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Existence results for nonlinear Fourier problems with variable exponent growth

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Abstract. We study nonlinear problems which use the generalized $\alpha(x)-$Laplacian operator under Fourier boundary conditions with L^1 data. Our research establishes both weak and entropy solutions through the framework of variable exponent Sobolev spaces. Our solution method uses monotone operator theory and appropriate approximation methods to develop a unified approach for dealing with nonlinearities that exhibit variable growth. These findings help expand knowledge about nonlinear Fourier-type problems while demonstrating how entropy formulations enable well-posedness for problems with minimal data requirements.

Keywords: Lebesgue and Sobolev spaces with variable exponent; Fourier boundary conditions; weak solution; entropy solution; nonlinear elliptic problem.

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1 Introduction

The study of nonlinear partial differential equations with variable exponents has garnered significant attention over the past few decades. This growing interest stems from both the challenging mathematical structures that equations present and their wide range of applications in real-world models. In particular, equations involving variable exponent growth naturally arise in the modeling of thermorheological fluids [5], electrorheological materials [23], image restoration problems [11], and in several areas of mathematical physics and economics [18]. For a comprehensive overview of recent developments in this field, we refer the reader to [1,2,6,7,8,9,10,13,15,20,21,22,24,25,26,27,28,29,30] and the references therein.

In this work, we are motivated by nonlinear Fourier boundary value problems of the form:

$$\begin{cases} -\operatorname{div}\Big(\mathcal{B}\left(x, \nabla\zeta - \Theta\left(\zeta\right)\right)\Big) + \mathcal{A}\left(x,\zeta\right) = f\left(x,\zeta\right) - b\left(\zeta\right) & \text{in } \Lambda, \\ \Big(\mathcal{B}\left(x, \nabla\zeta - \Theta\left(\zeta\right)\right)\Big).\eta + \lambda\zeta = g & \text{on } \partial\Lambda, \end{cases} \tag{1.1}$$

with

$$\lambda > 0 \text{ and } \mathcal{B}\left(x,\xi\right) = |\xi|^{\alpha(x)-2}\xi, \quad \forall\xi \in \mathbb{R}^N, \forall x \in \Lambda,$$

where $\Lambda \subseteq \mathbb{R}^N (N \geq 3)$ denotes a bounded open domain with a smooth boundary $\partial\Lambda$, η represents the outward unit normal vector on $\partial\Lambda$, and Θ is a real-valued function defined on $\mathbb{R}$ or $\mathbb{R}^N$, f is a Carathéodory function satisfy the Growth condition, $g \in L^1\left(\partial\Lambda\right)$ and Φ is the Leray-Lions operator.

A central difficulty in studying such problems lies in the multiplicity of weak solutions. While weak solutions exist in generalized Sobolev spaces $W^{1,\alpha(x)}\left(\Lambda\right)$, they do not always correspond to physically meaningful solutions. To address this issue, the notion of entropy solutions was introduced by Bénilan, Boccardo, Gallouët, Gariepy, Pierre, and Vázquez [10]. Entropy solutions refine the concept of weak solutions by incorporating additional stability and selection principles, thereby ensuring uniqueness and physical relevance.

Several contributions in the literature have addressed related problems. For example, Nyanquini and Ouaro [20] investigated Leray–Lions operators under L^1-data and established the existence and uniqueness of weak and entropy solutions. Other works studied the existence of entropy solutions when the source term f belongs to L^1 or is a Radon measure, under Neumann boundary conditions. Their methods relied on monotonicity arguments and a priori estimates, showing convergence of approximating sequences to entropy solutions. The body of literature on the variable exponent problem of $p(x)-$Laplacian operators has a long history, including papers on the entropy solutions of the Dirichlet or Neumann boundary value problems. The current paper is based on some well-known methods, especially monotonicity methods and truncation arguments used in the entropy approach.

The novelty of the present work lies in considering the case where $f = f\left(x,\zeta\right)$ depends explicitly on the solution variable ζ. This dependence forces us to impose additional growth conditions on f, employ refined a priori estimates compared to earlier works, and treat the problem in a variable-exponent environment. Our main objective is to establish the existence of both weak and entropy solutions for problem (1.1). So far, as far as we know, existence results for entropy solutions have not been established in this framework. The linkage of the interior $p(x)-$growth operator with the nonlinear boundary term needs fresh coercivity approximations and a precise adjustment of the functional scenario. Although some arguments are motivated by the existing work, the analysis cannot be directly extended to the previously known result because of the presence of the Fourier boundary condition and the variable exponent growth, which, at the same time, greatly increases the applicability of the theory. Such issues occur in all models of heat transfer with nonlinear boundary flux and in materials that have spatially heterogeneous diffusion characteristics (e.g., electrorheological materials). The variable exponent structure enables the modeling of nonhomogeneous material and the Fourier boundary condition allows the nonlinear exchange to be represented at the boundary.

For our study, we assume that our problem is submit the following assumptions:

(H1): $g \in L^1(\partial\Lambda)$.

(H2): $\Theta : \mathbb{R} \to \mathbb{R}^{\mathrm{N}}$ is Lipschitz function such that $\Theta(0) = 0$ and the Lipschitz constant c where:

$$0 < c \leq \min\Big((\alpha_-/2)^{\frac{1}{\alpha_-}}, (\alpha_-/2)^{\frac{1}{\alpha_+}} \Big).$$

Remark 1. The restriction on the Lipschitz constant c is imposed to guarantee the coercivity of the associated nonlinear operator, ensuring that the principal variable-exponent term dominates the lower-order contribution induced by Θ.

(H3): $\mathcal{B} : \Lambda \times \mathbb{R}^N \to \mathbb{R}^N$ is a function such that $\mathcal{B}(x,\xi) = |\xi|^{\alpha(x)-2}\xi$.

(H4): $b : \mathbb{R} \to \mathbb{R}$ is continuous, surjerctive and non-decreasing function such that $b(0) = 0$.

(H5): $f(.,.) : \Lambda \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e. $x \mapsto f(x,\zeta)$ is continuous for a.e. $x \in \Lambda$ and $x \mapsto f(x,\zeta)$ is measurable for every $\zeta \in \mathbb{R}$. Furthermore, f satisfy the growth condition, i.e. There exists $f_0 \in L^1(\Lambda)$ and $\delta \in L^{m(\cdot)}(\Lambda)$ such that :

$$|f(x,s)| \leq f_0(x) + \delta(x)\,|s|^{r(x)},\ \forall x \in \Lambda,\ \forall s \in \mathbb{R},$$

where $0 \leq r(.) < \alpha(.)$ and $m(.) > \alpha(.)/\big(\alpha(.) - r(.)\big)$.

Remark 2. The growth assumption (H5) is such as to provide the nonlinear term $f(x,s)$ to have subcritical growth with respect to the principal operator in $\alpha(x)$. The $r(x)$ inequality less than $\alpha(x)$ ensures that the reaction term does not prevail over the leading variable-exponent term, which is necessary in the arguments of coercivity and compactness. This $m(x)$ condition is added to prove that $\delta(x)\,|u|^{r(x)}$ is integrable and that the weak formulation is well-posed via the variable-exponent Holder inequality. The assumptions are adequate for the analysis we formulated in this paper; we do not purport them to be optimal.

The paper is organized as follows. In Section 2, we recall the fundamental tools on Lebesgue and Sobolev spaces with variable exponent, along with auxiliary lemmas. Section 3 is devoted to the proof of the existence of weak solutions to problem (1.1). Finally, in Section 4, we establish the existence of entropy solutions under the proposed framework.

2 Notations and basic properties

Since the exponent $\alpha(x)$ appearing in (1.1) depends on the variable x, it is necessary to employ Lebesgue and Sobolev spaces with variable exponents. We therefore impose the following assumptions on the function $\alpha(.)$:

$$\begin{cases} \alpha(.) : \overline{\Lambda} \to \mathbb{R} \quad \text{is continuous,} \\ 1 < \alpha_- \leq \alpha_+ < +\infty, \end{cases}$$

where $\alpha_- := ess \inf_{x \in \Lambda} \alpha(x)$ and $\alpha_+ := ess \sup_{x \in \Lambda} \alpha(x)$.

In the case when $\alpha_+ < +\infty$, we define the Lebesgue space with variable exponent $L^{\alpha(.)}(\Lambda)$ by:

$$L^{\alpha(.)}(\Lambda) = \left\{ \zeta : \Lambda \longrightarrow \mathbb{R} \text{ mesurable } / \ \rho_{\alpha(x)}(\zeta) := \int_\Lambda |\zeta|^{\alpha(x)} dx < +\infty \right\},$$

where $\rho_{\alpha(.)}(\zeta) := \int_\Lambda |\zeta|^{\alpha(x)} dx$ is the convex modular of u.

The space $L^{\alpha(.)}(\Lambda)$ is a separable Banach space endowed with the so-called Luxemburg norm

$$\|\zeta\|_{\alpha(.)} = \inf \left\{ \lambda > 0 / \rho_{\alpha(.)} \left(\zeta/\lambda \right) \leq 1 \right\}.$$

Moreover, if $1 < \alpha_- \leq \alpha_+ < +\infty$, then $L^{\alpha(.)}(\Lambda)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{\alpha'(x)}(\Lambda)$, where

$$\frac{1}{\alpha(x)} + \frac{1}{\alpha'(x)} = 1.$$

For any $u \in L^{\alpha(.)}(\Lambda)$ and $v \in L^{\alpha'(.)}(\Lambda)$, we have the following Hölder-type inequality:

$$\left| \int_\Lambda \zeta \ v dx \right| \leq \left(\frac{1}{\alpha_-} + \frac{1}{(\alpha')_-} \right) \|\zeta\|_{\alpha(.)} \|v\|_{\alpha'(.)}.$$

Next, we introduce the Sobolev space with variable exponent as

$$W^{1,\alpha(.)}(\Lambda) = \left\{ \zeta \in L^{\alpha(.)}(\Lambda) \ / \ |\nabla \zeta| \in L^{\alpha(.)}(\Lambda) \right\}.$$

This space is a Banach space when equipped with the norm:

$$\|\zeta\|_{1,\alpha(.)} = \|\zeta\|_{\alpha(.)} + \|\nabla \zeta\|_{\alpha(.)}.$$

This space $\left(W^{1,\alpha(.)}(\Lambda), \|.\|_{1,\alpha(.)} \right)$ is known to be separable and reflexive, see [14, 16].

