

The remainder of the paper is organized as follows. Section 2 presents preliminaries. Section 3 proves periodicity via coincidence degree theory. Section 4 establishes uniform exponential stability using Lyapunov functionals. A numerical example illustrates the theoretical findings.

2 Mathematical preliminaries

In the following, we set out the essential concepts and tools forming the basis of the subsequent analysis. We first present the core definitions, notation, and assumptions governing the recurrent neural network model. We then provide auxiliary lemmas and technical results used to establish periodic solvability through coincidence degree theory and to derive global exponential stability via Lyapunov-type constructions. These elements supply the rigorous foundation for the main theorems.

We examine recurrent neural architectures whose delay mechanism follows a generalized piecewise constant structure. For each index j with $m \geq j \geq 1$, the system evolves according to

$$x'_j(t) = -a_j(t)x_j(t) + \sum_{k=1}^{m}\Big(b_{jk}(t)f_k(x_k(t)) + c_{jk}(t)g_k(x_k(\beta(t)))\Big) + I_j(t), \ t \geq 0. \tag{2.1}$$

The components of (2.1) are interpreted as follows:

- A generalized piecewise constant delay is defined through an unbounded monotone sequence $\{t_l\}$ and the map $\beta(\cdot)$ for $t \in J_l = [t_l, t_{l+1})$, with $\mathbb{R}^+ = \bigcup_{l\in\mathbb{N}} J_l$. This construction represents the sample–and–hold behavior intrinsic to hybrid and event-driven computations.

- The strictly positive function $a_j(\cdot)$ specifies the natural decay rate of neuron j, governing its exponential return to equilibrium in the absence of interactions or external input.

- The term $f_k(x_k(\cdot))$ denotes the instantaneous activation of neuron k, while $g_k(x_k(\beta(\cdot)))$ captures its response to the sampled delayed signal. Together, they model both direct synaptic influence and the feedback produced by the delay mechanism.

- The coefficients $b_{jk}(\cdot)$ and $c_{jk}(\cdot)$ represent the weighted couplings between neurons, with $b_{jk}(\cdot)$ encoding the immediate effect of neuron k on neuron j, and $c_{jk}(\cdot)$ describing the sampled, piecewise constant contribution.

- The term $I_j(\cdot)$ denotes the external input acting on neuron j.

The RNN model studied here incorporates a generalized piecewise constant delay (DEGPCD), a feature that warrants particular attention. Unlike the classical DEPCD, which uses a fixed uniform partition of the time axis, the generalized formulation permits the sequence $\{t_l\}$ to be irregular, nonuniform, and influenced by structural parameters, thereby expanding the system's modeling scope. This flexibility captures scenarios where sampling or signal transmission occurs at uneven intervals—common in hybrid analog–digital mechanisms, asynchronous communication, and biological neural processes with variable synaptic latencies. The generalized framework is analytically more demanding because compactness, continuity, and contractive conditions must remain valid over intervals of differing lengths, and stability or existence criteria must depend on the supremum of varying segment widths rather than a single fixed step. Consequently, the analysis becomes more intricate and better aligned with realistic hybrid behaviors. Incorporating DEGPCD into RNNs thus extends classical delayed-network theory, bridging the gap between uniformly sampled models and genuinely irregular hybrid dynamics, and providing a versatile setting capable of representing deterministic and context-dependent delay mechanisms.

For subsequent analysis, we impose the following structural assumptions, which will be applied as needed.

($\mathcal{L}$) **Lipschitz Requirement.** The mappings f_k and g_k are required to satisfy the condition $f_k(0) = g_k(0) = 0$, for $k = 1, ..., m$. Moreover, they satisfy the uniform Lipschitz bounds

$$|f_k(v_1) - f_k(v_2)| \le \mathcal{L}_k^f |v_1 - v_2|, \qquad |g_k(v_1) - g_k(v_2)| \le \mathcal{L}_k^g |v_1 - v_2|,$$

for certain positive constants $\mathcal{L}_k^f, \mathcal{L}_k^g$ and for every pair of real numbers $v_1, v_2 \in \mathbb{R}$.

($\mathcal{E}$) **Existence Requirement.** For every $\zeta > 0$, one requires that

$$\varsigma = \max_{1\le j\le m}\left\{\max_{1\le l\le i(\zeta)}\int_{t_l}^{t_{l+1}} e^{\int_{t_l}^{r} a_j(u)\,du}\Big(\sum_{k=1}^{m}\mathcal{L}_k^f\,|b_{jk}(r)|+\mathcal{L}_k^g\,|c_{jk}(r)|\Big)\,dr\right\}<1$$

with the convention that $i(t) = l$ whenever $t \in J_l$.

($\mathcal{P}$) **Periodicity Requirement.** Let $\omega > 0$ be fixed. The functions associated with the model are assumed to be continuous and ω-periodic: a_j is uniformly positive, b_{jk} and c_{jk} represent periodic interconnections, while I_j corresponds to a periodic external drive. In addition, one can find $p \in \mathbb{N}$ such that the sequence $\{t_l\}_{l\in\mathbb{N}}$ satisfies the (ω, p)-property:

$$t_{l+p} = t_l + \omega, \qquad l \in \mathbb{N}.$$

This condition ($\mathcal{P}$) ensures consistency between the intrinsic periodicity of the system coefficients and the sampling structure induced by the generalized piecewise constant delay, thereby enabling the construction of periodic solutions.

An immediate generalization of the standard definition of a solution for DEPCA makes it possible to introduce the analogous concept for DEGPCD within the broader framework adopted in this work.

Definition 1. A function $x : \mathbb{R}^+ \to \mathbb{R}^m$ is said to be a solution of the DEGPCD system (2.1) provided the following hold:

i) x is continuous on $\mathbb{R}^+$;

ii) the derivative $x'(t)$ exists for all $t \in \mathbb{R}^+$ except at the sampling instants $t_l \in \mathbb{R}^+$, $l \in \mathbb{N}$, where one-sided derivatives are assumed to exist;

iii) on each interval (t_l, t_{l+1}), $l \in \mathbb{N}$, the function x satisfies the differential relation (2.1), and at the discontinuity points t_l the equation is understood in the sense of the right-hand derivative.

To analyze the nonlinear DEGPCD system, we adopt the method introduced in [19], which relies on reformulating the problem as an equivalent integral equation. The statement is articulated through the next proposition.

Proposition 1. *Take an initial state $(\zeta, x_0) \in \mathbb{R}^+ \times \mathbb{R}^m$. We call $x = (x_1(\cdot), \ldots, x_m(\cdot)) : \mathbb{R}^+ \to \mathbb{R}^m$ a solution of (2.1) if it fulfills the condition $x(\zeta) = x_0$, consistent with Definition 1. if and only if, for every $t \geq \zeta$, each component satisfies the integral identity*

$$x_j(t) = e^{-\int_\zeta^t a_j(s)ds} x_j(\zeta) + \int_\zeta^t e^{-\int_s^t a_j(u)du} \left(\sum_{k=1}^m b_{jk}(s) f_k(x_k(s)) + \sum_{k=1}^m c_{jk}(s) g_k(x_k(\beta(s))) + I_j(s) \right) ds, \quad \forall j \in \{1, \ldots, m\}.$$

The proof is omitted here, as it follows directly from arguments analogous to those used in Proposition 1 of [9], and Proposition 1 of [19].

Following the approach of [19], a key analytical element is a uniform bound derived through a Gronwall-type inequality tailored to DEGPCD systems. This framework systematically handles perturbations exhibiting exponential growth. By adapting the techniques in [9] and refined in [6], the next result follows.