In the study of generalized Lebesgue and Sobolev spaces, the modular $\rho_{\alpha(.)}$ in $L^{\alpha(.)}(\Lambda)$ plays a fundamental role. We have the following proposition:

Proposition 1. *[15] If $\zeta_n, \zeta \in L^{\alpha(.)}(\Lambda)$ and $\alpha_+ < +\infty$, then the following properties hold true:*

(i): $\|\zeta\|_{\alpha(.)} > 1 \Rightarrow \|\zeta\|_{\alpha(.)}^{\alpha_-} < \rho_{\alpha(.)}(\zeta) < \|\zeta\|_{\alpha(.)}^{\alpha_+}$,

(ii): $\|\zeta\|_{\alpha(.)} < 1 \Rightarrow \|\zeta\|_{\alpha(.)}^{\alpha_+} < \rho_{\alpha(.)}(\zeta) < \|\zeta\|_{\alpha(.)}^{\alpha_-}$,

(iii): $\|\zeta\|_{\alpha(.)} < 1$ *(resp.* $= 1; > 1$*)* $\Leftrightarrow \rho_{\alpha(.)}(\zeta) < 1$ *(resp.* $= 1; > 1$*),*

(iv): $\|\zeta_n\|_{\alpha(.)} \to 0$ *(resp.* $\to +\infty$*)* $\Leftrightarrow \rho_{\alpha(.)}(\zeta_n) \to 0$ *(resp.* $\to +\infty$*)* ,

(v): $\rho_{\alpha(.)}\left(\zeta/\|\zeta\|_{\alpha(.)}\right) = 1$.

For any measurable function $\zeta : \Lambda \to \mathbb{R}$, we define the notation:

$$\rho_{1,\alpha(.)}(\zeta) = \int_\Lambda |\zeta|^{\alpha(x)}\,dx + \int_\Lambda |\nabla\zeta|^{\alpha(x)}\,dx.$$

Proposition 2. *[26, 27] For every $\zeta \in W^{1,\alpha(.)}(\Lambda)$, the following statements hold:*

(i): $\|\zeta\|_{1,\alpha(.)} > 1 \Rightarrow \|\zeta\|_{1,\alpha(.)}^{\alpha_-} \le \rho_{1,\alpha(.)}(\zeta) \le \|\zeta\|_{1,\alpha(.)}^{\alpha_+}$,

(ii): $\|\zeta\|_{1,\alpha(.)} < 1 \Rightarrow \|\zeta\|_{1,\alpha(.)}^{\alpha_+} \le \rho_{1,\alpha(.)}(\zeta) \le \|\zeta\|_{1,\alpha(.)}^{\alpha_-}$,

(iii): $\|\zeta\|_{1,\alpha(.)} < 1$ *(resp.* $= 1; > 1$*)* $\Leftrightarrow \rho_{1,\alpha(.)}(\zeta) < 1$ *(resp.* $= 1; > 1$*).*

We denote

$$\alpha^\partial(x) := \left(\alpha(x)\right)^\partial := \begin{cases} \dfrac{(N-1)\,\alpha(x)}{N-\alpha(x)}, & \text{if } \alpha(x) < N, \\ \infty, & \text{if } \alpha(x) \ge N. \end{cases}$$

We recall the following compact embedding result.

Proposition 3. *[27] Let $\alpha(.) \in C\left(\overline{\Lambda}\right)$ and $\alpha_- > 1$. If $\beta(.) \in C(\partial\Lambda)$ satisfies*

$$1 \le \beta(x) < \alpha^\partial(x),\ \forall\, x \in \partial\Lambda,$$

then, the embedding

$$W^{1,\alpha(.)}(\Lambda) \hookrightarrow\hookrightarrow L^{\beta(.)}(\partial\Lambda).$$

is compact.
In particular, the embedding $W^{1,\alpha(.)}(\Lambda) \hookrightarrow\hookrightarrow L^{\alpha(.)}(\partial\Lambda)$ is compact.

Proposition 4. *[12] Let Λ be a bounded domain with Lipschitz boundary and $\alpha(.) \in \mathcal{P}\left(\mathbb{R}^N\right)$. Then $C^\infty\left(\overline{\Lambda}\right)$ is dense in $W^{1,\alpha(.)}(\Lambda)$.*

Let us introduce the following notation:
Given two bounded measurable functions $\alpha(.), \beta(.) : \Lambda \to \mathbb{R}$, we use the notation

$$\beta(x) \ll \alpha(x) \text{ to mean that } ess \inf_{x\in\Lambda} \big(\alpha(x) - \beta(x)\big) > 0.$$

In the sequel, we need the following technical lemmas:

Lemma 1. *Let $\xi, \eta \in \mathbb{R}^N$ and let $1 < \alpha < \infty$. We have*

$$\frac{1}{\alpha}|\xi|^\alpha - \frac{1}{\alpha}|\eta|^\alpha \le |\xi|^{\alpha-2}\xi.|\xi - \eta|.$$

Lemma 2. *[16, 25] Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions in Λ. If $(v_n)_{n\in\mathbb{N}}$ converges in measure to v and is uniformly bounded in $L^{\alpha(\cdot)}(\Lambda)$ for some $1 \ll \alpha(.) \in L^\infty(\Lambda)$, then $(v_n)_{n\in\mathbb{N}}$ strongly converges to v in $L^1(\Lambda)$.*

Lemma 3. *[8] Let $v \in L^{\alpha(\cdot)}\Lambda$ and $(v_n)_{n\in\mathbb{N}}$ be a sequence in $L^{\alpha(\cdot)}(\Lambda)$ with $\|v_n\|_{\alpha(.)} < +\infty$ for $1 < \alpha(.) < +\infty$.*
If $(v_n)_{n\in\mathbb{N}}$ converges to v a.e in Λ then $(v_n)_{n\in\mathbb{N}}$ weakly converges to v in $L^{\alpha(\cdot)}(\Lambda)$.

Lemma 4. *Let $(\zeta_n)_{n\in\mathbb{N}}$ be a sequence in $W^{1,\alpha(\cdot)}(\Lambda)$ such that $(\zeta_n)_n \rightharpoonup \zeta$ weakly in $W^{1,\alpha(\cdot)}(\Lambda)$.*
Assuming that the function $\mathcal{B}(x,\xi) = |\xi|^{\alpha(x)-2}\xi$, $\forall \xi \in \mathbb{R}^N$, $\forall x \in \Lambda$ satisfy:

$$\lim_{n\to\infty}\int_\Lambda \big[\mathcal{B}(x,\zeta_n) - \mathcal{B}(x,\zeta)\big](\zeta_n - \zeta)\ dx$$
$$+ \lim_{n\to\infty}\int_\Lambda \Big[\mathcal{B}\big(x, \nabla\zeta_n - \Theta(\zeta_n)\big) - \mathcal{B}\big(x, \nabla\zeta - \Theta(\zeta_n)\big)\Big](\nabla\zeta_n - \nabla\zeta)\,dx = 0.$$

Then, $(\zeta_n)_n \longrightarrow \zeta$ strongly in $W^{1,\alpha(\cdot)}(\Lambda)$, for a sub-sequence.

Lemma 5. *[17] If $a \ge 0$, $b \ge 0$ and $1 \le \alpha_- \le \alpha_+ < +\infty$, then, for $x \in \Lambda$ we have*

$$(a+b)^{p(x)} \le 2^{\alpha_+-1}\Big(a^{\alpha(x)} + b^{\alpha(x)}\Big).$$

3 Existence of weak solutions

In this section, we prove the existence of weak solutions of the problem (1.1).

Definition 1. A weak solution of problem (1.1) is a measurable function ζ such that

$$\zeta \in W^{1,\alpha(\cdot)}(\Lambda) \cap L^\infty(\partial\Lambda), b(\zeta) \in L^\infty(\Lambda),\ f(.,\zeta) \in L^1(\Lambda\times\mathbb{R}),\ g \in L^1(\partial\Lambda),$$

and

$$\int_\Lambda \mathcal{B}\big(x, \nabla\zeta - \Theta(\zeta)\big)\cdot\nabla\varphi\ dx + \int_\Lambda \mathcal{B}(x,\zeta)\varphi\ dx + \lambda\int_{\partial\Lambda}\zeta\varphi d\sigma$$
$$= \int_\Lambda f(x,\zeta)\varphi\ dx - \int_\Lambda b(\zeta)\varphi\ dx + \int_{\partial\Lambda} g\varphi d\sigma, \quad \forall\varphi \in W^{1,\alpha(\cdot)}(\Lambda).$$

The main result of this part is the following:

Theorem 1. *Assume that (H1)–(H5) hold trues. Then problem* (1.1) *has at least one weak solution.*

The proof proceeds in four steps :

Step 1. The approximate problem

Let us define, for $n \in \mathbb{N}^*$:

$$f_n(x, \zeta_n) = T_n\big(f(x, \zeta_n)\big), \quad g_n = T_n(g).$$

We have

$$\|g_n\|_{L^1(\partial\Lambda)} \leq \|g\|_{L^1(\partial\Lambda)}.$$

We consider the following approximated problem :

$$\begin{cases} -\operatorname{div}\mathcal{B}\big(x, \nabla\zeta_n - \Theta(\zeta_n)\big) + \mathcal{B}(x, \zeta_n) = f_n(x, \zeta_n) - T_n\big(b(\zeta_n)\big) \text{ in } \Lambda, \\ \mathcal{B}\big(x, \nabla\zeta_n - \Theta(\zeta_n)\big)\cdot\eta + \lambda T_n(\zeta_n) = g_n \text{ on } \partial\Lambda. \end{cases} \tag{3.1}$$

The variational formulation associate to problem (3.1) is the following : $\forall v \in W^{1,\alpha(.)}(\Lambda)$

$$\begin{aligned} &\int_\Lambda \mathcal{B}\big(x, \nabla\zeta_n - \Theta(\zeta_n)\big)\nabla v\ dx + \int_\Lambda \Phi(x, \zeta_n)\, v\ dx + \lambda\int_{\partial\Lambda} T_n(\zeta_n)\, v d\sigma \\ &= \int_\Lambda f_n(x, \zeta_n)\, v dx - \int_\Lambda T_n\big(b(\zeta_n)\big)\, v\ dx + \int_{\partial\Lambda} g_n v d\sigma. \end{aligned} \tag{3.2}$$

For the rest, we define the operator B_n as follow : $\forall\ \zeta, v \in W^{1,\alpha(.)}(\Lambda)$

$$\begin{aligned} \langle B_n\zeta, v\rangle = &\int_\Lambda \mathcal{B}\big(x, \nabla\zeta - \Theta(\zeta)\big)\nabla v\ dx + \int_\Lambda \mathcal{B}(x, \zeta)\, v\ dx + \int_\Lambda T_n\big(b(\zeta)\big)\, v\ dx \\ &+ \lambda\int_{\partial\Lambda} T_n(\zeta)\, v\ d\sigma - \int_\Lambda f_n(x, \zeta)\, v\ dx - \int_{\partial\Lambda} g_n v d\sigma. \end{aligned}$$

Proposition 5. *The operator* $B_n : W^{1,\alpha(.)}(\Lambda) \longrightarrow W^{-1,\alpha'(.)}(\Lambda)$ *is bounded, coercive and is of type (M).*

Proof. The proof of Proposition 5 is divided in three steps :
• **The operator B_n is bounded:**
We have :

$$\begin{aligned} |\langle B_n\zeta, v\rangle| \leq &\int_\Lambda \big|\mathcal{B}\big(x, \nabla\zeta - \Theta(\zeta)\big)\big|\ |\nabla v\ dx + \int_\Lambda |\mathcal{B}(x, \zeta)|\ |v|\ dx + \int_\Lambda |T_n(b(\zeta))| \\ &\times |v|\ dx + \lambda\int_{\partial\Lambda} |T_n(\zeta)|\ |v|\ d\sigma + \int_\Lambda |f_n(x, \zeta)|\ |v| dx + \int_{\partial\Lambda} |g_n|\ |v|\ d\sigma. \end{aligned}$$

We consider

$$I_1 = \int_\Lambda \big|\mathcal{B}\big(x, \nabla\zeta - \Theta(\zeta)\big)\big|\ |\nabla v|\ dx + \int_\Lambda |\mathcal{B}(x, \zeta)|\ |v|\ dx.$$

From (H2) and Lemma 5, we have

$$\begin{aligned} I_1 &\leq \int_{\Lambda} \left(|\nabla\zeta| + c\,|\zeta|\right)^{\alpha(x)-1}\ |\nabla v|\ dx + \int_{\Lambda} |\zeta|^{\alpha(x)-1}\ |v|\ dx \\ &\leq \int_{\Lambda} 2^{\alpha_+-2}\left(|\nabla\zeta|^{\alpha(x)-1} + c^{\alpha(x)-1}|\zeta|^{\alpha(x)-1}\right)\ |\nabla v|\ dx + \int_{\Lambda}|\zeta|^{\alpha(x)-1}\ |v|\ dx \\ &\leq 2^{\alpha_+-2}\int_{\Lambda}\left(|\nabla\zeta|^{\alpha(x)-1} + c_0\ |\zeta|^{\alpha(x)-1}\right)\ |\nabla v|\ dx + \int_{\Lambda}|\zeta|^{\alpha(x)-1}\ |v|\ dx, \end{aligned}$$

where c_0 is a positive constant. By Hölder's inequalities, we conclude

$$\begin{aligned} I_1 \leq\ & 2^{\alpha_+-2}\||\nabla\zeta|^{\alpha(.)-1}\|_{L^{\alpha'(\cdot)}(\Lambda)}\|\nabla v\|_{L^{\alpha(\cdot)}(\Lambda)} + 2^{\alpha_+-2}c_0\||\zeta|^{\alpha(.)-1}\|_{L^{\alpha'(\cdot)}(\Lambda)} \\ & \times \|\nabla v\|_{L^{\alpha(\cdot)}(\Lambda)} + \||\zeta|^{\alpha(.)-1}\|_{L^{\alpha'(\cdot)}(\Lambda)}\|v\|_{L^{\alpha(\cdot)}(\Lambda)} \\ & \leq \Big[2^{\alpha_+-2}\||\nabla\zeta|^{\alpha(.)-1}\|_{L^{\alpha'(\cdot)}(\Lambda)} + 2^{\alpha_+-2}c_0\||\zeta|^{\alpha(.)-1}\|_{L^{\alpha'(\cdot)}(\Lambda)} \\ & + \|\,|\zeta|^{\alpha(.)-1}\|_{L^{\alpha'(\cdot)}(\Lambda)}\Big] \times \|v\|_{1,\alpha(.)} \leq c_1\|v\|_{1,\alpha(.)}, \end{aligned}$$

where

$$\begin{aligned} c_1 =\ & 2^{\alpha_+-2}\||\nabla\zeta|^{\alpha(.)-1}\|_{L^{\alpha'(\cdot)}(\Lambda)} + 2^{\alpha_+-2}c_0\||\zeta|^{\alpha(.)-1}\|_{L^{\alpha'(\cdot)}(\Lambda)} \\ & + \|\,|\zeta|^{\alpha(.)-1}\|_{L^{\alpha'(\cdot)}(\Lambda)} > 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |\langle B_n\zeta, v\rangle| \leq\ & c_1\|v\|_{1,\alpha(.)} + \int_{\Lambda}|T_n\left(b\left(\zeta\right)\right)|\ |v|\ dx + \lambda\int_{\partial\Lambda}|T_n\left(\zeta\right)|\ |v|\ d\sigma \\ & + \int_{\Lambda}|T_n\left(f\left(x,\zeta\right)\right)|\ |v|dx + \int_{\partial\Lambda}|T_n\left(g\right)|\ |v|\ d\sigma. \end{aligned}$$

Then,

$$|\langle B_n\zeta, v\rangle| \leq c_1\|v\|_{1,\alpha(.)} + 2n\int_{\Lambda}|v|\ dx + n\left(1+\lambda\right)\int_{\partial\Lambda}|v|\ d\sigma.$$

As $W^{1,\alpha(.)}\left(\Lambda\right) \hookrightarrow L^1\left(\partial\Lambda\right)$, then $|\langle B_n\zeta, v\rangle| \leq c_2\|v\|_{1,\alpha(.)}$ Hence, the operator B_n is bounded.

• The operator B_n is coercive : We have

$$\begin{aligned} \langle B_n\zeta, u\rangle =\ & \int_{\Lambda}\mathcal{B}\left(\nabla\zeta - \Theta\left(\zeta\right)\right)\nabla\zeta\ dx + \int_{\Lambda}\mathcal{B}\left(\zeta\right)\zeta\ dx + \int_{\Lambda}T_n\left(b\left(\zeta\right)\right)\zeta\ dx \\ & + \lambda\int_{\partial\Lambda}T_n\left(\zeta\right)\zeta\ d\sigma - \int_{\Lambda}f_n\left(x,\zeta\right)\zeta dx - \int_{\partial\Lambda}g_n\zeta d\sigma. \end{aligned}$$

We put

$$I_2 = \int_{\Lambda}T_n\left(b\left(\zeta\right)\right)\zeta\ dx + \lambda\int_{\partial\Lambda}T_n\left(\zeta\right)u\ d\sigma - \int_{\Lambda}f_n\left(x,\zeta\right)\zeta dx - \int_{\partial\Lambda}g_n\zeta d\sigma.$$

From (H4), we conclude that $b(\zeta)$ and ζ have the same sign, then $b(\zeta)\,u$ is positive, then $T_n\big(b(\zeta)\big)\,\zeta \geq 0$, thus, $\int_\Lambda T_n\big(b(\zeta)\big)\,\zeta\; dx \geq 0$, we also have

$$\int_{\partial\Lambda} T_n(\zeta)\;\zeta\; d\sigma \geq 0,$$

we get $I_2 \geq -nk_1\|\zeta\|_{L^{\alpha(.)}(\Lambda)} - nk_2\|\zeta\|_{L^{\alpha(.)}(\partial\Lambda)}$, where k_1 and k_2 are a positive constants. Thus,

$$I_2 \geq -nk_3\|\zeta\|_{L^{\alpha(.)}(\Lambda)}.$$

Since $W^{1,\alpha(.)}(\Lambda) \hookrightarrow L^{\alpha(.)}(\Lambda)$, we deduce that

$$I_2 \geq -nc_4\|\zeta\|_{W^{1,\alpha(.)}(\Lambda)}.$$

Let $I_1 = \int_\Lambda |\nabla\zeta - \Theta(\zeta)|^{\alpha(x)-2}\big(\nabla\zeta - \Theta(\zeta)\big)\nabla\zeta\; dx + \int_\Lambda |\zeta|^{\alpha(x)}\; dx$. On another side, from the Lemma 1, we obtain:

$$I_1 \geq \int_\Lambda \frac{1}{\alpha(x)}|\nabla\zeta - \Theta(\zeta)|^{\alpha(x)}dx - \int_\Lambda \frac{1}{\alpha(x)}|\Theta(\zeta)|^{\alpha(x)}dx + \int_\Lambda |\zeta|^{\alpha(x)}\; dx.$$

From Lemma 5, it results

$$\frac{1}{2^{\alpha_+-1}}|\nabla\zeta|^{\alpha(x)} = \frac{1}{2^{\alpha_+-1}}|\nabla\zeta - \Theta(\zeta) + \Theta(\zeta)|^{\alpha(x)} \leq |\nabla\zeta - \Theta(\zeta)|^{\alpha(x)} + |\Theta(\zeta)|^{\alpha(x)},$$

then,

$$\frac{1}{2^{\alpha_+-1}}|\nabla\zeta|^{\alpha(x)} - |\Theta(\zeta)|^{\alpha(x)} \leq |\nabla\zeta - \Theta(\zeta)|^{\alpha(x)}.$$

The last inequality of I_1 implies that

$$\begin{aligned}
I_1 &\geq \int_\Lambda \frac{1}{\alpha(x)}\left[\frac{1}{2^{\alpha_+-1}}|\nabla\zeta|^{\alpha(x)} - |\Theta(\zeta)|^{\alpha(x)}\right]dx - \int_\Lambda \frac{1}{\alpha(x)}|\Theta(\zeta)|^{\alpha(x)}dx \\
&\quad + \int_\Lambda |\zeta|^{\alpha(x)}dx \\
&\geq \int_\Lambda \frac{1}{\alpha(x)}\frac{1}{2^{\alpha_+-1}}|\nabla\zeta|^{\alpha(x)}dx - \int_\Lambda \frac{2}{\alpha(x)}|\Theta(\zeta)|^{\alpha(x)}dx + \int_\Lambda |\zeta|^{\alpha(x)}dx \\
&\geq \int_\Lambda \frac{1}{\alpha(x)}\frac{1}{2^{\alpha_+-1}}|\nabla\zeta|^{\alpha(x)}dx - \int_\Lambda \frac{2}{\alpha(x)}c^{\alpha(x)}|\zeta|^{\alpha(x)}dx + \int_\Lambda |\zeta|^{\alpha(x)}dx \\
&\geq \int_\Lambda \frac{1}{\alpha(x)}\frac{1}{2^{\alpha_+-1}}|\nabla\zeta|^{\alpha(x)}dx + \int_\Lambda \left(1 - \frac{2}{\alpha(x)}c^{\alpha(x)}\right)|\zeta|^{\alpha(x)}dx \\
&\geq \int_\Lambda \frac{1}{\alpha_+}\frac{1}{2^{\alpha_+-1}}|\nabla\zeta|^{\alpha(x)}dx + \int_\Lambda \left(1 - \frac{2}{\alpha_-}c^{\alpha(x)}\right)|\zeta|^{\alpha(x)}dx.
\end{aligned}$$