Lemma 1. *Assume that $v : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and satisfies the inequality*

$$v(t) \;\leq\; \alpha \;+\; \left| \int_\zeta^t \Big(\varrho_1(s)\, v(s) + \varrho_2(s)\, v(\beta(s)) \Big)\, ds \right|,$$

for some constant $\alpha \geq 0$, where the coefficient functions ϱ_1, ϱ_2 are defined on $\mathbb{R}^+$, take nonnegative values, and are piecewise continuous. Under these assumptions, the following estimates are valid:

- *For all $t \geq \zeta$,*

$$v(t) \;\leq\; \alpha \,\exp\!\left(\int_\zeta^t \big(\varrho_1(s) + \varrho_2(s)\big)\, ds \right).$$

- *For $0 \leq t \leq \zeta$,*

$$v(t) \;\leq\; \alpha \,\exp\!\left(\int_t^\zeta \left(\varrho_1(s) + \frac{\varrho_2(s)}{1-\sigma} \right) ds \right),$$

where

$$\sigma := \max_{1 \leq l \leq i(\zeta)} \int_{t_l}^{t_{l+1}} \big(\varrho_1(s) + \varrho_2(s)\big)\, ds, \qquad \sigma \leq \kappa < 1.$$

To proceed, it is necessary to establish the global well-posedness of the nonlinear DEGPCD system (2.1). The forthcoming theorem states sufficient requirements under which solutions exist uniquely throughout the domain $\mathbb{R}^+$.

Theorem 1. *[9] Under hypotheses* $(\mathcal{E})$ *and* $(\mathcal{L})$*, it follows that for every prescribed pair* $(\zeta, x_0) \in \mathbb{R}^+ \times \mathbb{R}^m$*, the differential system* (2.1) *possesses a single well-defined solution* $x(\cdot) = x(\cdot, \zeta, x_0)$ *in accordance with Definition 1, and this solution satisfies the initial requirement* $x(\zeta) = x_0$.

3 On the presence of periodic orbits

In this section, we apply Mawhin's continuation theorem to obtain sufficient conditions ensuring an ω-periodic orbit for the DEGPCD system (2.1). The subsequent section examines global convergence toward this periodic solution.

Let $\omega > 0$ be fixed. Define $\mathbb{P}_\omega$ as the collection of all $\mathbb{R}^m$-valued continuous functions that are ω-periodic in t. Equipped with the supremum norm, the pair $(\mathbb{P}_\omega, \|\cdot\|)$ forms a Banach space, where

$$\|x\| = \max_{1\le k\le m} \|x_j\| = \max_{1\le k\le m}\left[\sup_{t\in\mathbb{R}^+} |x_k(t)|\right] = \max_{1\le k\le m}\left[\sup_{t\in[\zeta,\zeta+\omega]} |x_k(t)|\right].$$

We begin by recalling several operator-theoretic concepts that will be central to the application of Mawhin's continuation theorem. Take two normed spaces X and Y. We focus on a linear transformation $L : \mathrm{Dom}(L) \subset X \to Y$ and a continuous function $N : X \to Y$.
The mapping L is classified as a Fredholm mapping of index 0 whenever the following requirements are satisfied:

$$\dim(\ker L) = \operatorname{codim}(\operatorname{Im} L) < \infty, \qquad \operatorname{Im} L \text{ is a closed subspace of } Y.$$

Suppose further that L is identified as a Fredholm operator of index 0. Whenever bounded projectors $P : X \to X$ and $Q : Y \to Y$ exist with the properties

$$\operatorname{Im} P = \ker L, \qquad \ker Q = \operatorname{Im} L = \operatorname{Im}(I - Q),$$

the restriction

$$L\big|_{\mathrm{Dom}\, L\cap\ker P} : (I-P)X \longrightarrow \operatorname{Im} L$$

acts as an isomorphism, and we denote its inverse by K_P.
Consider an open bounded region $\Omega \subset X$. The mapping N is termed L-compact on $\bar{\Omega}$ whenever $QN(\bar{\Omega})$ is bounded in Y and the operator $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ and $\ker L$ are isomorphic, we may introduce the isomorphism $J : \operatorname{Im} Q \to \ker L$.

With these definitions in place, we can invoke the Mawhin's fundamental continuation theorem [12, p. 40], which furnishes a fundamental tool for establishing existence results of nonlinear operator equations in Banach spaces.

Lemma 2. *Let* $\Omega \subset X$ *be a bounded open subset, and let* $N : X \to Y$ *be continuous and* L*-compact on* $\bar{\Omega}$*. Assume the following hold:*

i) For every $\lambda \in (0,1)$ *and every* $x \in \partial\Omega \cap \operatorname{Dom} L$*, the equation* $Lx = \lambda Nx$ *does not hold;*

ii) For all $x \in \partial\Omega \cap \ker L$*, it holds that* $QNx \neq 0$*;*

iii) The topological degree $\deg(JNQ,\ \Omega \cap \ker L,\ 0)$ *is nonzero.*

Then $Lx = Nx$ *admits at least one solution in* $\bar{\Omega} \cap \operatorname{Dom} L$*.*

Theorem 2. *Suppose that assumptions* ($\mathcal{L}$)*,* ($\mathcal{E}$)*, and* ($\mathcal{P}$) *are satisfied. Then the RNNs model with the DEGPCD system* (2.1) *possesses an* ω*-periodic solution.*

Proof. For the purpose of applying Mawhin's continuation theorem in the context of coincidence degree framework and demonstrate that the RNNa model given by the DEGPCD system (2.1) admits an ω-periodic solution, we formulate the subsequent functional framework:

$$X = Y = \Big\{ x \in C(\mathbb{R}, \mathbb{R}^m) : x(t+\omega) = x(t) \text{ for all } t \in \mathbb{R} \Big\}.$$

Consider the norm on this space given by

$$\|x\| = \max_{1 \le k \le m} \Big(\sup_{t \in [\zeta, \zeta+\omega]} |x_k(t)| \Big).$$

With respect to this norm, X is a complete normed linear space, that is, a Banach space. Define the linear operator

$$L : \operatorname{Dom}(L) \cap X \to X, \qquad Lx = \frac{d}{dx} x(t), \quad x \in X,$$

with domain

$$\operatorname{Dom}(L) = \{x \in C^1(\mathbb{R}, \mathbb{R}^m)\}.$$

The corresponding nonlinear mapping $N : X \to X$ is specified in a componentwise manner as

$$(Nx)_j(t) \;=\; -a_j(t)\, x_j(t) + \sum_{k=1}^{m} b_{jk}(t)\, f_k(x_k(t)) + \sum_{k=1}^{m} c_{jk}(t)\, g_k(x_k(\beta(t))) + I_j(t),$$

for each index $j = 1, \ldots, m$.

Next, consider the projectors $P : X \to X$ and $Q : X \to X$, specified by

$$Px = Qx = \frac{1}{\omega} \int_{\zeta}^{\zeta+\omega} x(s)\, ds, \qquad x \in X.$$

From this construction, we obtain

$$\ker L = \mathbb{R}^m, \operatorname{Im}(L) = \Big\{ (x_1, \ldots, x_m)^T \in X : \int_{\zeta}^{\zeta+\omega} x_j(t)\, dt = 0,\ j = 1, \ldots, m \Big\},$$

where $\mathrm{Im}(L)$ is a closed subspace of X. It follows that

$$\dim\ker L = \operatorname{codim}\mathrm{Im}(L) = n,$$

hence L qualifies as a Fredholm operator with index 0.

In addition, arguments parallel to those employed in [18, Theorem 1] establish that N is L-compact on $\bar{\Omega}$ whenever $\Omega \subset X$ is bounded and open. Hence, for $\lambda \in (0,1)$, the operator relation $Lx = \lambda Nx$ can be rewritten in the form of an equivalent system

$$\frac{dx_j}{dt} = \lambda\Big(-a_j(t)\,x_j(t) + \sum_{k=1}^{m}\Big(b_{jk}(t)\,f_k(x_k(t)) + c_{jk}(t)\,g_k(x_k(\beta(t)))\Big) + I_j(t)\Big), \tag{3.1}$$

for $j = 1,\ldots,m$.

Let $x = (x_1,\ldots,x_m)^T \in X$ be a solution of system (3.1) corresponding to some fixed $\lambda \in (0,1)$. For each component j, choose a point $\xi_j \in [\zeta,\zeta+\omega]$ satisfying

$$x_j(\xi_j) = \max_{t\in[\zeta,\zeta+\omega]} x_j(t).$$

At this extremal point it holds that $x_j'(\xi_j) = 0$. Inserting this condition into (3.1) gives

$$\lambda\Big(-a_j(\xi_j)\,x_j(\xi_j) + \sum_{k=1}^{m}\Big(b_{jk}(\xi_j)\,f_k(x_k(\xi_j)) + c_{jk}(\xi_j)\,g_k(x_k(\beta(\xi_j)))\Big) + I_j(\xi_j)\Big) = 0.$$

Consequently, we obtain the estimate

$$\begin{aligned} x_j(\xi_j) \le{}& \frac{1}{a_j(\xi_j)}\Big(\sum_{k=1}^{m}|b_{jk}(\xi_j)|\,|f_k(x_k(\xi_j))| + \sum_{k=1}^{m}|c_{jk}(\xi_j)| \\ &\times |g_k(x_k(\beta(\xi_j)))| + |I_j(\xi_j)|\Big) \le \underline{a}_j^{-1}\Big(m(\overline{b}\,\overline{f} + \overline{c}\overline{g}) + \overline{I}\Big) =: A_1,\ j = 1,\ldots,m, \end{aligned} \tag{3.2}$$

where

$$\overline{f} = \sup_{u\in\mathbb{R}}\max_{1\le j\le m}|f_k(u)|,\ \ \overline{g} = \sup_{u\in\mathbb{R}}\max_{1\le j\le m}|g_k(u)|,\ \ \overline{I} = \sup_{t\in[\zeta,\zeta+\omega]}\max_{1\le j\le m}|I_j(t)|,$$
$$\overline{b} = \sup_{t\in[\zeta,\zeta+\omega]}\max_{1\le j,k\le m}|b_{jk}(t)|,\ \ \overline{c} = \sup_{t\in[\zeta,\zeta+\omega]}\max_{1\le j,k\le m}|c_{jk}(t)|.$$

Thus, one can find a constant $A_1 > 0$ satisfying

$$x_j(\xi_j) \le A_1, \qquad j = 1,\ldots,m.$$

Similarly, let $\kappa_j \in [\zeta,\zeta+\omega]$ be such that

$$x_j(\kappa_j) = \min_{t\in[\zeta,\zeta+\omega]} x_j(t).$$

At this point we obtain $x_j'(\kappa_j) = 0$, and substituting into (3.1) gives

$$\lambda\Big(- a_j(\kappa_j)x_j(\kappa_j) + \sum_{k=1}^{m} b_{jk}(\kappa_j)f_k(x_k(\kappa_j)) + \sum_{k=1}^{m} c_{jk}(\kappa_j)g_k(x_k(\beta(\kappa_j))) + I_j(\kappa_j)\Big) = 0.$$

From this relation, one deduces the lower bound

$$x_j(\kappa_j) \geq -\frac{1}{a_j(\kappa_j)}\Big(\sum_{k=1}^{m} |b_{jk}(\kappa_j)|\,|f_k(x_k(\kappa_j))| + |c_{jk}(\kappa_j)|\,|g_k(x_k(\beta(\kappa_j)))| + |I_j(\kappa_j)|\Big) \geq -\underline{a}_j^{-1}\Big(m(\overline{b}\,\overline{f} + \overline{c}\,\overline{g}) + \overline{I}\Big) =: -A_2, \quad j = 1,\dots,m,$$

where the constants $\overline{b}, \overline{c}, \overline{f}, \overline{g}$, and $\overline{I}$ are defined as in (3.2).
As a consequence, there exists a constant $A_2 > 0$ for which

$$x_j(\kappa_j) \geq -A_2, \qquad j = 1,\dots,m.$$

We then introduce

$$D = \max\{A_1, A_2\},$$

which does not depend on λ, and define the bounded open set

$$\Omega = \{\, x \in X : \ \|x\| < D \,\}.$$

The above construction guarantees that criterion (i) of Lemma 2 is fulfilled.
Next, let us examine an element $x \in \partial\Omega \cap \ker L = \partial\Omega \cap \mathbb{R}^m$. In this case,

$$\begin{aligned} x^T QNx &= \frac{1}{\omega}\sum_{j=1}^{m} x_j \int_{\zeta}^{\zeta+\omega} \Big[- a_j(s)x_j(s) + \sum_{k=1}^{m} b_{jk}(s)f_k(x_k(s)) \\ &\quad + \sum_{k=1}^{m} c_{jk}(s)g_k(x_k(\beta(s))) + I_j(s)\Big]\, ds \\ &\leq -\frac{1}{\omega}\sum_{j=1}^{m} x_j \int_{\zeta}^{\zeta+\omega} \Big(a_j(s)x_j(s) - m(\overline{b}\,\overline{f} + \overline{c}\,\overline{g})) - \overline{I}\Big)\, ds \ < \ 0, \end{aligned}$$

for $j = 1,\dots,m$. When necessary, D can be increased to ensure

$$-\frac{1}{\omega}\sum_{j=1}^{m} x_j \int_{\zeta}^{\zeta+\omega} \Big(a_j(s)\,x_j(s) - n\big(\overline{b}\,\overline{f} + \overline{c}\,\overline{g}\big)\big) - \overline{I}\Big)\, ds \ < \ 0.$$

As a consequence, for every $x \in \partial\Omega \cap \ker L$, one obtains $QNx \neq 0$. Thus, requirement (ii) of Lemma 2 is satisfied.