Therefore, from the choice of c in (H2), we obtain a positive constant c_2 such that

$$\begin{aligned} I_1 \geq & \frac{1}{\alpha_+} \frac{1}{2^{\alpha_+ -1}} \int_{\Lambda} |\nabla\zeta|^{\alpha(x)} dx + c_2 \int_{\Lambda} |\zeta|^{\alpha(x)} \\ \geq & \min\left\{ \frac{1}{\alpha_+} \frac{1}{2^{\alpha_+ -1}}, c_2 \right\} \left(\int_{\Lambda} |\nabla\zeta|^{\alpha(x)} dx + \int_{\Lambda} |\zeta|^{\alpha(x)} \right) \\ \geq & \min\left\{ \frac{1}{\alpha_+} \frac{1}{2^{\alpha_+ -1}}, c_2 \right\} \rho_{1,\alpha(.)}(\zeta). \end{aligned}$$

The above inequality and the last inequality of I_2 imply that

$$\langle B_n\zeta, \zeta\rangle \geq c_3 \rho_{1,\alpha(.)}(\zeta) - nc_4 \|\zeta\|_{W^{1,\alpha(.)}(\Lambda)} \text{ where } c_3 = \min\left\{ \frac{1}{\alpha_+} \frac{1}{2^{\alpha_+ -1}}, c_2 \right\} > 0.$$

Then, from Proposition 2, it follows that

$$\frac{\langle B_n\zeta, \zeta\rangle}{\|\zeta\|_{W^{1,\alpha(.)}(\Lambda)}} \geq c_3 \frac{\rho_{1,\alpha(.)}(\zeta)}{\|\zeta\|_{W^{1,\alpha(.)}(\Lambda)}} - nc_4 \geq c_3 \|\zeta\|^{\alpha_- -1}_{W^{1,\alpha(.)}(\Lambda)} - nc_4.$$

Since $\alpha_- > 1$, then $\lim\limits_{\|\zeta\|_{W^{1,\alpha(.)}(\Lambda)} \to +\infty} \dfrac{\langle B_n\zeta, \zeta\rangle}{\|\zeta\|_{W^{1,\alpha(.)}(\Lambda)}} = +\infty$, hence B_n is an operator coercive.

• **We show that the operator B_n is of type (M) :**
To this end, let $(\zeta_k)_{k\in\mathbb{N}}$ be a sequence in $W^{1,\alpha(.)}(\Lambda)$ such that

$$\begin{cases} \zeta_k \rightharpoonup \zeta \quad \text{weakly in } W^{1,\alpha(.)}(\Lambda), \\ B_n\zeta_k \rightharpoonup \chi_n \text{ weakly in } W^{-1,\alpha'(.)}(\Lambda), \\ \limsup\limits_{k\to\infty} \langle B_n\zeta_k, \zeta_k\rangle \leq \langle \chi_n, \zeta\rangle. \end{cases} \tag{3.3}$$

We proof that: $B_n\zeta = \chi_n$.
From Proposition 3, we have $W^{1,\alpha(.)}(\Lambda) \hookrightarrow\hookrightarrow L^{\alpha(.)}(\Lambda)$ and $W^{1,\alpha(.)}(\Lambda) \hookrightarrow\hookrightarrow L^1(\partial\Lambda)$, and since $u_k \rightharpoonup u$ weakly in $W^{1,\alpha(.)}(\Lambda)$, then $(\zeta_k)_k$ is bounded in $W^{1,\alpha(.)}(\Lambda)$, we conclude $(\zeta_k)_k \longrightarrow \zeta$ in $L^{\alpha(.)}(\Lambda)$ and $(\zeta_k)_k \longrightarrow \zeta$ in $L^1(\partial\Lambda)$, thus $(\zeta_k)_k \longrightarrow \zeta$ a.e. in Λ and $(\zeta_k)_k \longrightarrow \zeta$ a.e. in $\partial\Lambda$. By the growth condition of Φ (see [22]), we deduce that $\mathcal{B}(x,\zeta)$ and $\mathcal{B}\big(x, \nabla\zeta - \theta(\zeta)\big)$ are bounded in $L^{\alpha'(.)}(\Lambda)$. Therefore, for a sub-sequence denoted again $(\zeta_k)_k$, there exists a functions Ψ and ϕ in $L^{\alpha'(.)}(\Lambda)$ such that

$$\mathcal{B}(x,\zeta_k)_n \rightharpoonup \mathcal{B} \text{ and } \mathcal{B}\big(x, \nabla\zeta_k - \theta(\zeta_k)\big)_n \rightharpoonup \Psi \text{ in } L^{\alpha'(.)}(\Lambda). \tag{3.4}$$

In the same way, by (H5), $(\zeta_k)_k \longrightarrow \zeta$ in $L^{\alpha(.)}(\Lambda)$ and in view of Lebesgue's dominated convergence theorem, we conclude that

$$\big(f_n(x,\zeta_k)\big)_k \longrightarrow f_n(x,\zeta) \quad \text{in } L^{\alpha'(.)}(\Lambda). \tag{3.5}$$

Since $(\zeta_k)_k \rightharpoonup \zeta$ in $L^1(\partial\Lambda)$, $g_n \in L^1(\partial\Lambda)$, $T_n(.)$ in $L^\infty(\partial\Lambda)$, $b(\zeta) \in L^\infty(\Lambda)$, $(\zeta_k)_k \longrightarrow \zeta$ *a.e.* in Λ and in view of Lebesgue dominated convergence theorem, we conclude that

$$(g_n\ \zeta_k)_k \longrightarrow g_n\ \zeta \quad \text{in} \quad L^1(\partial\Lambda), \qquad \big(T_n(\zeta_k)\big)_k \longrightarrow T_n(\zeta) \quad \text{in} \quad L^1(\partial\Lambda),$$
$$T_n\big(b(\zeta_k)\big)_k \longrightarrow T_n\big(b(\zeta)\big) \quad \text{in} \quad L^{\alpha'(.)}(\Lambda).$$

Then,

$$(g_n\ \zeta_k)_k \longrightarrow g_n\ \zeta \quad \text{in} \quad L^{\alpha'(.)}(\partial\Lambda), \tag{3.6}$$
$$\big(T_n(\zeta_k)\big)_k \longrightarrow T_n(\zeta) \quad \text{in} \quad L^{\alpha'(.)}(\partial\Lambda), \tag{3.7}$$
$$T_n\big(b(\zeta_k)\big)_k \longrightarrow T_n\big(b(\zeta)\big) \quad \text{in} \quad L^{\alpha'(.)}(\Lambda). \tag{3.8}$$

Let $v \in W^{1,\alpha(.)}(\Lambda)$, from (3.3) we have

$$\begin{aligned}\langle \chi_n, v\rangle &= \lim_{k\to+\infty}\langle B_n\zeta_k, v\rangle = \lim_{k\to+\infty}\int_\Lambda \mathcal{B}\big(x, \nabla\zeta_k - \Theta(\zeta_k)\big)\nabla v\ dx \\ &+ \lim_{k\to+\infty}\int_\Lambda \mathcal{B}(x,\zeta_k)\, v\ dx + \lim_{k\to+\infty}\int_\Lambda T_n\big(b(\zeta_k)\big)\, v\ dx \\ &+ \lambda \lim_{k\to+\infty}\int_{\partial\Lambda} T_n(\zeta_k)\, v d\sigma - \lim_{k\to+\infty}\int_\Lambda f_n(x,\zeta_k)\, v dx - \lim_{k\to+\infty}\int_{\partial\Lambda} g_n v d\sigma,\end{aligned}$$

from (3.4)–(3.8), we get

$$\begin{aligned}\langle \chi_n, v\rangle = &\int_\Lambda \Psi\nabla v\ dx + \int_\Lambda \mathcal{B}v\ dx + \int_\Lambda T_n\big(b(\zeta)\big)\, v\ dx \\ &+ \lambda\int_{\partial\Lambda} T_n(\zeta)\, v\ d\sigma - \int_\Lambda f_n(x,\zeta)\, v dx - \int_{\partial\Lambda} g_n v d\sigma.\end{aligned}$$

By (3.3), we conclude that

$$\begin{aligned}\limsup_{k\to+\infty}\langle B_n\zeta_k, v\rangle \le &\int_\Lambda \Psi\nabla v\ dx + \int_\Lambda \mathcal{B}v\ dx + \int_\Lambda T_n\big(b(\zeta)\big)\, v\ dx \\ &+ \lambda\int_{\partial\Lambda} T_n(\zeta)\, v\ d\sigma - \int_\Lambda f_n(x,\zeta)\, v dx - \int_{\partial\Lambda} g_n\ v d\sigma.\end{aligned}$$

Thanks to (3.5) and (3.8), we obtain

$$\begin{aligned}&\limsup_{k\to+\infty}\int_\Lambda \Big[\mathcal{B}\big(x, \nabla\zeta_k - \Theta(\zeta_k)\big)\nabla\zeta_k\ dx + \mathcal{B}(x,\zeta_k)\,\zeta_k\ dx\Big] \\ &\quad \le \int_\Lambda \Psi\nabla v\ dx + \int_\Lambda \mathcal{A}v\ dx.\end{aligned} \tag{3.9}$$

On other hand, by the monotony of Φ, we get

$$\Big[\mathcal{B}\big(x, \nabla\zeta_k - \Theta(\zeta_k)\big) - \mathcal{B}\big(x, \nabla\zeta - \Theta(\zeta_k)\big)\Big](\nabla\zeta_k - \nabla\zeta) \ge 0,$$
$$\big[\mathcal{B}(x,\zeta_k) - \mathcal{B}(x,\zeta)\big](\zeta_k - \zeta) \ge 0.$$

Then,

$$\begin{aligned}&\int_{\varLambda}\mathcal{B}\left(x,\nabla\zeta_k-\varTheta\left(\zeta_k\right)\right)\nabla\zeta_k\ dx+\int_{\varLambda}\mathcal{B}\left(x,\zeta_k\right)\zeta_k dx\\ &\geq\int_{\varLambda}\mathcal{B}\left(x,\nabla\zeta_k-\varTheta\left(\zeta_k\right)\right)\nabla\zeta dx+\int_{\varLambda}\mathcal{B}\left(x,\nabla\zeta-\varTheta\left(\zeta_k\right)\right)\left(\nabla\zeta_k-\nabla\zeta\right)dx\\ &\quad+\int_{\varLambda}\mathcal{B}\left(x,\zeta_k\right)\zeta dx+\int_{\varLambda}\mathcal{B}\left(x,\zeta\right)\left(\zeta_k-\zeta\right)dx.\end{aligned}\tag{3.10}$$

In view of Lebesgue's dominated convergence theorem, we have

$$\left(\varTheta\left(\zeta_k\right)\right)_k\longrightarrow\varTheta(\zeta)\quad\text{in}\quad\left(L^{\alpha(.)}(\varLambda)\right)^N,$$

so,
$\mathcal{B}\left(x,\nabla\zeta-\varTheta\left(\zeta_k\right)\right)\longrightarrow\mathcal{B}\left(x,\nabla\zeta-\varTheta\left(\zeta\right)\right)$ in $L^{\alpha'(.)}(\varLambda)$. Then,

$$\lim_{k\to+\infty}\int_{\varLambda}\mathcal{B}\left(x,\nabla\zeta-\varTheta\left(\zeta_k\right)\right)\left(\nabla\zeta_k-\nabla\zeta\right)\ dx=0.\tag{3.11}$$