Finally, introduce the mapping

$$\psi(\nu; x) = -\nu x + (1-\nu)QNx, \qquad \nu \in [0,1],\ x \in \partial\Omega \cap \ker L.$$

For every such x and ν, one has $x^T\psi(\nu;x) < 0$. Therefore,

$$\deg(JNQ,\ \Omega\cap\ker L,\ 0) = \deg(-x,\ \Omega\cap\ker L,\ 0)\ \neq\ 0.$$

By Lemma 2, it follows that there exists at least one $x \in X$ satisfying $Lx = Nx$. Consequently, the RNNs model with the DEGPCD system (2.1) possesses at least one ω-periodic solution. □

4 Theorem on global exponential stability

Stability plays a central role in the analysis and design of cellular neural networks, particularly when delays or deviating arguments introduce destabilizing effects. In what follows, we derive explicit conditions ensuring that the unique periodic orbit x^* of the DEGPCD system (2.1) is globally exponentially stable. To facilitate the analytical treatment, system (2.1) can be reformulated in terms of the deviation variables

$$y_j(t) = x_j(t) - x_j^*(t), \qquad \hat{f}_k(y_k(t)) = f(x_k(t) + x_k^*(t)) - f(x_k^*(t)),$$

$$\hat{g}_k(y_k(\beta(t))) = g(x_k(\beta(t)) + x_k^*(\beta(t))) - g(x_k^*(\beta(t))).$$

With these notations, the RNNs model with DEGPCD dynamics can be written as

$$y_j'(t) = -a_j(t)y_j(t) + \sum_{k=1}^{m}\left[b_{jk}(t)\hat{f}_k(y_k(t)) + c_{jk}(t)\hat{g}_k(y_k(\beta(t)))\right],\ j=1,\dots,m. \tag{4.1}$$

The auxiliary mappings $\hat{f}_j(\cdot)$ preserve the structural characteristics of the original activations $f_j(\cdot)$. In particular, they satisfy

$$\hat{f}_j(0) = 0, \qquad |\hat{f}_j(v_2) - \hat{f}_j(v_1)|\ \leq\ \mathcal{L}_k^f\,|v_2 - v_1|, \qquad j = 1,\dots,m,$$

for all $v_1, v_2 \in \mathbb{R}^+$.

In a similar manner, the auxiliary mappings $\hat{g}_j(\cdot)$ preserve the structural properties of the original activation functions $g_j(\cdot)$. Specifically, they satisfy

$$\hat{g}_j(0) = 0, \qquad |\hat{g}_j(v_2) - \hat{g}_j(v_1)|\ \leq\ \mathcal{L}_k^g\,|v_2 - v_1|, \qquad j = 1,\dots,m,$$

for every pair $v_1, v_2 \in \mathbb{R}^+$.

We now establish the notation to be used in the subsequent sections:

$$\|x(t)\| = \max_{1\leq j\leq m}|x_j(t)|, \qquad a_* = \min_{1\leq j\leq m}\ \inf_{t\in\mathbb{R}^+}|a_j(t)|, \qquad \hat{a}_j = \sup_{t\in\mathbb{R}^+}|a_j(t)|,$$

$$|\hat{b}_{jk}| = \max_{1\leq j\leq m}\ \sup_{t\in\mathbb{R}^+}|b_{jk}(t)|, \qquad |\hat{c}_{jk}| = \max_{1\leq j\leq m}\ \sup_{t\in\mathbb{R}^+}|c_{jk}(t)|,$$

In order to analyze the stability characteristics of the RNNs described by the DEGPCD framework (2.1), we now impose the following assumption:

($\mathcal{S}$) **Stability Condition.** Suppose that the subsequent requirement holds

$$\rho := a_* - \sum_{k=1}^{m} \Big(\mathcal{L}_k^f |\hat{b}_{jk}| + \mu \mathcal{L}_k^g |\hat{c}_{jk}| \Big) > 0,$$

where

$$\mu = \max_{1\le j\le m} \Big[e^{\hat{a}_j\theta} / \Big(1 - e^{\hat{a}_j\theta}\theta \Big(\sum_{k=1}^{m} \Big(\mathcal{L}_k^f |\hat{b}_{jk}|\, e^{\lambda\theta} + \mathcal{L}_k^g |\hat{c}_{jk}| \Big)\Big)\Big)\Big],$$

with

$$\lambda = \max_{1\le j\le m} \Big[- a_* + \sum_{k=1}^{m} \mathcal{L}_k^f |\hat{b}_{jk}| + \sum_{k=1}^{m} e^{\theta} \mathcal{L}_k^g |\hat{c}_{jk}| \Big],$$

and

$$\max_{1\le j\le m} \Big[e^{\hat{a}_j\theta}\theta \Big(\sum_{k=1}^{m} \mathcal{L}_k^f |\hat{b}_{jk}|\, e^{\lambda\theta} + \sum_{k=1}^{m} \mathcal{L}_k^g |\hat{c}_{jk}| \Big)\Big] := \upsilon < 1.$$

Remark 1. Condition ($\mathcal{S}$) summarizes the balance among leakage, instantaneous couplings, and delayed effects in DEGPCD dynamics. The parameter ρ acts as an effective decay rate, ensuring dissipation outweighs both continuous and sampled delayed excitation, while μ and υ describe how the maximal slab length θ and exponential coefficient weights influence stability. These factors are crucial because variable delay intervals require controlling worst-case growth across irregular segments.

The condition is sharp: when $\rho \le 0$ or $\upsilon \ge 1$, bounded systems may exhibit nonconvergent or divergent behavior. Unlike classical delay-dependent criteria for Hopfield or Cohen–Grossberg models, ($\mathcal{S}$) unifies instantaneous and piecewise-constant delayed contributions into a single verifiable inequality. Consequently, it offers both the theoretical foundation for global exponential stability and a practical tool for assessing robustness in recurrent networks influenced by hybrid sampling.

Remark 2. Although assumptions ($\mathcal{E}$) and ($\mathcal{S}$) involve supremum operations over sampling intervals and the auxiliary parameter λ, their verification can be carried out in a constructive manner. In practice, the coefficients $a_j(t)$, $b_{jk}(t)$, and $c_{jk}(t)$ are typically bounded and piecewise continuous, so that the required suprema over each interval $[t_k, t_{k+1})$ can be evaluated either analytically or numerically. The parameter λ is defined explicitly in terms of these bounds and the Lipschitz constants of the activation functions, and therefore can be computed once the system parameters and the maximal sampling length θ are specified.

From a computational viewpoint, one may first determine uniform upper bounds for the coefficients on each sampling interval, then evaluate λ and the corresponding constants appearing in assumptions ($\mathcal{E}$) and ($\mathcal{S}$). Since these conditions are sufficient rather than necessary, a certain level of conservativeness is expected. However, this conservativeness ensures robustness with respect to variations in sampling intervals and parameter uncertainties, which is desirable in sampled-data and hybrid neural systems. The numerical example in Section 4 illustrates that the proposed conditions remain verifiable and effective for representative parameter choices.

4.1 Global exponential stability of the DEGPCD solutions

We start by introducing a definition that will serve as a key ingredient in establishing the stability properties of solutions to the RNNs governed by the DEGPCD system (2.1).

Definition 2. Let $x^*(t)$ denote a solution of the RNNs model described by the DEGPCD system (2.1). We call $x^*(t)$ *globally exponentially attractive* provided that there exist constants $\alpha > 0$ and $\lambda > 0$ such that, for any solution $x(t)$ of (2.1), the estimate

$$\|x(t) - x^*(t)\| \ \leq \ \alpha \, \|x(\kappa) - x^*(\kappa)\| \, e^{-\lambda(t-\kappa)}, \qquad t \geq \kappa,$$

is valid for every initial instant $\kappa \geq 0$.

The next lemma establishes an auxiliary estimate that serves as a key ingredient in demonstrating the global exponential stability of the DEGPCD neural network model (2.1). It ensures the existence of globally exponentially stable solutions and constitutes a cornerstone for applying the Lyapunov approach.