By (3.4), (3.10) and (3.11), we obtain

$$\begin{aligned}&\liminf_{k\to+\infty}\int_{\varLambda}\left[\mathcal{B}\left(x,\nabla\zeta_k-\varTheta\left(\zeta_k\right)\right)\nabla\zeta_k\ dx+\mathcal{B}\left(x,\zeta_k\right)\zeta_k\right]\ dx\\ &\quad\geq\int_{\varLambda}\varPsi\nabla\zeta\ dx+\int_{\varLambda}\mathcal{B}\zeta\ dx.\end{aligned}\tag{3.12}$$

From (3.9) and (3.12), we deduce that

$$\begin{aligned}&\lim_{k\to+\infty}\int_{\varLambda}\left[\mathcal{B}\left(x,\nabla\zeta_k-\varTheta\left(\zeta_k\right)\right)\nabla\zeta_k\ dx+\mathcal{B}\left(x,\zeta_k\right)\zeta_k\ dx\right]\\ &\quad=\int_{\varLambda}\varPsi\nabla\zeta\ dx+\int_{\varLambda}\mathcal{B}\zeta\ dx.\end{aligned}\tag{3.13}$$

By (3.13), we can prove that

$$\begin{aligned}&\lim_{k\to+\infty}\int_{\varLambda}\left[\mathcal{B}\left(x,\zeta_k\right)-\mathcal{B}\left(x,\zeta\right)\right]\left(\zeta_k-\zeta\right)\ dx\\ &+\lim_{k\to+\infty}\int_{\varLambda}\left[\mathcal{B}\left(x,\nabla\zeta_k-\varTheta\left(\zeta_k\right)\right)-\mathcal{B}\left(x,\nabla\zeta-\varTheta\left(\zeta_k\right)\right)\right]\left(\nabla\zeta_k-\nabla\zeta\right)dx=0.\end{aligned}$$

In view of Lemma 4 , we conclude $(\zeta_k)_k\longrightarrow\zeta$ in $W^{1,\alpha(.)}(\varLambda)$,
then $(\nabla\zeta_k)_k\longrightarrow\nabla\zeta$ in $\left(L^{\alpha(.)}(\varLambda)\right)^N$ and so $(\nabla\zeta_k)_k\longrightarrow\nabla\zeta$ a.e. in $\varLambda$. Then,

$$\mathcal{B}\left(x,\zeta_k\right)_n\longrightarrow\mathcal{B}\left(x,\zeta\right),\mathcal{B}\left(x,\nabla\zeta_k-\varTheta\left(\zeta_k\right)\right)_n\longrightarrow\mathcal{B}\left(x,\nabla\zeta-\varTheta(\zeta)\right)\ \text{in}\ L^{\alpha'(.)}(\varLambda).$$

The above convergences, (3.5) and (3.8) imply

$$\lim_{k\to+\infty}\langle B_n\zeta_k,v\rangle=\langle B_n\zeta,v\rangle.$$

This means that $(B_n\zeta_k)_k \rightharpoonup B_n\zeta$ in $W^{1,\alpha(.)}(\Lambda)$, hence $B_n\zeta = \chi_n$. Therefore, the operator B_n is of type (M). Finally, the proof of the proposition is concluded. □

From Minty-Browder theorem [19], there exists at least one weak solution ζ_n for the approximate problem (3.1).

Step 2. A priori estimates:

Proposition 6. *The sequence* $(\zeta_k)_{k\in\mathbb{N}}$ *converges weakly in* $W^{1,\alpha(.)}(\Lambda)$ *to some function* ζ.

Proof. Using $\zeta_k \in W^{1,\alpha(.)}(\Lambda)$ as a test function in (3.2), we have

$$\int_\Lambda \mathcal{B}\left(x, \nabla\zeta_k - \Theta(\zeta_n)\right)\cdot\nabla\zeta_k dx + \int_\Lambda \mathcal{B}(x,\zeta_k)\zeta_k dx + \lambda\int_{\partial\Lambda} T_n(\zeta_k)\zeta_k d\sigma$$
$$= \int_\Lambda f_n(x,\zeta_k)\zeta_k dx - \int_\Lambda T_n\left(b(\zeta_k)\right)\zeta_k dx + \int_{\partial\Lambda} g_n\zeta_k d\sigma.$$

The same argument of the coerciveness proof of B_n yields

$$\min\left\{\frac{1}{\alpha_+}\frac{1}{2^{\alpha_+-1}}, c_1\right\}\rho_{1,\alpha(.)}(\zeta_k) \le \zeta k_1\|\zeta_k\|_{L^{\alpha(.)}(\Lambda)} + nk_2\|\zeta_k\|_{L^{\alpha(.)}(\partial\Lambda)}.$$

It follows that

$$\rho_{1,\alpha(.)}(\zeta_k) \le c_2\left[\|\zeta_k\|_{L^{\alpha(.)}(\Lambda)} + \|\zeta_k\|_{L^{\alpha(.)}(\partial\Lambda)}\right] \le c_3\|\zeta_k\|_{L^{\alpha(.)}(\Lambda)}.$$

Thus,

$$\rho_{1,\alpha(.)}(\zeta_k) \le c_4,$$

where $c_4 > 0$ is a constant independent of k. We use Proposition 2: If $\|\zeta_k\|_{1,\alpha(.)} \ge 1$, we have

$$\|\zeta_k\|_{1,\alpha(.)}^{\alpha-} \le \rho_{1,\alpha(.)}(\zeta_k) \le c_4,$$

else

$$\|\zeta_k\|_{1,\alpha(.)}^{\alpha+} \le \rho_{1,\alpha(.)}(\zeta_k) \le \|\zeta_k\|_{1,\alpha(.)}^{\alpha-} \le 1.$$

Then, $\|\zeta_k\|_{1,\alpha(.)} \le \max\left(1, (c_4)^{\frac{1}{\alpha-}}\right)$, so $\|\zeta_k\|_{W^{1,\alpha(.)}(\Lambda)} \le c_5$. Furthermore, $(\zeta_k)_n$ is a uniformly bounded sequence in a reflexive space $W^{1,\alpha(.)}(\Lambda)$, thus it exists a sub-sequence noted $(\zeta_k)_k$ of $(\zeta_k)_k$ such as

$$(\zeta_k)_k \rightharpoonup \zeta \text{ in } W^{1,\alpha(.)}(\Lambda).$$

□

From Proposition 3, we have $W^{1,\alpha(.)}(\Lambda) \hookrightarrow\hookrightarrow L^{\alpha(.)}(\Lambda)$, $W^{1,\alpha(.)}(\Lambda) \hookrightarrow\hookrightarrow L^1(\partial\Lambda)$. Then $(\zeta_k)_k \longrightarrow \zeta$ in $L^{\alpha(.)}(\Lambda)$ and $(\zeta_k)_k \longrightarrow \zeta$ in $L^1(\partial\Lambda)$, so $(\zeta_k)_k \longrightarrow \zeta$ a.e. in Λ, and $(\zeta_k)_k \longrightarrow \zeta$ a.e. in $\partial\Lambda$. We deduce that

$$\left(\mathcal{B}(x,\zeta_k)\right)_k \longrightarrow \mathcal{B}(x,\zeta) \text{ in } L^{\alpha'(.)}(\Lambda). \tag{3.14}$$

In the other side, we can prove that $\left(f_n\left(x,\zeta_k\right)\right)_k \longrightarrow f_n\left(x,\zeta\right)$ in $L^1\left(\Lambda\right)$. Indeed, for any measurable subset $E \subset \Lambda$, we can write

$$\int_E \left|f_n\left(x,\zeta_k\right)\right| \, dx \le n \text{ meas}(E),$$

then for all $\epsilon > 0$, there exists $h = \epsilon/n > 0$ such that meas$(E) < h$, we have

$$\int_E \left|f_n\left(x,\zeta_k\right)\right| \, dx \le \epsilon.$$

This means $\left(f_n\left(x,\zeta_k\right)\right)_n$ is uniformly equi-integrable.
As $\left(f\left(x,\zeta_k\right)\right)_k \longrightarrow f\left(x,\zeta\right)$ a.e. in Λ and from Vitali's theorem, we conclude:

$$\left(f\left(x,\zeta_k\right)\right)_k \longrightarrow f\left(x,\zeta\right) \text{ in } L^1\left(\Lambda\right). \tag{3.15}$$

Proposition 7. *The sequence* $\left(\nabla\zeta_n\right)_{n\in\mathbb{N}}$ *converges a.e. to* $\nabla\zeta$.

Proof. By using $(\zeta_k - \zeta) \in W^{1,\alpha(.)}\left(\Lambda\right)$ as a test function in (3.2), we have

$$\begin{aligned} &\int_\Lambda \mathcal{B}\left(x, \nabla\zeta_k - \Theta\left(\zeta_k\right)\right)\cdot\left(\nabla\zeta_k - \nabla\zeta\right) \, dx + \int_\Lambda T_n\left(b\left(\zeta_k\right)\right)\left(\zeta_k - \zeta\right) dx \\ &+ \int_\Lambda \mathcal{B}\left(x,\zeta_k\right)\left(\zeta_k - \zeta\right) dx + \lambda\int_{\partial\Lambda} T_n\left(\zeta_k\right)\left(\zeta_k - \zeta\right) d\sigma \\ = &\int_\Lambda f_n\left(x,\zeta_k\right)\left(\zeta_k - \zeta\right) dx + \int_{\partial\Lambda} g_n\left(\zeta_k - \zeta\right) d\sigma. \end{aligned}$$

Then,

$$\begin{aligned} &\int_\Lambda \left|\mathcal{B}\left(x,\nabla\zeta_k - \Theta\left(\zeta_k\right)\right) - \mathcal{B}\left(x,\nabla\zeta - \Theta\left(\zeta_k\right)\right)\right|\cdot\left|\nabla\zeta_k - \nabla\zeta\right| \, dx \\ &+ \int_\Lambda \left|T_n\left(b\left(\zeta_k\right)\right) + \Phi\left(x,\zeta_k\right)\right| \cdot\left|\zeta_k - \zeta\right| \, dx + \lambda\int_{\partial\Lambda}\left|T_n\left(\zeta_k\right)\right|\cdot\left|\zeta_k - \zeta\right| \, d\sigma \\ \le &\int_\Lambda \left|f_n\left(x,\zeta_k\right)\right|\cdot\left|\zeta_k - \zeta\right| \, dx + \int_{\partial\Lambda}\left|g_n\right|\cdot\left|\zeta_k - \zeta\right| \, d\sigma \\ &+ \int_\Lambda \left|\mathcal{B}\left(x,\nabla\zeta - \Theta\left(\zeta_k\right)\right)\right|\cdot\left|\nabla\zeta_k - \nabla\zeta\right| \, dx. \end{aligned}$$