Lemma 3. *Assume that the condition*

$$\max_{1\leq j\leq m} \left\{ e^{\hat{a}_j\theta} \Big(\sum_{k=1}^{m} \mathcal{L}_k^f |\hat{b}_{jk}| \, e^{\lambda\theta} + \sum_{k=1}^{m} \mathcal{L}_k^g |\hat{c}_{jk}| \Big) \theta \right\} := \upsilon < 1, \tag{4.2}$$

is satisfied, where

$$\lambda = \max_{1\leq j\leq m} \Big\{ -a_* + \sum_{k=1}^{m} \mathcal{L}_k^f |\hat{b}_{jk}| + \sum_{k=1}^{m} e^{\theta} \mathcal{L}_k^g |\hat{c}_{jk}| \Big\}, \ \ \theta = \sup_{k\in\mathbb{N}} \{t_{k+1} - t_k\}.$$

Consequently, for any solution $y(t)$ that fulfills the DEGPCD system (4.1), *the bound below is obtained*

$$\big\| y(\beta(t)) \big\| \ \leq \ \mu \, \big\| y(t) \big\| ,$$

where

$$\mu = \max_{1\leq j\leq m} \left\{ \frac{e^{\hat{a}_j\theta}}{1 - e^{\hat{a}_j\theta} \Big(\sum_{k=1}^{m} \big(\mathcal{L}_k^f |\hat{b}_{jk}| \, e^{\lambda\theta} + \mathcal{L}_k^g |\hat{c}_{jk}| \big) \theta \Big)} \right\}.$$

Proof. Let $t \in \mathbb{R}^+$ be arbitrary. Then there exists $k \in \mathbb{N}$ such that $t \in [t_k, t_{k+1})$. On this interval, the deviating argument $\beta(t)$ remains constant, which allows the delayed terms to be estimated uniformly.

For $j = 1, \dots, m$, the solution $y_j(t)$ of system (2.1) admits the integral representation

$$y_j(t) = e^{-\int_{t_k}^t a_j(r)\,dr}\, y_j(t_k) + \int_{t_k}^t e^{-\int_r^t a_j(u)\,du} \left[\sum_{k=1}^m b_{jk}(r)\, \hat{f}_k(y_k(r)) + \sum_{k=1}^m c_{jk}(r)\, \hat{g}_k(y_k(\beta(r))) \right] dr.$$

Taking absolute values and using the Lipschitz continuity of $\hat{f}_k$ and $\hat{g}_k$, we obtain

$$|y_j(t)| \le e^{-\int_{t_k}^t a_*(r)\,dr}\, |y_j(t_k)| + \int_{t_k}^t e^{-\int_r^t a_*(u)\,du} \left[\sum_{k=1}^m \mathcal{L}_k^f |\hat{b}_{jk}(r)|\, |y_j(r)| + \sum_{k=1}^m \mathcal{L}_k^g |\hat{c}_{jk}(r)|\, |y_j(\beta(r))| \right] dr,$$

where

$$a_*(r) = \min_{1\le j\le m} a_j(r), \quad \hat{b}_{jk}(r) = \max_{1\le j\le m} |b_{jk}(r)|, \quad \hat{c}_{jk}(r) = \max_{1\le j\le m} |c_{jk}(r)|.$$

Introduce the transformed variable

$$u_j(t) = e^{\int_{t_k}^t a_*(r)\,dr}\, y_j(t), \qquad j = 1, \dots, m.$$

Then, for all $t \in [t_k, t_{k+1})$, it follows that

$$\max_{1\le j\le m} |u_j(t)| \le \max_{1\le j\le m} |u_j(t_k)| + \int_{t_k}^t \left[\sum_{k=1}^m \mathcal{L}_k^f |\hat{b}_{jk}(r)| \max_{1\le j\le m} |u_j(r)| + e^{\int_{t_k}^r a_*(u)\,du} \sum_{k=1}^m \mathcal{L}_k^g |\hat{c}_{jk}(r)| \max_{1\le j\le m} |u_j(\beta(r))| \right] dr.$$

Since $\beta(r)$ takes values in a finite set of sampling instants on each interval, a Gronwall type inequality adapted to generalized piecewise constant delays can be applied. Consequently,

$$\max_{1\le j\le m} |u_j(t)| \le \max_{1\le j\le m} |u_j(t_k)| \exp\left(\int_{t_k}^t \Phi(r)\,dr \right),$$

where

$$\Phi(r) = \sum_{k=1}^m \mathcal{L}_k^f |\hat{b}_{jk}(r)| + e^{\int_{t_k}^r a_*(u)\,du} \sum_{k=1}^m \mathcal{L}_k^g |\hat{c}_{jk}(r)|.$$

Reverting to $y_j(t)$ and taking the supremum over $t \in [t_k, t_{k+1})$, we obtain

$$\|y(t)\| \le \|y(t_k)\| \exp(\lambda_k \theta), \tag{4.3}$$

with

$$\lambda_k = \sup_{r\in[t_k, t_{k+1}]} \left\{ -a_*(r) + \Phi(r) \right\}.$$

Taking the supremum over k yields

$$\|y(t)\| \leq \|y(t_k)\| \exp(\lambda\theta).$$

Similarly, we have

$$\begin{aligned}|y_j(t_k)| \leq & e^{\int_{t_k}^{t} a_j(r)\,dr} |y_j(t)| \\ & + \int_{t_k}^{t} e^{\int_{t_k}^{r} a_j(u)\,du} \left[\sum_{k=1}^{m} \mathcal{L}_k^f |\hat{b}_{jk}(r)|\, |y_j(r)| + \sum_{k=1}^{m} \mathcal{L}_k^g |\hat{c}_{jk}(r)|\, |y_j(t_k)| \right] dr.\end{aligned}$$

Taking the maximum over $1 \leq j \leq m$ and combining with (4.3), we obtain

$$\begin{aligned}\max_{1\leq j\leq m} |y_j(t_k)| \leq & e^{\int_{t_k}^{t} \hat{a}_j(r)\,dr} \max_{1\leq j\leq m} |y_j(t)| \\ & + \int_{t_k}^{t} e^{\int_{t_k}^{r} \hat{a}_j(u)\,du} \left[\sum_{k=1}^{m} \mathcal{L}_k^f |\hat{b}_{jk}(r)| \max_{1\leq j\leq m} |y_j(t_k)|\, e^{\lambda_k\theta} \right. \\ & \left. + \sum_{k=1}^{m} \mathcal{L}_k^g |\hat{c}_{jk}(r)| \max_{1\leq j\leq m} |y_j(t_k)| \right] dr.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}&\max_{1\leq j\leq m} |y_j(t_k)| \leq e^{\int_{t_k}^{t_{k+1}} \hat{a}_j(r)\,dr} \max_{1\leq j\leq m} |y_j(t)| \\ &+ \left\{ \int_{t_k}^{t_{k+1}} e^{\int_{t_k}^{r} \hat{a}_j(u)\,du} \left[\sum_{k=1}^{m} \mathcal{L}_k^f |\hat{b}_{jk}(r)|\, e^{\lambda_k\theta} + \sum_{k=1}^{m} \mathcal{L}_k^g |\hat{c}_{jk}(r)| \right] dr \right\} \max_{1\leq j\leq m} |y_j(t_k)|.\end{aligned}$$

As a direct implication of condition (4.2), one obtains the estimate

$$\|y(\beta(t))\| \;\leq\; \mu\, \|y(t)\|,$$

where

$$\mu := \max_{1\leq j\leq m} \left\{ e^{\hat{a}_j\theta} \Big/ \Big(1 - e^{\hat{a}_j\theta} \Big(\sum_{k=1}^{m} \big(\mathcal{L}_k^f |\hat{b}_{jk}|\, e^{\lambda\theta} + \mathcal{L}_k^g |\hat{c}_{jk}| \big) \theta \Big) \Big) \right\}.$$

□

Remark 3. The constant μ in Lemma 3 depends monotonically on the Lipschitz bounds $\mathcal{L}_k^f$ and $\mathcal{L}_k^g$. Larger values of these constants increase the coupling contribution in the denominator and hence enlarge μ. Its sensitivity is mainly determined by the margin in condition (4.2): when υ is sufficiently below 1, μ varies moderately, whereas υ close to 1 leads to rapid growth of μ, which is typical for sufficient conditions derived from Gronwall-type estimates.