Since $\left(\Theta\left(\zeta_k\right)\right)_k \longrightarrow \Theta\left(\zeta\right)$ in $\left(L^{\alpha(.)}\left(\Lambda\right)\right)^N$ and $\left(\zeta_k\right)_k \longrightarrow \zeta$ in $L^{\alpha(.)}\left(\Lambda\right)$. Then, $\left(\mathcal{B}\left(x,\nabla\zeta - \Theta\left(\zeta_k\right)\right)\right)_k \longrightarrow \mathcal{B}\left(x,\nabla\zeta - \Theta\left(\zeta\right)\right)$ in $L^{\alpha'(.)}\left(\Lambda\right)$. Moreover, $\left(\zeta_k\right)_k \rightharpoonup \zeta$ in $W^{1,\alpha(.)}\left(\Lambda\right)$ implies $\left(\nabla\zeta_k\right)_k \rightharpoonup \nabla\zeta$ in $L^{\alpha(.)}\left(\Lambda\right)$, consequently,

$$\int_\Lambda \left|\mathcal{B}\left(x,\nabla\zeta - \Theta\left(\zeta_k\right)\right)\right|\cdot\left|\nabla\zeta_k - \nabla\zeta\right| \, dx \longrightarrow 0. \tag{3.16}$$

On another side, from (3.8) and (3.15) we have

$$\int_{\Lambda} |T_n\left(b\left(\zeta_k\right)\right) + \mathcal{B}\left(x,\zeta_k\right)| \cdot |\zeta_k - \zeta| \; dx + \lambda \int_{\partial\Lambda} |T_n\left(\zeta_k\right)| \cdot |\zeta_k - \zeta| \; d\sigma \longrightarrow 0, \quad (3.17)$$

$$\int_{\Lambda} |f_n\left(x,\zeta_k\right)| \cdot |\zeta_k - \zeta| dx \longrightarrow 0. \quad (3.18)$$

As $(\zeta_k)_k \rightharpoonup \zeta$ in $L^1(\partial\Lambda)$ and $g_n \in L^\infty(\partial\Lambda)$, then,

$$\int_{\partial\Lambda} |g_n| \cdot |\zeta_k - \zeta| dx \longrightarrow 0. \quad (3.19)$$

From (3.14) and (3.16)–(3.19), we obtain

$$\lim_{k\to+\infty} \int_{\Lambda} \left[\mathcal{B}\left(x,\zeta_k\right) - \mathcal{B}\left(x,\zeta\right)\right]\left(\zeta_k - \zeta\right) \; dx$$
$$+ \lim_{k\to+\infty} \int_{\Lambda} \left[\mathcal{B}\left(x, \nabla\zeta_k - \Theta\left(\zeta_k\right)\right) - \mathcal{B}\left(x, \nabla\zeta - \Theta\left(\zeta_k\right)\right)\right]\left(\nabla\zeta_k - \nabla\zeta\right) \; dx = 0.$$

Since $(\zeta_k)_k \rightharpoonup \zeta$ in $W^{1,\alpha(.)}(\Lambda)$, by Lemma 4, we get $(\zeta_k)_k \longrightarrow \zeta$ in $W^{1,\alpha(.)}(\Lambda)$, so $(\nabla\zeta_k)_k \longrightarrow \nabla\zeta$ in $\left(L^{\alpha(.)}(\Lambda)\right)^N$. Finally,

$$(\nabla\zeta_k)_k \longrightarrow \nabla\zeta \text{ a.e. in } \Lambda. \quad (3.20)$$

□

Step 3. Passage to the limit

From (3.16) and (3.20), we have

$$\left(\mathcal{B}\left(x, \nabla\zeta_k - \Theta\left(\zeta_k\right)\right)\right)_k \longrightarrow \mathcal{B}\left(x, \nabla\zeta - \Theta\left(\zeta\right)\right) \text{ a.e. in } \Lambda,$$
$$\mathcal{B}\left(x, \nabla\zeta_k - \Theta\left(\zeta_k\right)\right), \; \mathcal{B}\left(x, \nabla\zeta - \Theta\left(\zeta\right)\right) \text{ in } L^{\alpha'(.)}(\Lambda),$$

and thanks to the growth condition of Φ we have

$$\|\mathcal{B}\left(x, \nabla\zeta_k - \Theta\left(\zeta_k\right)\right)\|_{\alpha'(.)} < +\infty,$$

by using Lemma 3, we conclude that

$$\left(\mathcal{B}\left(x, \nabla\zeta_k - \Theta\left(\zeta_k\right)\right)\right)_k \rightharpoonup \mathcal{B}\left(x, \nabla\zeta - \Theta(\zeta)\right) \text{ in } L^{\alpha'(.)}(\Lambda). \quad (3.21)$$

By passing to the limit in the variational formulation (3.2) and from (3.7), (3.14)–(3.15) and (3.21), we conclude

$$\int_{\Lambda} \mathcal{B}\left(x, \nabla\zeta - \Theta(\zeta)\right) \cdot \nabla v \; dx + \int_{\Lambda} \mathcal{B}\left(x,\zeta\right) v \; dx + \lambda \int_{\partial\Lambda} \zeta \; v d\sigma$$
$$= \int_{\Lambda} f\left(x,\zeta\right) \; v dx - \int_{\Lambda} b\left(\zeta\right) \; v \; dx + \int_{\partial\Lambda} g \; v d\sigma, \; \forall v \in W^{1,\alpha(.)}(\Lambda).$$

This completes the proof of Theorem 1.

4 Existence of entropy solutions

In this section, we study the existence of entropy solutions to problem (1.1).
For any $\zeta \in W^{1,\alpha(.)}(\Lambda)$, we denote by $\tau(\zeta)$ the trace of ζ on $\partial\Lambda$ in the usual sense.
In the sequel, we will identify at the boundary ζ and $\tau(\zeta)$.
We define the set

$$\mathcal{T}^{1,\alpha(.)}(\Lambda) = \left\{\zeta : \Lambda \to \mathbb{R},\ \text{measurable}/T_k(\zeta) \in W^{1,\alpha(.)}(\Lambda)\ \text{for all}\ k > 0\right\}.$$

Following the approach in [10], we have the following result:

Proposition 8. *Let $\zeta \in \mathcal{T}^{1,\alpha(.)}(\Lambda)$. Then there exists a unique measurable function $v : \Lambda \to \mathbb{R}^N$ such that $\nabla T_k(\zeta) = v\chi_{\{|\zeta|<k\}}$, for all $k > 0$, and we denote this function by $\nabla\zeta$.*
Moreover if $\zeta \in W^{1,\alpha(.)}(\Lambda)$ then $v \in \left(L^{\alpha(.)}(\Lambda)\right)^N$ and coincides with the classical gradient, that is, $v = \nabla\zeta$ in the usual sense.

In accordance with [3, 4, 20, 21, 22], we define $\mathcal{T}^{1,\alpha(.)}_{tr}(\Lambda)$ as the collection of functions $\zeta \in \mathcal{T}^{1,\alpha(.)}(\Lambda)$ for which there exists a sequence $(\zeta_n)_{n\in\mathbb{N}} \subset W^{1,\alpha(.)}(\Lambda)$ satisfying the following properties:

(C_1) $\zeta_n \to \zeta$ a.e. in Λ.

(C_2) $\nabla T_k(\zeta_n) \to \nabla T_k(\zeta)$ in $\left(L^1(\Lambda)\right)^N$ for any $k > 0$.

(C_3) There exists a measurable function v defined on $\partial\Lambda$, such that $\zeta_n \to v$ almost everywhere on $\partial\Lambda$.

The function v is referred to as the trace of ζ in the generalized framework described in [3, 4]. In what follows, the trace of any element v is the trace of $\zeta \in \mathcal{T}^{1,\alpha(.)}_{tr}(\Lambda)$ on $\partial\Lambda$ will be denoted by $tr(\zeta)$.
If $\zeta \in W^{1,\alpha(.)}(\Lambda)$, this trace coincides with the classical one, denoted by $\tau(\zeta)$.
Furthermore, for every $\zeta \in \mathcal{T}^{1,\alpha(.)}_{tr}(\Lambda)$ and for all $k > 0$, we have

$$\tau\left(T_k(\zeta)\right) = T_k\left(tr(\zeta)\right).$$

If $\varphi \in W^{1,\alpha(.)}(\Lambda) \cap L^\infty(\Lambda)$, then $(\zeta - \varphi) \in \mathcal{T}^{1,\alpha(.)}_{tr}(\Lambda)$ and

$$tr(\zeta - \varphi) = tr(\zeta) - tr(\varphi).$$

We are now ready to introduce the concept of an entropy solution to problem (1.1).

Definition 2. A measurable function ζ is said to be an entropy solution of problem (1.1), if $\zeta \in \mathcal{T}^{1,\alpha(.)}_{tr}(\Lambda)$, $\mathcal{B}(.,\zeta) \in L^1(\Lambda)$ and

$$\begin{aligned}&\int_\Lambda \mathcal{B}\left(x, \nabla\zeta - \Theta(\zeta)\right)\nabla T_k(\zeta - \varphi)\,dx + \int_\Lambda b(\zeta)(\zeta - \varphi)\,dx\\&\quad + \int_\Lambda \mathcal{B}(x,\zeta)(\zeta - \varphi)\,dx + \lambda\int_{\partial\Lambda} \zeta T_k(\zeta - \varphi)\,d\sigma\\&\le \int_\Lambda f(x,\zeta)\,T_k(\zeta - \varphi)\,dx + \int_{\partial\Lambda} gT_k(\zeta - \varphi)\,d\sigma,\end{aligned} \tag{4.1}$$

for every $\varphi \in W^{1,\alpha(.)}(\Lambda) \cap L^\infty(\Lambda)$ and for every $k > 0$.

Remark 3. It is worth noting that all the integrals appearing in the above definition are well-defined. Indeed, since $\varphi \in W^{1,\alpha(.)}(\Lambda) \cap L^{\infty}(\Lambda)$ we have $\zeta - \varphi \in \mathcal{T}_{tr}^{1,\alpha(.)}(\Lambda)$. Consequently, $T_k(\zeta - \varphi) \in W^{1,\alpha(.)}(\Lambda) \cap L^{\infty}(\Lambda)$ which ensures that the first, second, third, and fifth integrals in (4.1) are properly defined.
In addition, for the fourth and sixth integrals, we use the fact that the trace of any function $g \in W^{1,\alpha(.)}(\Lambda)$ on $\partial\Lambda$ is well defined in $L^{\alpha(.)}(\partial\Lambda)$.

Our main result is the following:

Theorem 2. *Assume that (H1)–(H5) hold trues. Then problem* (1.1) *admits at least one entropy solution.*

The proof of Theorem 2 is divided into three main steps.

Step 1. The approximate problem

We define $f_n = T_n\big(f(., \zeta_n)\big)$ and $g_n = T_n(g)$ so that the sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are bounded in $L^1(\Lambda)$ and $L^1(\partial\Lambda)$ respectively, and converge strongly to $f(., \zeta)$ and g respectively. Moreover, we have $\|f_n\|_{L^1(\Lambda)} \leq \|f\|_{L^1(\Lambda)}$ and $\|g_n\|_{L^1(\partial\Lambda)} \leq \|g\|_{L^1(\partial\Lambda)}$ for all $n \in \mathbb{N}^*$.

From Theorem 1, we know that the approximated problem (3.1) admit at least one weak solution $\zeta_n \in W^{1,\alpha(.)}(\Lambda)$ given by Definition (3.2).

Our aim is to prove that these approximated solutions ζ_n tend, as n goes to infinity, to a measurable function ζ which is an entropy solution of (1.1).

Step 2. A priori estimates

To start with, we prove the following lemma:

Lemma 6. *Let assumptions (H1)–(H4) hold true. Then for any $k > 0$:*

$$\|T_k(\zeta_n)\|_{1,\alpha(.)} \leq c_6,$$

where $c_6 = \text{const}\big(k, r(.), \alpha(.), c, g, \alpha_-, \alpha_+, meas(\Lambda)\big)$ *is a positive constant and c is the Lipschitz constant of $\Theta(.)$.*

Proof. By taking $\varphi = T_k(\zeta_n)$ in (3.2), we get

$$\begin{aligned}
&\int_{\Lambda} \mathcal{B}\big(x, \nabla\zeta_n - \Theta(\zeta_n)\big) \cdot \nabla T_k(\zeta_n)\,dx + \int_{\Lambda} T_n\big(b(\zeta_n)\big)\, T_k(\zeta_n)\,dx \\
&\quad + \int_{\Lambda} \mathcal{B}(x, \zeta_n)\, T_k(\zeta_n)\,dx + \lambda \int_{\partial\Lambda} T_n(\zeta_n)\, T_k(\zeta_n)\,d\sigma \\
&= \int_{\Lambda} f_n T_k(\zeta_n)\,dx + \int_{\partial\Lambda} g_n T_k(\zeta_n)\,d\sigma.
\end{aligned}$$

Since the second and fourth terms in the left-hand side of above equality are non-negatives, we obtain

$$\begin{aligned}
&\int_{\Lambda} \mathcal{B}\big(x, \nabla\zeta_n - \Theta(\zeta_n)\big) \nabla T_k(\zeta_n)\,dx + \int_{\Lambda} \mathcal{B}(x, \zeta_n)\, T_k(\zeta_n)\,dx \\
&\leq \int_{\Lambda} |f_n|\, |T_k(\zeta_n)| dx + \int_{\partial\Lambda} |g_n|\, |T_k(\zeta_n)| d\sigma.
\end{aligned}$$

As $\|f_n\|_{L^1(\Lambda)} \leq \|f\|_{L^1(\Lambda)}$ and $\|g_n\|_{L^1(\partial\Lambda)} \leq \|g\|_{L^1(\partial\Lambda)}$ and $|T_k(\zeta_n)| \leq k$, the above inequality implies

$$\int_\Lambda \mathcal{B}\left(x, \nabla\zeta_n - \Theta(\zeta_n)\right) \nabla T_k(\zeta_n)\, dx + \int_\Lambda \mathcal{B}(x, \zeta_n)\, T_k(\zeta_n)\, dx \leq kc_3, \tag{4.2}$$

where $c_3 = \text{const}\left(r(.), \alpha(.), g, \text{meas}(\Lambda)\right) > 0$. Since

$$\int_\Lambda |\zeta_n|^{\alpha(x)-2}\zeta_n T_k(\zeta_n)\, dx = \int_{\{|\zeta_n|\leq k\}} |(\zeta_n)|^{\alpha(x)} dx + \int_{\{|\zeta_n|>k\}} k\ |(\zeta_n)|^{\alpha(x)-1} dx$$
$$\geq \int_{\{|\zeta_n|\leq k\}} |T_k(\zeta_n)|^{\alpha(x)} dx + \int_{\{|\zeta_n|>k\}} k^{\alpha(x)} dx \geq \int_\Lambda |T_k(\zeta_n)|^{\alpha(x)} dx,$$

we deduce from (4.2) that

$$\int_\Lambda \mathcal{B}\Big(x, \nabla T_k(\zeta_n) - \Theta\left(T_k(\zeta_n)\right)\Big) \nabla T_k(\zeta_n)\ \ dx + \int_\Lambda |T_k(\zeta_n)|^{\alpha(x)} dx \leq k\ c_3.$$

Following the same proof as that of the coerciveness of B_n, there exists $c_4 = \text{const}(c, \alpha_-, \alpha_+) > 0$ where c is the Lipschitz constant of $\Theta(.)$ such that

$$\rho_{1,\alpha(.)}\left(T_k(\zeta_n)\right) \leq k\ c_3\ c_4,$$

then,

$$\rho_{1,\alpha(.)}\left(T_k(\zeta_n)\right) \leq k\ c_5, \tag{4.3}$$

where $c_5 = \text{const}\left(c, g, r(.), \alpha(.), \alpha_-, \alpha_+, \text{meas}(\Lambda)\right)$. From Proposition 2 we have: If $\|T_k(\zeta_n)\|_{1,\alpha(.)} \geq 1$, we get

$$\|T_k(\zeta_n)\|_{1,\alpha(.)}^{\alpha-} \leq \rho_{1,\alpha(.)}\left(T_k(\zeta_n)\right) \leq kc_5,$$

else

$$\|T_k(\zeta_n)\|_{1,\alpha(.)}^{\alpha+} \leq \rho_{1,\alpha(.)}\left(T_k(\zeta_n)\right) \leq \|T_k(\zeta_n)\|_{1,\alpha(.)}^{\alpha-} \leq 1.$$

Then, $\|T_k(\zeta_n)\|_{1,\alpha(.)} \leq \max\left(1, (kc_5)^{1/\alpha_-}\right)$, thus

$$\|T_k(\zeta_n)\|_{1,\alpha(.)} \leq c_6, \tag{4.4}$$

where $c_6 = \text{const}\left(k, r(.), \alpha(.), c, g, \alpha_-, \alpha_+, \text{meas}(\Lambda)\right) > 0$.

Finally, the proof of this Lemma is completed. □

It follows that, for any $k > 0$, the sequence $\left(T_k(\zeta_n)\right)_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,\alpha(.)}(\Lambda)$. Then, up to a sub-sequence, we can assume that for any $k > 0$,

$$T_k(\zeta_n) \rightharpoonup v_k \qquad \text{in} \quad W^{1,\alpha(.)}(\Lambda)$$

and by the compact embedding $W^{1,\alpha(.)}(\Lambda) \hookrightarrow\hookrightarrow L^{\alpha(.)}(\Lambda)$, we obtain

$$T_k(\zeta_n) \to v_k \qquad \text{in} \quad L^{\alpha(.)}(\Lambda) \qquad \text{and } T_k(\zeta_n) \to v_k \quad \text{a.e. in } \Lambda.$$

We will now prove the following proposition:

Proposition 9. *The sequence $(\zeta_n)_{n\in\mathbb{N}}$ converges in measure to some function ζ.*

To prove this, it's enough to prove that ζ_n is a Cauchy sequence in measure.

Proof. Let $k > 0$ be large enough. Using $T_k(\zeta_n)$ as a test function in (3.2), from (4.3), we get

$$\rho_{1,\alpha(.)}\left(T_k(\zeta_n)\right) \leq k\ c_5.$$

As

$$\begin{aligned}\rho_{1,\alpha(.)}\left(T_k(\zeta_n)\right) &\geq \int_{\{|\zeta_n|>k\}} |T_k(\zeta_n)|^{\alpha(x)}dx + \int_{\{|\zeta_n|>k\}} |\nabla T_k(\zeta_n)|^{\alpha(x)}dx \\ &= \int_{\{|\zeta_n|>k\}} k^{\alpha(x)}dx, \quad \text{then} \int_{\{|\zeta_n|>k\}} k^{\alpha(x)}dx \leq kc_5.\end{aligned}$$

It follows that

$$\text{meas}\left\{|\zeta_n| > k\right\} \leq kc_7^{1-\alpha_-}.$$

Since $1-\alpha_- < 0$ and for every fixed $n > 0$, we deduce that

$$\text{meas}\left\{|\zeta_n| > k\right\} \to 0 \text{ as } k \to +\infty.$$

Let $\epsilon > 0$, we choose $k > k(\epsilon)$, we conclude that

$$\text{meas}\left(\left\{|\zeta_n| > k\right\}\right) \leq \frac{\epsilon}{3} \qquad \text{and} \qquad \text{meas}\left(\left\{|\zeta_m| > k\right\}\right) \leq \frac{\epsilon}{3}. \tag{4.5}$$

Moreover, for every fixed $n \in \mathbb{N}^*$ and every positive $k > 0$, we know that

$$\left\{|\zeta_n - \zeta_m| > t\right\} \subset \left\{\left\{|\zeta_n| > k\right\} \cup \left\{|\zeta_m| > k\right\} \cup \left\{|T_k(\zeta_n) - T_k(\zeta_m)| > t\right\}\right\}$$

and hence

$$\begin{aligned}\text{meas}\ \left(\left\{|\zeta_n - \zeta_m| > t\right\}\right) \leq &\text{meas}\ \left(\left\{|\zeta_n| > k\right\}\right) + \text{meas}\ \left(\left\{|\zeta_m| > k\right\}\right) \\ &+ \text{meas}\ \left(\left\{|T_k(\zeta_n) - T_k(\zeta_m)| > t\right\}\right).\end{aligned} \tag{4.6}$$

Since $T_k(\zeta_n)$ converges strongly in the Banach space $L^{\alpha(\cdot)}(\Lambda)$, then $T_k(\zeta_n)$ is a Cauchy sequence in $L^{\alpha(\cdot)}(\Lambda)$.

So, $\forall\epsilon > 0, \forall t > 0, \exists n_0(t,\epsilon) \in \mathbb{N}^*, \forall n, m \geq n_0(t,\epsilon)$

$$\text{meas}\ \left(\left\{|T_k(\zeta_n) - T_k(\zeta_m)| > t\right\}\right) \leq \frac{1}{t^{\alpha_-}} \int_\Lambda |T_k(\zeta_n) - T_k(\zeta_m)|^{\alpha_-}dx \leq \frac{\epsilon}{3}. \tag{4.7}$$

Finally, from (4.5), (4.6) and (4.7), it follows that

$$\text{meas}\left(\left\{|\zeta_n - \zeta_m| > t\right\}\right) \leq \epsilon \quad \forall n, m \geq n_0(t,\epsilon).$$

This shows that the sequence $(\zeta_n)_{n\in\mathbb{N}}$ is Cauchy in measure and therefore converges almost everywhere to a measurable function u. □

Therefore,

$$T_k(\zeta_n) \rightharpoonup T_k(\zeta) \quad \text{in } W^{1,\alpha(.)}(\Lambda),$$

$$T_k(\zeta_n) \to T_k(\zeta) \quad \text{in} \quad L^{\alpha(.)}(\Lambda) \text{ and } T_k(\zeta_n) \to T_k(\zeta) \quad \text{a.e. in} \quad \Lambda.$$

We can prove the two following propositions:

Proposition 10. *The sequence* $(\nabla\zeta_n)_{n\in\mathbb{N}}$ *converges in measure to the weak gradient* $\nabla\zeta$.