For practical verification, $\mathcal{L}_k^f$ and $\mathcal{L}_k^g$ can be chosen as global Lipschitz bounds of the activation functions, obtained from derivative bounds or slope estimates. Once uniform bounds of $a_j(t)$, $b_{jk}(t)$, and $c_{jk}(t)$ on sampling intervals are determined, one computes θ, evaluates λ, and then directly calculates υ and μ. This provides a constructive procedure to verify the stability condition.

Before presenting the main results, we outline the analytical strategy. Periodicity is established via coincidence degree theory by reformulating the boundary value problem as an operator equation, with a priori bounds derived from the hold structure of the generalized piecewise constant delay. Exponential stability is then obtained through Lyapunov functionals tailored to the retarded segments, yielding explicit decay rates. This approach separates existence and stability while maintaining a unified analytical framework.

Theorem 3. *Suppose that assumptions* $(\mathcal{E})$, $(\mathcal{L})$, *and* $(\mathcal{S})$ *hold. Then the RNNs model with the DEGPCD system* (2.1) *admits a solution which is globally exponentially stable.*

Proof. Define the maximum norm

$$\|y(t)\| := \max_{1\le j\le m} |y_j(t)|$$

and the Lyapunov functional

$$V(y(t)) := \frac{1}{2}\|y(t)\|^2.$$

Step 1. Justification of the nonsmooth Lyapunov functional. The mapping $y \mapsto \|y\|$ is convex and globally Lipschitz in $\mathbb{R}^m$, hence V is locally Lipschitz. Therefore, along any absolutely continuous trajectory $y(t)$, the upper right Dini derivative

$$D^+V(y(t)) := \limsup_{h\to 0^+} \frac{V(y(t+h)) - V(y(t))}{h}$$

exists for all t. Let

$$\mathcal{I}(t) := \big\{ j \in \{1,\dots,m\} : \ |y_j(t)| = \|y(t)\| \big\}$$

be the active index set. A standard property of the maximum norm yields

$$D^+\|y(t)\| \le \max_{j\in\mathcal{I}(t)} \operatorname{sgn}(y_j(t))\, y_j'(t) \quad \text{for all } t \ne t_k,$$

and consequently,

$$D^+V(y(t)) = \|y(t)\|\, D^+\|y(t)\| \le \|y(t)\| \max_{j\in\mathcal{I}(t)} \operatorname{sgn}(y_j(t))\, y_j'(t), \quad t \ne t_k.$$

Hence it is valid to estimate the Lyapunov variation using the componentwise dynamics, even though V is not everywhere differentiable.

Step 2. Estimating the Dini derivative along solutions. For $t \ne t_k$, the jth component of system (4.1) satisfies

$$y_j'(t) = -a_j(t)y_j(t) + \sum_{k=1}^{m} b_{jk}(t)\, \hat{f}_k(y_k(t)) + \sum_{k=1}^{m} c_{jk}(t)\, \hat{g}_k(y_k(\beta(t))).$$

Using the Lipschitz properties of $\hat{f}_k$ and $\hat{g}_k$, and taking absolute values, we obtain for any $j \in \mathcal{I}(t)$,

$$\operatorname{sgn}(y_j(t))\, y_j'(t) \le -a_*(t)\|y(t)\| + \sum_{k=1}^{m} \mathcal{L}_k^f |\hat{b}_{jk}(t)|\, \|y(t)\| + \sum_{k=1}^{m} \mathcal{L}_k^g |\hat{c}_{jk}(t)|\, \|y(\beta(t))\|.$$

Taking the maximum over $j \in \mathcal{I}(t)$ and multiplying by $\|y(t)\|$, we arrive at

$$\begin{aligned} D^+V(y(t)) \le &- a_*(t)\|y(t)\|^2 + \sum_{k=1}^{m} \mathcal{L}_k^f |\hat{b}_{jk}(t)|\, \|y(t)\|^2 \\ &+ \sum_{k=1}^{m} \mathcal{L}_k^g |\hat{c}_{jk}(t)|\, \|y(t)\|\, \|y(\beta(t))\|, \quad t \ne t_k. \end{aligned}$$

Step 3. Using Lemma 3 to eliminate the delayed term. By Lemma 3, we have $\|y(\beta(t))\| \le \mu \|y(t)\|$. Substituting this bound yields

$$D^+V(y(t)) \le -\Big(a_*(t) - \sum_{k=1}^{m} \mathcal{L}_k^f |\hat{b}_{jk}(t)| - \mu \sum_{k=1}^{m} \mathcal{L}_k^g |\hat{c}_{jk}(t)|\Big) \|y(t)\|^2, \quad t \ne t_k.$$

Assumption $(\mathcal{S})$ implies that there exists $\rho > 0$ such that

$$a_*(t) - \sum_{k=1}^{m} \mathcal{L}_k^f |\hat{b}_{jk}(t)| - \mu \sum_{k=1}^{m} \mathcal{L}_k^g |\hat{c}_{jk}(t)| \ge 2\rho \quad \text{for all } t,$$

and thus

$$D^+V(y(t)) \le -2\rho\, \|y(t)\|^2 = -4\rho\, V(y(t)), \quad t \ne t_k.$$

Equivalently,

$$D^+V(y(t)) \le -2\rho\, V(y(t)), \quad t \ne t_k,$$

after rescaling ρ if desired. For simplicity, we keep the notation $\rho > 0$ so that

$$D^+V(y(t)) \le -2\rho\, V(y(t)), \quad t \ne t_k.$$

Step 4. Exponential decay. Consider $W(t) := e^{2\rho t} V(y(t))$. Using the Dini derivative product rule for absolutely continuous functions,

$$D^+W(t) = e^{2\rho t} \left(2\rho V(y(t)) + D^+V(y(t))\right) \le 0, \quad t \ne t_k.$$

Hence $W(t)$ is nonincreasing on each interval (t_k, t_{k+1}), and by continuity of $y(t)$ and $V(y(t))$, we conclude that

$$V(y(t)) \le V(y(t_0)) e^{-2\rho(t-t_0)} \quad \text{for all } t \ge t_0.$$

Therefore,

$$\|y(t)\|^2 \le 2V(y(t)) \le 2V(y(t_0)) e^{-2\rho(t-t_0)} \le \|y(t_0)\|^2 e^{-2\rho(t-t_0)},$$

which implies

$$\|y(t)\| \le \|y(t_0)\| e^{-\rho(t-t_0)}.$$

This proves that the solution is globally exponentially stable. □

Remark 4. Theorem 3 shows that the deviation system is globally exponentially stable whenever $\rho > 0$, where ρ reflects the net decay after subtracting delayed and instantaneous couplings from the leakage. This condition condenses the hybrid dynamics into a single computable inequality, and the estimate $\|y(t)\| \leq \sqrt{m}\|y(t_0)\|e^{-\rho(t-t_0)}$ holds uniformly for all initial data. The requirement $\rho > 0$ is sharp, since $\rho = 0$ may yield marginal or oscillatory behavior. The theorem extends classical delay-system stability results to DEGPCD settings with discontinuities and offers a practical guideline: leakage must exceed total effective gain, including slab-wise delayed amplification.

4.2 Exponential attractiveness of the periodic orbit

This part of the analysis formulates criteria ensuring that the periodic trajectory of the RNNs model governed by generalized piecewise constant delay attracts all other solutions at a uniform exponential rate.

Theorem 4. *Suppose that assumptions* $(\mathcal{E})$, $(\mathcal{L})$, $(\mathcal{P})$, *and* $(\mathcal{S})$ *are satisfied. Then every solution of the RNNs governed by the DEGPCD system* (2.1) *converges exponentially to the unique* ω*-periodic orbit.*

Proof. By Theorem 2, the DEGPCD-based RNNs model (2.1) admits an ω-periodic solution; we denote this solution by

$$x^*(t) = \big(x_1^*(t), \dots, x_m^*(t)\big)^T.$$

Take any solution $x(t) = \big(x_1(t), \dots, x_m(t)\big)^T$ of the DEGPCD system (2.1), and define the deviation function as

$$z(t) = x(t) - x^*(t) = \big(x_1(t) - x_1^*(t), \dots, x_m(t) - x_m^*(t)\big)^T.$$

Substituting into the original system, we derive the following equivalent representation:

$$z_j'(t) = -a_j(t)z_j(t) + \sum_{k=1}^{m}\Big(b_{jk}(t)\,\tilde{f}_j(z_j(t)) + c_{jk}(t)\,\tilde{g}_j(z_j(\beta(t)))\Big), \qquad j = 1, \dots, m, \tag{4.4}$$

where the perturbed nonlinearities are defined by

$$\tilde{f}_j(z_j(t)) = f_k\big(z_j(t) + x_j^*(t)\big) - f_k(x_k^*(t)),$$

and

$$\tilde{g}_j(z_j(\beta(t))) = g_k\big(z_j(\beta(t)) + x_j^*(\beta(t))\big) - g_k(x_k^*(\beta(t))).$$

Employing the variation-of-constants formula, the solution of (4.4) for each component $j = 1, \dots, m$ can be reformulated as

$$\begin{aligned} z_j(t) =& e^{-\int_{t_0}^{t} a_j(r)\,dr}\, z_j(t_0) \\ &+ \int_{t_0}^{t} e^{-\int_{r}^{t} a_j(u)\,du}\left[\sum_{k=1}^{m} b_{jk}(r)\,\tilde{f}_k(z_k(r)) \sum_{k=1}^{m} c_{jk}(r)\,\tilde{g}_k(z_k(\beta(r)))\right] dr. \end{aligned}$$

Moreover, assumption $(\mathcal{L})$ ensures the estimate

$$|z_j(t)| \le e^{-\int_{t_0}^t a_j(r)\,dr}\,|z_j(t_0)| + \int_{t_0}^t e^{-\int_r^t a_j(u)\,du}\left[\sum_{k=1}^m \mathcal{L}_k^f |b_{jk}(r)|\,|z_k(r)| + \sum_{k=1}^m \mathcal{L}_k^g |c_{jk}(r)|\,|z_k(\beta(r))|\right] dr.$$

Proceeding in analogy with the arguments used in Lemma 3 and Theorem 3, we derive the estimate

$$\max_{1\le j\le m} |x_j(t) - x_j^*(t)| \le \sqrt{m} \max_{1\le j\le m} |x_j(t_0) - x_j^*(t_0)| e^{-\rho(t-t_0)},$$

which demonstrates exponential convergence.
Hence, under assumption $(\mathcal{S})$, the periodic solution $x^*(t)$ exhibits global exponential attractivity. This ensures that all trajectories of the RNNs model with DEGPCD system (2.1) asymptotically approach this unique periodic solution at an exponential rate. The proof is thus concluded. □

Remark 5. Theorem 4 shows that every trajectory converges exponentially to the unique periodic orbit, unifying existence and stability in a single result. This is the strongest conclusion: a periodic solution not only exists but also attracts the entire phase space. The proof integrates Theorem 2 with Lyapunov-based arguments, ensuring consistency between topological and stability analyses. The attractivity property guarantees that DEGPCD-driven recurrent neural networks reliably synchronize to a stable oscillatory regime, independent of initial conditions—an essential feature in rhythmic processing, associative memory, and biological oscillations. It also secures asymptotic uniqueness, since exponential attraction renders any other potential periodic orbit irrelevant. Thus, Theorem 4 delivers both theoretical closure and practical predictability.

5 Illustrative example and numerical experiments

This section presents a representative example with numerical simulations to demonstrate the effectiveness of the proposed approach and theoretical results.

Example 1. Consider a 3-dimensional recurrent neural network subject to a generalized piecewise constant delay specified by $(\omega, p) = (3, 4)$, yielding $\theta = \frac{\omega}{p} = 0.75$. The time-varying coefficients are given by

$$a_j(t) = 0.64 + 0.10\cos\left(\tfrac{2\pi}{3}t\right), \qquad j = 1, 2, 3,$$

and the nonlinear responses are defined as

$$f_k(u) = \alpha\tanh(u), \qquad g_k(u) = \alpha\tanh(u), \qquad \alpha = 0.40,$$

so that $\mathcal{L}_k^f = \mathcal{L}_k^g = \alpha = 0.40$. Furthermore, the weighted interconnections and the external forcing term take the form

$$B(t) = 0.65\cos\left(\tfrac{2\pi}{3}t\right) M_b, C(t) = 0.01\sin\left(\tfrac{2\pi}{3}t\right) M_c, I(t) = 2 + 0.05\cos\left(\tfrac{2\pi}{3}t\right) v,$$

with

$$M_b = \begin{pmatrix} 0.62 & 0.55 & 0.53 \\ 0.40 & 0.71 & 0.45 \\ 0.58 & 0.36 & 0.72 \end{pmatrix}, \quad M_c = \begin{pmatrix} 0.52 & 0.47 & 0.61 \\ 0.69 & 0.28 & 0.33 \\ 0.41 & 0.57 & 0.42 \end{pmatrix}, \quad v = \begin{pmatrix} 1.2 \\ 0.7 \\ 0.9 \end{pmatrix}.$$

By direct computation, we obtain $a_* = 0.54$, $\hat{a} = 0.74$, $\theta = 0.75$. The corresponding bounds are

$$\max_{1\le k\le 3} \sum_{k=1}^{3} \mathcal{L}_k^f |\hat{b}_{jk}| = 0.442 \quad \max_{1\le k\le 3} \sum_{k=1}^{3} \mathcal{L}_k^g |\hat{c}_{jk}| = 0.0064.$$

From this, we infer that:

(a)

$$\max_{1\le k\le 3} \left\{ \max_{1\le k\le i(\zeta)} \int_{t_k}^{t_{k+1}} e^{\int_{t_k}^{s} a_j(u)du} \Big(\sum_{k=1}^{3} \mathcal{L}_k^f |b_{jk}(s)| + \mathcal{L}_k^g |c_{jk}(s)| \Big) ds \right\}$$
$$\approx 0.9246 < 1.$$

Hence, condition $(\mathcal{E})$ is satisfied.

(b) All coefficients are 3-periodic and the sequence $\{t_l\}$ satisfies the $(3,4)$-property. Therefore, condition $(\mathcal{P})$ is fulfilled.