Proposition 11. *The sequence* $(\zeta_n)_{n\in\mathbb{N}}$ *converges a.e. on* $\partial\Lambda$ *to some function* v.

Step 3. Passage to limit

Proposition 12. *The function* ζ *is an entropy solution of the problem* (1.1).

Proof. Since the sequence $\left(\nabla T_k(\zeta_n)\right)_{n\in\mathbb{N}}$ converges in measure to $\nabla T_k(\zeta)$, then by (4.4) and Lemma 2, we get

$$\nabla T_k(\zeta_n) \to \nabla T_k(\zeta) \quad \text{in } \left(L^1(\Lambda)\right)^N. \tag{4.8}$$

Consequently, from Propositions 9, 11 and (4.8), we get that $\zeta \in \mathcal{T}_{tr}^{1,\alpha(.)}(\Lambda)$. Let $\varphi \in W^{1,\alpha(.)}(\Lambda) \cap L^\infty(\Lambda)$, we take $v = T_k(\zeta_n - \varphi)$ as test function in (3.2) to get

$$\begin{aligned} &\int_\Lambda \mathcal{B}\left(x, \nabla\zeta_n - \Theta(\zeta_n)\right) \nabla T_k(\zeta_n - \varphi)\, dx + \int_\Lambda \mathcal{B}(x, \zeta_n) T_k(\zeta_n - \varphi)\, dx \\ &+ \int_\Lambda T_n\left(b(\zeta_n)\right) T_k(\zeta_n - \varphi)\, dx + \lambda \int_{\partial\Lambda} T_n(\zeta_n) T_k(\zeta_n - \varphi)\, d\sigma \\ = &\int_\Lambda T_n\left(f(x, \zeta_n)\right) T_k(\zeta_n - \varphi)\, dx + \int_{\partial\Lambda} T_n(g) T_k(\zeta_n - \varphi)\, d\sigma. \end{aligned} \tag{4.9}$$

Let $\overline{k} = k + \|\varphi\|_\infty$, we have

$$\begin{aligned} &\int_\Lambda \mathcal{B}\left(x, \nabla\zeta_n - \Theta(\zeta_n)\right) \nabla T_k(\zeta_n - \varphi)\, dx \\ = &\int_\Lambda \mathcal{B}\left(x, \nabla T_{\overline{k}}(\zeta_n) - \Theta\left(T_{\overline{k}}(\zeta_n)\right)\right) \nabla T_k\left(T_{\overline{k}}(\zeta_n) - \varphi\right) \chi_{\{|\zeta_n| \le \overline{k}\}}\, dx \\ = &\int_\Lambda \mathcal{B}\left(x, \nabla T_{\overline{k}}(\zeta_n) - \Theta(\zeta_n)\right) \nabla T_{\overline{k}}(\zeta_n) \chi_{\Lambda(n,\overline{k})}\, dx \\ &- \int_\Lambda \mathcal{B}\left(x, \nabla T_{\overline{k}}(\zeta_n) - \Theta(\zeta_n)\right) \nabla\varphi \chi_{\Lambda(n,\overline{k})}\, dx, \end{aligned}$$

where $\chi_{\Lambda(n,\overline{k})}$ is the characteristic function of the measurable set

$$\Lambda\left(n, \overline{k}\right) = \left\{|T_{\overline{k}}(\zeta_n) - \varphi| \le k\right\}.$$

The inequality (4.9) can be written as

$$\begin{aligned}
&\int_\Lambda \left(\mathcal{B}\left(x, \nabla T_{\overline{k}}\left(\zeta_n\right) - \Theta\left(T_{\overline{k}}\left(\zeta_n\right)\right)\right) \nabla T_{\overline{k}}\left(\zeta_n\right) + \frac{1}{\alpha(x)} \left|\Theta\left(T_{\overline{k}}\left(\zeta_n\right)\right)\right|^{\alpha(x)} \right) \chi_{\Lambda\left(n,\overline{k}\right)} dx \\
&\quad - \int_\Lambda \mathcal{B}\left(x, \nabla T_{\overline{k}}\left(\zeta_n\right) - \Theta\left(T_{\overline{k}}(\zeta_n)\right)\right) \nabla\varphi \ \chi_{\Lambda\left(n,\overline{k}\right)} \ dx \\
&\quad + \int_\Lambda |\zeta_n|^{\alpha(x)-2} \zeta_n T_k\left(\zeta_n - \varphi\right) \ dx + \int_\Lambda T_n\left(b(\zeta_n)\right) T_k\left(\zeta_n - \varphi\right) \ dx \\
&\quad + \lambda \int_{\partial\Lambda} T_n\left(\zeta_n\right) T_k\left(\zeta_n - \varphi\right) \ d\sigma \\
&= \int_\Lambda T_n\left(f\left(x, \zeta_n\right)\right) T_k\left(\zeta_n - \varphi\right) \ dx + \int_{\partial\Lambda} T_n\left(g\right) T_k\left(\zeta_n - \varphi\right) \ d\sigma \\
&\quad + \int_\Lambda \frac{1}{\alpha(x)} \left|\Theta\left(T_{\overline{k}}\left(\zeta_n\right)\right)\right|^{\alpha(x)} \chi_{\Lambda\left(n,\overline{k}\right)} \ dx. \qquad (4.10)
\end{aligned}$$

Since

$$\begin{aligned}
&\nabla T_{\overline{k}}\left(\zeta_n\right) \rightharpoonup \nabla T_{\overline{k}}(\zeta) \ \text{ in } \ \left(L^{\alpha(.)}(\Lambda)\right)^N, \\
&\Theta\left(T_{\overline{k}}\left(\zeta_n\right)\right) \longrightarrow \Theta\left(T_{\overline{k}}(\zeta)\right) \ \text{ in } \ \left(L^{\alpha(.)}(\Lambda)\right)^N, \qquad (4.11)
\end{aligned}$$

then,

$$\nabla T_{\overline{k}}\left(\zeta_n\right) - \Theta\left(T_{\overline{k}}\left(\zeta_n\right)\right) \rightharpoonup \nabla T_{\overline{k}}(\zeta) - \Theta\left(T_{\overline{k}}(\zeta)\right) \ \text{ in } \ \left(L^{\alpha(.)}(\Lambda)\right)^N.$$

Thus,

$$\mathcal{B}\left(x, \nabla T_{\overline{k}}\left(\zeta_n\right) - \Theta\left(T_{\overline{k}}\left(\zeta_n\right)\right)\right) \rightharpoonup \mathcal{B}\left(x, \nabla T_{\overline{k}}\left(\zeta\right) - \Theta\left(T_{\overline{k}}(\zeta)\right)\right) \ \text{ in } L^{\alpha'(.)}\left(\Lambda\right). \qquad (4.12)$$

Furthermore, $\nabla\varphi\chi_{\Lambda\left(n,\overline{k}\right)} \longrightarrow \nabla\varphi\chi_{\Lambda\left(\overline{k}\right)}$ in $L^{\alpha(.)}(\Lambda)$ with $\Lambda\left(\overline{k}\right) = \left\{|T_{\overline{k}}(\zeta) - \varphi| \leq k\right\}$. From this and (4.12), we obtain

$$\begin{aligned}
&\lim_{n\to+\infty} \int_\Lambda \mathcal{B}\left(x, \nabla T_{\overline{k}}(\zeta_n) - \Theta\left(T_{\overline{k}}(\zeta_n)\right)\right) \nabla\varphi\chi_{\Lambda(n,\overline{k})} \ dx \\
&= \int_\Lambda \mathcal{B}\left(x, \nabla T_{\overline{k}}(\zeta) - \Theta\left(T_{\overline{k}}(\zeta)\right)\right) \nabla\varphi\chi_{\Lambda\left(\overline{k}\right)} \ dx.
\end{aligned}$$

According to (H_2) and the properties of the truncation function, we get

$$\left|\Theta\left(T_{\overline{k}}(\zeta_n)\right)\right|^{\alpha(x)} \leq \left(c\overline{k}\right)^{\alpha(x)}.$$

Using (4.11) and the Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to+\infty} \int_\Lambda \frac{1}{\alpha(x)} \left|\Theta\left(T_{\overline{k}}(\zeta_n)\right)\right|^{\alpha(x)} \chi_{\Lambda\left(n,\overline{k}\right)} \ dx = \int_\Lambda \frac{1}{\alpha(x)} \left|\Theta\left(T_{\overline{k}}(\zeta)\right)\right|^{\alpha(x)} \chi_{\Lambda\left(\overline{k}\right)} \ dx.$$

Now, since

$$\left(\mathcal{B}\left(x, \nabla T_{\overline{k}}(\zeta_n) - \Theta\left(T_{\overline{k}}(\zeta_n)\right)\right) \nabla T_{\overline{k}}(\zeta_n) + \frac{1}{\alpha(x)} |\Theta\left(T_{\overline{k}}(\zeta_n)\right)|^{\alpha(x)}\right) \chi_{\Lambda(n,\overline{k})} \geq 0$$

a.e. in Λ,

we obtain by using Fatou's lemma

$$\begin{aligned}&\int_\Lambda \left(\mathcal{B}(x, \nabla T_{\overline{k}}(\zeta) - \Theta(T_{\overline{k}}(\zeta)))\nabla T_{\overline{k}}(\zeta) + \frac{1}{\alpha(x)} |\Theta(T_{\overline{k}}(\zeta))|^{\alpha(x)}\right) \chi_{\Lambda(\overline{k})}\, dx \\ &\leq \liminf_{n\to\infty} \left(\int_\Lambda \left(\mathcal{B}(x, \nabla T_{\overline{k}}(\zeta_n) - \Theta(T_{\overline{k}}(\zeta_n)))\nabla T_{\overline{k}}(\zeta_n)\right.\right. \\ &\quad \left.\left. + \frac{1}{\alpha(x)} |\Theta(T_{\overline{k}}(\zeta_n))|^{\alpha(x)}\right) \chi_{\Lambda(n,\overline{k})}\, dx\right).\end{aligned}$$

Letting $n \to \infty$ in the equality (4.10) to conclude that ζ is an entropy solution of problem (1.1). □

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