(c)

$$v = \max_{1\le k\le 3} \left\{ e^{\hat{a}_j\theta}\theta \Big(\sum_{k=1}^{3} \mathcal{L}_k^f |\hat{b}_{jk}| e^{\lambda\theta} + \sum_{k=1}^{3} \mathcal{L}_k^g |\hat{c}_{jk}| \Big) \right\} \approx 0.5504 < 1,$$

$$\lambda = \max_{1\le k\le 3} \left\{ -a_* + \sum_{k=1}^{3} \mathcal{L}_k^f |\hat{b}_{jk}| + \sum_{k=1}^{3} e^{\theta} \mathcal{L}_k^g |\hat{c}_{jk}| \right\} \approx -0.0845 < 0,$$

$$\mu = \max_{1\le k\le 3} \left\{ \frac{e^{\hat{a}_j\theta}}{1 - e^{\hat{a}_j\theta}} \Big(\sum_{k=1}^{3} (\mathcal{L}_k^f |\hat{b}_{jk}|\, e^{\lambda\theta} + \mathcal{L}_k^g |\hat{c}_{jk}|)\theta \Big) \right\} \approx 3.874,$$

$$\rho := a_* - \sum_{k=1}^{m} \Big(\mathcal{L}_k^f |\hat{b}_{jk}| + \mu \mathcal{L}_k^g |\hat{c}_{jk}| \Big) \approx 0.0732 > 0.$$

Consequently, condition $(\mathcal{S})$ holds.

We note that all requirements $(\mathcal{E})$, $(\mathcal{L})$, $(\mathcal{P})$, and $(\mathcal{S})$ in Theorem 4 are fulfilled for the RNNs governed by the DEGPCD system in Example 1. Hence, every trajectory of the RNNs with DEGPCD system in Example 1 converges exponentially to the unique 3-periodic solution. Numerical simulations illustrating this behavior are presented in Figures 1–3.

We impose the condition

$$\varrho := \max_{1\le j\le m} \left\{ c_G \sup_{t\in\mathbb{R}^+} \int_t^{t+\omega} \Big(\sum_{k=1}^{m} \mathcal{L}_k^f \,|b_{jk}(s)| + \mathcal{L}_k^g \,|c_{jk}(s)| \Big)\, ds \right\} < 1,$$

which, according to the framework of [9], guarantees the existence of a unique solution for the RNNs with the DEGPCD system in Example 1. A direct calculation, however, gives $\varrho \approx 1.0035 > 1$, showing that the above criterion is not satisfied in this case. Therefore, the classical approach cannot be applied to establish the uniqueness of a 3-periodic solution. Nevertheless, by exploiting the stability properties proved in this work, we can still obtain such a periodic solution. This highlights that our results broaden the applicability of the theory to scenarios where previous methods are ineffective.

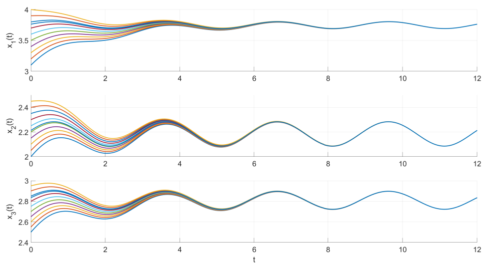

Figure 1. Ten distinct trajectories demonstrating exponential convergence toward the 3-periodic orbit of the RNNs model described by the DEGPCD framework in Example 1.

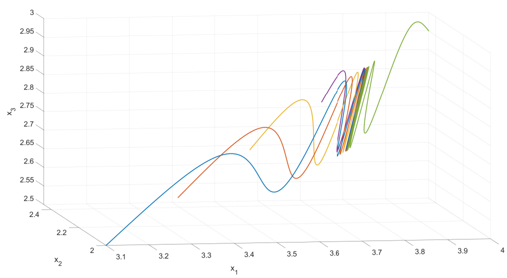

Figure 2. State components $x(t)$ depicted in the phase plane for RNNs corresponding to the DEGPCD system in Example 1.

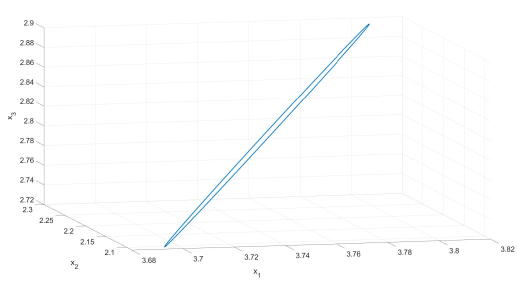

Figure 3. State evolution of $x(t)$ shown in the phase plane for the RNNs governed by the DEGPCD system in Example 1, with initial point $x(0) = (3.7632, 2.21322, 2.83486)^T$.

The numerical simulations presented in this section provide concrete evidence supporting the theoretical results on periodicity and global exponential convergence. Figure 1 depicts the time evolution of the neural network states from a given initial condition. After a transient phase, all state variables settle into a bounded oscillatory pattern, which numerically confirms the existence of a 3-periodic solution as predicted by the coincidence degree analysis.

To further examine the influence of initial conditions, Figure 2 illustrates the state trajectories corresponding to different initial values. Despite the diversity of initial states, all trajectories converge to the same 3-periodic orbit. This behavior demonstrates that the periodic solution is unique and globally attractive, and it is consistent with the Lyapunov-based stability framework developed in the theoretical analysis.

Figure 3 focuses on the convergence rate toward the periodic regime. The exponential decay of the transient components is clearly observed, indicating that convergence occurs at a uniform exponential rate. This numerical result supports the global exponential stability established in Theorem 3 and confirms that the derived Lyapunov functional and associated differential inequality provide realistic estimates of the convergence behavior under generalized piecewise constant delay.

Overall, the numerical experiments verify that the proposed criteria are effective in guaranteeing the existence, uniqueness, and global exponential convergence of periodic solutions. Moreover, they illustrate that the theoretical conditions yield robust convergence behavior even in the presence of generalized piecewise constant delay effects.

Although only representative numerical examples are presented, the proposed criteria are formulated in a general and verifiable manner and are applicable to a broad class of hybrid neural systems. In particular, the results may be relevant to engineering applications involving sampled-data neural controllers and digital neural implementations, where piecewise constant delay effects naturally arise.

6 Conclusions

This paper established the existence of periodic solutions and their global exponential stability for recurrent neural networks with generalized piecewise constant delay. Periodicity was obtained through a coincidence degree framework adapted to the sample-and-hold structure, while stability was derived via a Lyapunov functional combined with interval-wise growth estimates. The resulting criteria are explicit and verifiable in terms of coefficient bounds, Lipschitz constants, and the maximal sampling length.

The analysis integrates degree-theoretic arguments with stability estimates suited to nonuniform sampling intervals. In particular, the inequality linking the delayed state to the current state plays a central role in bridging the hybrid delay mechanism with the network dissipation structure.

Extending the present framework to fully discrete-time neural networks with irregular update instants would require substantial additional work. Such an extension involves nonautonomous difference systems with variable step sizes, discrete Lyapunov techniques, and the reconstruction of an appropriate Fredholm operator in a sequence space to apply coincidence degree methods. Developing a corresponding periodicity and stability theory for irregular discrete-time models remains an interesting direction for future research.

Acknowledgements

The research was supported by ANID–Fondecyt Regular N°1231256 and DI-UMCE 01-2025-ID. The author expresses deep appreciation to the anonymous referees for their careful evaluation of the original submission and for their constructive observations, which have contributed substantially to enhancing the precision and presentation of the results.

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